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ON THE HODGE GROUPS OF SOME ABELIAN VARIETIES

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In this paper we shall consider the Hodge group $Hg(A)$ of an abelian variety A which is a product of elliptic curves and give the following result:
For non-isogenous elliptic curves E_i $(i=1, 2, \cdots, n)$,

$$Hg(E_1\times \cdots \times E_n)=Hg(E_1)\times \cdots \times Hg(E_n).$$

Secondly we shall show that the Serre's conjecture about the relation between Hodge groups and l-adic representations is true if the abelian variety concerned are of the above type (under some condition concerning l-adic representations).

1. Let $A=\boldsymbol{C}^g/L$ be a complex torus of dimension g. Denoting by V the $\boldsymbol{Q}$-linear span of L, we obtain the following data:

(i) a $2g$-dimensional $\boldsymbol{Q}$-vector space V;
(ii) a complex structure on $V_{\boldsymbol{R}}=V\otimes_{\boldsymbol{Q}}\boldsymbol{R}$;
(iii) a lattice $L\subset V$.

The datum (ii) is equivalent to either of the following data:

(ii′) an endomorphism $J: V_{\boldsymbol{R}}\to V_{\boldsymbol{R}}$ such that $J^2=-1$;
(ii″) a homomorphism of algebraic groups $\phi: T\to GL(V)$ defined over $\boldsymbol{R}$ where T is the compact 1-dimensional torus over $\boldsymbol{R}$, i.e., $T_{\boldsymbol{R}}=\{z\in\boldsymbol{C} \mid |z|=1\}$; and such that ϕ, as a representation of $\boldsymbol{G}_m$, has weights $+1$ and -1, each with multiplicity g.

Now we define the Hodge group of A, written $Hg(A)$, as the smallest subgroup of $GL(V)$ defined over $\boldsymbol{Q}$ and containing $\phi(T)$.

From this definition we have that for any complex tori $A_1, A_2, Hg(A_1\times A_2)\subset Hg(A_1)\times Hg(A_2)$, and that the projection to the i-th factor $Hg(A_1)\times Hg(A_2)\to Hg(A_i)$ is surjective when restricted to $Hg(A_1\times A_2)$ $(i=1, 2)$.

It is known that $Hg(A)$ is always connected reductive, with compact center, and its semi-simple part is of Hermitian type if A is an abelian variety (cf. [1]).

2. In this section we describe the Hodge group of an abelian variety which is a product of elliptic curves.

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Let E be an elliptic curve. It is well known that $Hg(E)$ is a 1-dimensional torus if E is of CM-type (i. e., $\phi(T)=Hg(E)$ is defined over $\boldsymbol{Q}$), and that $Hg(E)=SL_2$ if E is not of CM-type (cf. [2], § 2).

Now our result is:

PROPOSITION. *Let $E_i=V_{i\boldsymbol{R}}/L_i$ $(i=1, 2, \cdots, n)$ be non-isogenous elliptic curves, then*

$$Hg(E_1\times\cdots\times E_n)=Hg(E_1)\times\cdots\times Hg(E_n).$$

Proof. Put $H_i=Hg(E_i)$ $(i=1, 2, \cdots, n)$, $H=Hg(E_1\times\cdots\times E_n)$, and let $\phi_i: T\rightarrow GL(V_i)$ be the map which gives the complex structure on E_i $(i=1, 2, \cdots, n)$.

We devide the proof into three cases.

Firstly we suppose that all E_i are of CM-type with CM-field $K_i=\boldsymbol{Q}(\sqrt{-d_i})$ (d_i is a square-free positive integer). We suppose that $\phi_i(\sqrt{-1})$ is given as the multiplication by $\sqrt{-d_i}\otimes 1/\sqrt{d_i}$ on $V_i\otimes\boldsymbol{R}=K_i\otimes\boldsymbol{R}$ $(i=1, \cdots, n)$ since other cases where some $\phi_i(\sqrt{-1})$ is the multiplication by $-\sqrt{-d_i}\otimes 1/\sqrt{d_i}$ can be treated similarly (cf. [6], Chapter 5). Then taking 1, $\sqrt{-d_i}$ as a $\boldsymbol{Q}$-basis of K_i, we can represent $\phi_i(z)$ $(z=a+b\sqrt{-1},\ |z|=1,\ a, b\in\boldsymbol{Q})$ as:

$$\phi_i(z)=\begin{pmatrix} a & b\sqrt{1/d_i} \\ -bd_i\sqrt{1/d_i} & a \end{pmatrix}=A_i^{-1}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}A_i$$

where A_i is a constant matrix.

As $H\subset H_1\times\cdots\times H_n\cong\boldsymbol{G}_m^n$, H is defined by equations of following form if we identify H as a subgroup of $\boldsymbol{G}_m^n$ (via conjugation by A_i on the i-th component):

$$(*)\qquad \prod_{i=1}^{n} X_i^{e_i}=1 \qquad (e_i\in\boldsymbol{Z})$$

where X_i is the i-th coordinate of $\boldsymbol{G}_m^n$.

We want to prove $e_i=0$ $(i=1, \cdots, n)$.

As $(z, \cdots, z)\in\boldsymbol{G}_m^n$ (corresponding to $(\phi_1(z), \cdots, \phi_n(z))\in H$) satisfies (*), we have $\sum_{i=1}^{n} e_i=0$.

As H is defined over $\boldsymbol{Q}$, $x^\sigma\in H$ for all $x\in H$ and for all $\sigma\in\mathrm{Aut}(\boldsymbol{C})$. Take primes $p_1, \cdots, p_s$ which devide some d_i and take $\sigma\in\mathrm{Aut}(\boldsymbol{C})$ such that $\sigma(\sqrt{p_j})=-\sqrt{p_j}$ $(j=1, \cdots, s)$, $\sigma(\sqrt{q})=\sqrt{q}$ (for other primes q). Putting the point of $\boldsymbol{G}_m^n$ corresponding to $(\phi_1(z)^\sigma, \cdots, \phi_n(z)^\sigma)\in H$ in (*) (with $z=a+b\sqrt{-1},\ a, b\in\boldsymbol{Q},\ |z|=1$, hence $\phi_i(z)^\sigma=\phi_i(z)^{\varepsilon_i}$ where $\varepsilon_i=\pm 1$), we have:

$$(**)\qquad \sum_{d_i\in P_0} e_i=0$$

where $P=\{p_1, \cdots, p_s\}$ and $P_0=\{d_i \mid \sum_{p\in P}\mathrm{ord}_p(d_i): \text{odd}\}$. Suppose $p_1, \cdots, p_t$ are all the distinct primes appearing in some d_i and suppose integers e_i are given so as to satisfy (**) for any subset $P\subset\{p_1, \cdots, p_t\}$. We use the induction on t to prove $e_i=0$ $(i=1, \cdots, n)$.

The case $t=1$ is trivial.

Let $d_1, \cdots, d_m$ be all d_ι such that $p_t \nmid d_i$ (by renumbering d_ι, if necessary). In order to use the induction hypothesis we must prove the following equation:

$$\sum_{\substack{d_i \in P_0 \\ 1 \leqq i \leqq m}} e_i = 0$$

where P is a subset of $\{p_1, \cdots, p_{t-1}\}$.

But this follows easily from the following equations:

$$\sum_{d_i \in \{p_t\}_e} e_i = \sum_{1 \leqq i \leqq m} e_i = \sum_{\substack{d_i \in P_0 \\ 1 \leqq i \leqq m}} e_\iota + \sum_{\substack{d_i \in P_e \\ 1 \leqq i \leqq m}} e_i = 0\,,$$

$$\sum_{d_i \in P_0} e_i = \sum_{\substack{d_i \in P_0 \\ 1 \leqq i \leqq m}} e_\iota + \sum_{\substack{d_i \in P_0 \\ m+1 \leqq i \leqq n}} e_i = 0\,,$$

$$\sum_{d_i \in (P \cup \{p_t\})_e} e_i = \sum_{\substack{d_i \in P_e \\ 1 \leqq i \leqq m}} e_\iota + \sum_{\substack{d_i \in P_0 \\ m+1 \leqq i \leqq n}} e_i = 0$$

where $P_e = \{d_\iota \mid \sum_{p \in P} \mathrm{ord}_p(d_\iota) : \text{even}\}$.

Therefore we have $e_1 = \cdots = e_m = 0$.

Applying the same argument to other $p_\jmath$, we see that the proposition is true in this case.

Secondly suppose that all E_i are not of CM-type. Then $H_i = SL(V_\iota)$ $(i=1, \cdots, n)$. We use the induction on n and suppose that the proposition is true for $n-1$.

Suppose $H \neq H_1 \times \cdots \times H_n$ and write $H = H' \cdot D$ (almost direct product) where H' is a semi-simple subgroup and D is a central torus both defined over $\boldsymbol{Q}$. Let $p: H \to H_1 \times \cdots \times H_{n-1}$ be the projection to $1 \times \cdots \times n-1$ factor, then $p(H') = p([H, H]) = [p(H), p(H)] = H_1 \times \cdots \times H_{n-1}$ since $p(H) = Hg(E_1 \times \cdots \times E_{n-1}) = H_1 \times \cdots \times H_{n-1}$ by induction hypothesis. As the ranks of H' and $H_1 \times \cdots \times H_{n-1}$ are equal and as $H_1 \times \cdots \times H_{n-1}$ is simply-connected, we see that $p|H'$ is an isomorphism. Let

$$\gamma: \ H_1 \times \cdots \times H_{n-1} \xrightarrow{(p|H')^{-1}} H' \subset H_1 \times \cdots \times H_n \xrightarrow{pr_n} H_n\,.$$

Then γ is a non-trivial homomorphism ($pr_n(H') = H_n$ similarly as above) and H' contains the following group (which is equal to H' from dimension counting):

$$\{(x_1, \cdots, x_{n-1}, \gamma(x_1, \cdots, x_{n-1})) \mid x_i \in H_\iota,\ i=1, \cdots, n-1\}\,.$$

As D centralizes H' and as D is contained in $SL(V_1) \times \cdots \times SL(V_n)$, we see $D = \{1\}$. Put $\gamma_i = \gamma | H_\iota$, then we see that γ_ι is either a trivial homomorphism or an isomorphism as both H_ι and H_n are isomorphic to SL_2. Let $\gamma_1, \cdots, \gamma_m$ be non-trivial ones (by renumbering E_i if necessary). As $\gamma(x_1, 1, \cdots, 1, x_\iota, 1, \cdots, 1) = \gamma_1(x_1)\gamma_i(x_\iota) = \gamma_i(x_\iota)\gamma_1(x_1)$ and as $\gamma_i(x_\iota)$ runs over all the elements of $SL(V_\iota)$ for $i \leqq m$, we see $m=1$. Hence:

$$H = \{(x_1, \cdots, x_{n-1}, \gamma_1(x_1)) \mid x_i \in H_\iota,\ i=1, \cdots, n-1\}\,.$$

As $H_1 \cong H_n \cong SL_2$, and as $\mathrm{Aut}(SL_2)=\mathrm{Inn}\cdot\mathrm{aut}(SL_2)$, there exists an isomorphism $\alpha: V_1 \to V_n$ defined over $\boldsymbol{Q}$ (as γ_1 is defined over $\boldsymbol{Q}$) such that γ_1 is induced from α.

We have $\gamma_1 \circ pr_1 = pr_n$ from the above presentation of H. Applying this formula to $(\phi_1(\sqrt{-1}), \cdots, \phi_n(\sqrt{-1}))$, we have $\phi_n(\sqrt{-1})=\gamma_1(\phi_1(\sqrt{-1}))=\alpha\circ\phi_1(\sqrt{-1})\circ\alpha^{-1}$, i.e., α is $\boldsymbol{C}$-linear. Therefore some integral multiple of α defines an isogeny $E_1 \to E_n$. This is a contradiction.

Lastly suppose $E_1, \cdots, E_m$ are of CM-type and $E_{m+1}, \cdots, E_n$ are not of CM-type. Write $H=H'\cdot D$ as before.

Denote by p (q resp.) the projection to $1\times \cdots \times m$ factor ($m+1\times \cdots \times n$ factor resp.). Then $p(D)=p(H)=Hg(E_1\times \cdots \times E_m)=H_1\times \cdots \times H_m$ as H' has no non-trivial character and as $H_i \cong \boldsymbol{G}_m$ ($i \leqq m$), and $q(H')=q([H, H])=[q(H), q(H)]= E_{m+1}\times \cdots \times E_n$ as $q(H)=Hg(E_{m+1}\times \cdots \times E_n)=H_{m+1}\times \cdots \times H_n$.

Hence $\dim H=\dim D+\dim H' \geqq \sum_{i=1}^{n} \dim (H_i)$.

Therefore we have $H=H_1\times \cdots \times H_n$.

3. Remarks.

If E_1 and E_2 are isogenous elliptic curves, the Hodge group of $E_1\times E_2$ is obtained as:

$$Hg(E_1\times E_2)=\{(x, \tilde{\lambda}(x)) \mid x \in Hg(E_1)\}$$

where $\lambda: E_1 \to E_2$ is an isogeny viewed as a map $V_1 \to V_2$, and $\tilde{\lambda}: GL(V_1) \to GL(V_2)$ is the map induced from λ. To see this, we first observe that the right hand side contains the group $\phi(T)$ which gives the complex structure on $E_1\times E_2$ and is defined over $\boldsymbol{Q}$. Hence the left hand side is contained in the right hand side. On the other hand as $\dim Hg(E_1\times E_2) \geqq \dim Hg(E_1)$, which follows from the surjectivity of pr_1 restricted to $Hg(E_1\times E_2)$, we have the desired equality.

For the product of elliptic curves (isogenous or not), its Hodge group can be obtained as follows:

Let $E_j^{(i)}$ ($i=1, \cdots, n$, $j=1, \cdots, m_i$) be elliptic curves such that $E_j^{(i)}$, $E_{j'}^{(i)}$ are isogenous and $E_j^{(i)}$, $E_{j'}^{(i')}$ ($i \neq i'$) are non-isogenous. Then,

$$Hg(\prod_{i,j} E_j^{(i)}) \cong \prod_{i=1}^{n} \Delta^{m_i}(Hg(E_1^{(i)}))$$

where

$$\Delta^m(H)=\text{the diagonal subgroup of } H^m.$$

We outline the proof of this and we use induction on the number of elliptic curves. Let $E_j^{(i)}$ be as above and let E be another elliptic curve. If E is isogenous to some $E_j^{(i)}$, say $E_1^{(n)}$, then one can prove by the similar argument as before that $Hg(\prod_{j=1}^{m_n} E_j^{(n)}\times E) \cong \Delta^{m_n+1}(Hg(E_1^{(n)}))$. From the fact $Hg(\prod_{i,j} E_j^{(i)}\times E) \subset Hg(\prod_{1\leqq i\leqq n-1, j} E_j^{(i)}) \times Hg(\prod_j E_j^{(n)}\times E)$, $\dim Hg(\prod_{i,j} E_j^{(i)}\times E) \geqq \dim Hg(\prod_{i,j} E_j^{(i)})$, and from the induction hypothesis on $\prod_{i,j} E_j^{(i)}$ we have disired isomorphism by dimension counting. In the case when E is non-isogenous to any $E_j^{(i)}$, the case when $m_i=1$ $(i=1, \cdots, n)$ has been proved in the proposition and the case when some $m_i>1$ may be reduced to the above case by interchanging $E_1^{(i)}$ and E.

There is a conjecture due to Serre, which states the relation between the Hodge group and l-adic representation as follows:

Let A be an abelian variety of dimension g defined over an algebraic number field K of finite degree. Let

$$\rho_l: \quad G=\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Aut}(T_l(A)) \cong GL_{2g}(\boldsymbol{Z}_l)$$

be the l-adic representation attached to A where $\bar{K}$ is the algebraic closure of K and $T_l(A)$ is the Tate module of A.

Let $\mathfrak{g}_l$ be the Lie algebra of l-adic Lie group $\rho_l(G)$, and let $\mathfrak{h}$ be the Lie algebra of $Hg(A)$, then

$$\mathfrak{g}_l \cap \mathfrak{sl}_{2g} = \mathfrak{h} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \quad (?) .$$

In the case when A is the product of two elliptic curves (with some restriction concerning l-adic representations) this conjecture is verified easily by the proposition of this paper and by the theorems 6, 7 of [5], § 6.

In the case when A is the product of elliptic curves, this conjecture is also true as the proposition and the following statement show (in what follows all elliptic curves are assumed to be defined over an algebraic number field K).

(i) Let E_i $(i=1, \cdots, n)$ be elliptic curves not of CM-type such that their associated l-adic representations $\rho_l^{(i)}$ are not equivalent over any finite extension of K. Denoting by $\mathfrak{g}_l$ the Lie algebra of Galois group represented by $E_1 \times \cdots \times E_n$, and by $\mathfrak{g}_l^{(i)}$ the Lie algebra of $\rho_l^{(i)}(G)$, then

$$\mathfrak{g}_l = \{(x_1, \cdots, x_n) \mid x_i \in \mathfrak{g}_l^{(i)}, \ \mathrm{Tr}\, x_i = \mathrm{Tr}\, x_j\} .$$

(ii) Let E_i $(i=1, \cdots, n)$ be non-isogenous CM-type elliptic curves. By using the same notations as in (i), we have the same result as in (i).

The proof of (i) is almost the same as that of the proposition. In fact using induction on n, under the hypothesis that (i) is false for n, the projection $\mathfrak{g}_l \to \{(x_1, \cdots, x_{n-1}) \mid x_i \in \mathfrak{g}_l^{(i)}, \ \mathrm{Tr}\, x_i = \mathrm{Tr}\, x_j\}$ is an isomorphism by the similar argument when restricted to the semi-simple parts and on the radical parts it may be verified easily by taking traces on both sides (cf. [3] Chapter 1, § 1, 2). From this one can prove that some $\mathfrak{g}_l^{(i)}$ is isomorphic to $\mathfrak{g}_l^{(n)}$ by the same argument as in the proof of the proposition. This contradicts to the hypothesis in (i), hence we have the disired result.

For the proof of (ii) we use the result of complex multiplication. Let $K_i = \boldsymbol{Q}(\sqrt{-d_i})$ be the CM-field corresponding to E_i $(i=1, \cdots, n)$. Put $L = K_1 \cdots K_n$. Identify K_i with the Cartan subalgebra of $\mathfrak{gl}_2$ defined by K_i. It is known that $\mathfrak{g}_l = \{(x_1, \cdots, x_n) \mid x_i = \mathrm{Tr}_{L_l/K_{il}} x$, for some $x \in L_l\}$ where $L_l = L \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l$ (cf. [5], § 4, 5, Théorèm 5). Since the trace of $\mathfrak{gl}_2$ coincides with $\mathrm{Tr}_{K_i/\boldsymbol{Q}}$ on K_i it is enough to prove that for any $x_i \in K_i$ such that $\mathrm{Tr}_{K_i/\mathbf{Q}} x_i = \mathrm{Tr}_{K_j/\mathbf{Q}} x_j$ $(i, j=1, \cdots, n)$, there exists $x \in L$ such that $\mathrm{Tr}_{L/K_i} x = x_i$. But this is elementary and can be proved easily.

Let $E_1, \cdots, E_m$ be elliptic curves as in (i), and let $E_{m+1}, \cdots, E_n$ be as in (ii), then we can prove the same result as in (i) using exactly the same argument as in the proof of proposition.

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