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ON AUTOMORPHISM GROUPS OF QUATERNION KÄHLER MANIFOLDS

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It is a well-known result that the group of isometries $I(M)$ of an n-dimensional Riemannian manifold M is of dimension at most $\frac{1}{2}n(n+1)$. And if $\dim I(M)=\frac{1}{2}n(n+1)$, then M is isometric to one of the following spaces of constant curvature: (a) an n-dimensional Euclidean space R^n; (b) an n-dimensional sphere S^n; (c) an n-dimensional projective space $Pn(R)$; (d) an n-dimensional simply connected hyperbolic space. In 1947, Wang [11] showed that the group of isometries of an n-dimensional Riemannian manifold with $n\neq 4$ has no closed subgroup of dimension r for $\frac{1}{2}n(n-1)+1<r<\frac{1}{2}n(n+1)$ (See also Yano [13]). And in 1954, Ishihara [5] proved that in a Kähler manifold M the group of automorphisms $A(M)$ of a $2m$-dimensional Kähler manifold M with $m\geqq 3$, $m\neq 4$ contains no closed subgroup of dimension r for $m^2+2<r<m^2+2m-1$. On the other hand, recently, quaternion Kähler manifolds have been studied by several authors (Alekseevski [1], [2], Gray [4], Ishihara [6], [7] Ishihara and Konishi [8] and Wolf [12]). The purpose of this paper is to prove for quaternion Kähler manifolds a theorem stated in the last part of § 5 which is similar to the Wang's theorem for Riemannian case. If M is a $4m$-dimensional quaternion Kähler manifold, then the maximum dimension of the automorphism group $A(M)$ is $2m^2+5m+3$, as will be seen in Lemma 2.1. And it is known that if the maximum dimension of the automorphism group is attained, i. e., the isotropy subgroup is $Sp(m)\cdot Sp(1)=Sp(m)\times Sp(1)/\{\pm 1\}$, then M is isomorphic to one of the following spaces: (a) a $4m$-dimensional Euclidean space Q^m; (b) a quaternion projective space $P^m(Q)$; (c) a quaternion hyperbolic space form [2].

In § 1 and § 2, we recall definitions and some properties of quaternion Kähler manifolds and its automorphisms. In § 3, we recall some algebraic lemmas for later use. § 4 and § 5 are devoted to prove our main results which will be stated in § 5. Manifolds, mappings, tensor fields and other geometric objects we discuss are assumed to be differentiable and of class C^∞.

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§ 1. Quaternion Kähler manifolds.

Let M be a differentiable manifold of dimension n and assume that there is a subbundle V of the tensor bundle of type (1.1) over M such that V satisfies the following condition:

(a) In any coordinate neighborhood U of M, there is a local base $\{F, G, H\}$ of V such that

$$
\begin{aligned}
&F^2=-I, \qquad G^2=-I, \qquad H^2=-I, \\
&GH=-HG=F, \qquad HF=-FH=G, \qquad FG=-GF=H.
\end{aligned} \tag{1.1}
$$

I denoting the identity tensor field of type (1.1) in M. Such a local base $\{F, G, H\}$ is called a *canonical local base* of the bundle V in U. Thus the bundle V is 3-dimensional as a vector bundle. Such a bundle V is called an *almost quaternion structure* and the pair (M, V) an *almost quaternion manifold.* An almost quaternion manifold is orientable and of dimension $n=4m$ $(m\geqq 1)$ (See [6]).

For an almost quaternion manifold (M, V), the tensor field

$$\Lambda=F\otimes F+G\otimes G+H\otimes H \tag{1.2}$$

of type (2.2) determines in M a global tensor field, which will be denoted also by Λ (See [6]).

Next, let there be given an almost quaternion structure V in a Riemannian manifold (M, g) and assume that, for any canonical local base $\{F, G, H\}$ of V, all of F, G and H are almost Hermitian with respect to g. Moreover, we suppose that the set (M, g, V) satisfies the following condition:

(b) If ϕ is a cross-section of the bundle V, then $\nabla_X\phi$ is also a cross-section of V for any vector field X in M, where ∇ denotes the Riemannian connection of the Riemannian manifold (M, V). Such a set (M, g, V) is called a *quaternion Kähler manifold* and the set $\{g, V\}$ a *quaternion Kähler structure* in M.

§ 2. Q-transformations and automorphisms.

Let (M, V) be an almost quaternion manifold. If a transformation $f: M\rightarrow M$ leaves the bundle V invariant, then f is called a *Q-transformation* of (M, V). Let $\{F, G, H\}$ be a canonical local base of V in a coordinate neighborhood V of M. Moreover let (M, g, V) be a quaternion Kähler manifold. If a transformation $f: M\rightarrow M$ is a Q-transformation of (M, V) and at the same time an isometry of (M, g), then f is called an *automorphism* of (M, g, V). An isometry f of (M, g) is an automorphism of (M, g, V) if and only if f leaves the tensor field Λ defined by (1.2) invariant (See [6]).

Let A be the group of all automorphisms of (M, g, V) and A_P the isotropy subgroup for a point P of M, i. e., the subgroup consisting of all automorphisms leaving P fixed. Then, as is well known, A is a Lie group and A_P is a closed subgroup of A. It is easily seen that A_P leaves Λ_P invariant, where Λ_P denotes the value of Λ at P. Thus A_P is isomorphic to a subgroup of $Sp(m)\cdot Sp(1)=Sp(m)\times Sp(1)/\{\pm 1\}$ and hence $\dim A_P \leqq 2m^2+m+3$ is established. On the other hand, we have $4m=\dim M \geqq \dim A/A_P$ and hence $\dim A=\dim A/A_P+\dim A_P \leqq 2m^2+5m+3$. Thus we have

LEMMA 2.1. *Let M be a m-dimensional quaternion Kähler manifold. Then the maximum dimension of the group of automorphism is $2m^2+5m+3$.*

§ 3. Algebraic preliminaries.

In the present section, we recall some algebraic lemmas for later use.

Let $\mathfrak{G}$ be a subalgebra of the Lie algebra $\mathfrak{Gl}(V)$ of all linear endomorphisms of V, where V is a finite dimensional vector space over R (real number field). For any $X\in\mathfrak{G}$, we define a linear endomorphism $\hat{X}$ on the complexification V^c of V by

$$\hat{X}(u+iv)=Xu+i(Xv) \qquad u, v\in V, \quad i^2=-1\,.$$

Then the set of all such $\hat{X}$'s form a linear Lie algebra over R acting on V^c. We denote this Lie algebra by $\hat{\mathfrak{G}}$. If $\mathfrak{G}$ is irreducible (resp. reducible) on V, we say that $\mathfrak{G}$ is *R-irreducible* (resp. *R-reducible*). If $\hat{\mathfrak{G}}$ is irreducible (resp. reducible) on V^c, we say that $\mathfrak{G}$ is *C-irreducible* (resp. *C-reducible*). The R-irreducibility (resp. R-reducibility) and the C-irreducibility (resp. C-reducibility) of a linear group is defined in a similar way as above. We here state the following Lemma 3.1 without proof (See Wakakuwa [10]).

LEMMA 3.1. *Let $\mathfrak{G}$ be a subalgebra of $\mathfrak{Gl}(n, R)$ acting irreducibly on $V=R^n$ but reducibly on $V^c=C^n$. Then n is even. If a proper subspace V_1 $(\neq\{0\})\subset V^c$ is $\hat{\mathfrak{G}}$-invariant, then $\bar{V}_1$ is so. In this case, $V^c=V_1+\bar{V}_1$ (direct sum), $\dim_c V_1=\dim_c \bar{V}_1=\frac{n}{2}$ and $\hat{\mathfrak{G}}$ acts on V_1 (resp. $\bar{V}_1$) irreducibly.*

In Lemma 3.1, $\bar{V}_1$ denotes the subspace of V^c obtained from V_1 by the conjugation σ in V^c, i. e., $\sigma(u+iv)=u-iv$ for any $u, v\in V$. Let $\mathfrak{G}$ be a Lie algebra satisfying conditions of Lemma 3.1, $\hat{\mathfrak{G}}$ induces a real linear Lie algebra on V_1 (resp. $\bar{V}_1$), which is denoted by $\hat{\mathfrak{G}}|V_1$ (resp. $\hat{\mathfrak{G}}|\bar{V}_1$). Let $\{W_{(\alpha)}\}\,\alpha=1, 2, \cdots, \frac{n}{2}$ be a complex basis of V_1, then for any $\hat{X}\in\hat{\mathfrak{G}}$, $\hat{X}W_{(\beta)}=\sum_\alpha A_{\alpha\beta}W_{(\alpha)}$. The complex $\left(\frac{n}{2}\times\frac{n}{2}\right)$-matrix $A=(A_{\alpha\beta})$ gives, with respect to the basis $\{W_{(\alpha)}\}$, an endomorphism induced by $\hat{X}$ on V_1. Since $\{\overline{W}_{(\alpha)}\}$ is a basis of $\bar{V}_1$, the matrix $\hat{X}$ is of the form $\begin{pmatrix} A & 0 \\ 0 & \bar{A}\end{pmatrix}$ with respect to $\{W_{(\alpha)}, \overline{W}_{(\alpha)}\}$.

Put $W_{(\alpha)}=u_{(\alpha)}+iv_{(\alpha)}$, $(u_{(\alpha)}, v_{(\alpha)}\in V)$ and $A=P+iQ$, where P, Q are real $\left(\frac{n}{2}\times\frac{n}{2}\right)$-matrices. Then $\{u_{(\alpha)}, v_{(\alpha)}\}$ forms a basis of V and the matrix $\hat{X}$ is of the form $\begin{pmatrix} P & -Q \\ Q & P\end{pmatrix}$ with respect to $\{u_{(\alpha)}, v_{(\alpha)}\}$. The two matrices above are equivalent, that is,

$$I^{-1}\begin{pmatrix} A & 0 \\ 0 & \bar{A}\end{pmatrix}I=\begin{pmatrix} P & -Q \\ Q & P\end{pmatrix} \quad \text{where} \quad I=\begin{pmatrix} E & iE \\ E & iE\end{pmatrix},$$

E denoting the identity matrix.

Next, we state some lemmas giving certain properties of R-irreducible Lie algebra. The proofs of the following Lemmas 3.2 and 3.3 are stated in [10].

LEMMA 3.2. *Let $\mathfrak{G}$ be an R-irreducible subalgebra of $\mathfrak{SD}(n)$. Then, if $\mathfrak{G}$ is C-irreducible, it is semi-simple and, if $\mathfrak{G}$ is C-reducible, $\mathfrak{G}$ is semi-simple or $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{C}$ (direct sum), where $\mathfrak{G}_1$ is a semi-simple ideal and $\mathfrak{C}$ is the centre of $\mathfrak{G}$ such that $\mathfrak{C}=\{b\phi\}$ $(b\in R,\ \phi^2=-I)$.*

LEMMA 3.3. *Let $\mathfrak{G}$ be an R-irreducible subalgebra of $\mathfrak{Gl}(n:R)$, then $\mathfrak{G}$ decomposes into the form*

$$\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2 \quad (\textit{direct sum})$$

where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are ideals of $\mathfrak{G}$. We can regard that $\mathfrak{G}_1$ is semi-simple. Then, with respect to a suitable real basis in V, one of the following three cases can occur:

(1) *Any element X of $\mathfrak{G}$ is, with respect to the direct sum $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2$ uniquely written in the form*

$$X=X\times I_{n_2}+I_{n_1}\times X_2\,, \qquad n_1n_2=n\,,$$

$\times$ denoting the Kronecker product, where each X_i $(i=1,2)$ is a real matrix of degree n_i and I_{n_i} denotes the unit matrix of degree n_i. Each $\{X_i\,|\,X\in\mathfrak{G}\}$ forms an R-irreducible Lie subalgebra of $\mathfrak{Gl}(n_i:R)$ isomorphic to $\mathfrak{G}_i$.

(2) *Any element X of $\mathfrak{G}$ is, with respect to the direct sum $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2$, uniquely written in the form*

$$X=X_1\times I_{n_2}+I_{n_1}\times X_2+F_{n_1}\times Z\,, \qquad n_1n_2=n\,,$$

$$X_1\times I_{n_2}\in\mathfrak{G}_1\,, \qquad I_{n_1}\times X_2+F_{n_1}\times Z\in\mathfrak{G}_2$$

where F_{n_1} is a real fixed matrix of degree n_1 such that $F_{n_1}{}^2=-I_{n_1}$ and Z is a real matrix of degree n_2, the other being the same as in (1). In this case, n_1 is even and the set $\{X_1\,|\,X\in\mathfrak{G}\}$ forms an R-irreducible Lie algebra of the real representation of $\mathfrak{Gl}(m_1:C)$ $(n_1=2m_1)$ and is isomorphic to $\mathfrak{G}_1$.

(3) *Any element X of $\mathfrak{G}$ is, with respect to the direct sum $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2$, uniquely written in the form*

$$X=X_1\times I_{n_2}+I_{n_1}\times X_2+F_{n_1}\times Y+G_{n_1}\times Z+H_{n_1}\times W, \qquad n_1n_2=n\,;$$

$$X_1\times I_{n_2}\in G_1\,, \qquad I_{n_1}\times X_2+F_{n_1}\times Y+G_{n_1}\times Z+H_{n_1}\times W\in G_2\,,$$

where F_{n_1}, G_{n_1} *and* H_{n_1} *are real fixed matrices of degree* n_1 *such that* $F_{n_1}^2=G_{n_1}^2=H_{n_1}^2=-I_{n_1}$, $F_{n_1}G_{n_1}=-G_{n_1}F_{n_1}=H_{n_1}$ *and* Y, Z, W *are real matrices of degree* n_2, *the others being the same as in* (2). *In this case the set* $\{X_1 | X\in\mathfrak{G}\}$ *forms an R-irreducible subalgebra of the real representation of* $\mathfrak{Gl}(l_1 : Q)$ *(quaternion general linear Lie algebra)* $(n_1=4l_1)$, *and is isomorphic to* $\mathfrak{G}_1$.

§ 4. Subgroups of $Sp(m)$.

We denote by $Sp(m)$ the real representation of the symplectic group. In this section, we shall prove

PROPOSITION 4.1. *Let G be a connected and closed proper subgroup of $Sp(m)$. If* $\dim G\geqq 2m^2-3m+4$ *and* $m\geqq 3$, *then G is R-reducible.*

To prove Proposition 4.1, we need the following

LEMMA 4.2. *Let G be a proper, connected and closed subgroup of $Sp(m)$. Assume that G is R-irreducible and* $\dim G\geqq 2m^2-3m+4$. *If we write the Lie algebra $\mathfrak{G}$ of G, then $\tilde{\mathfrak{G}}$ is simple as a complex Lie algebra, where $\tilde{\mathfrak{G}}$ denotes* $(\hat{\mathfrak{G}}| V_1)^c$.

Proof. G is naturally considered to be a transformation group of a real vector space of $4m$-dimension. Since $G\subset Sp(m)\subset SO(4m)$, $(\mathfrak{G}\subset\mathfrak{Sp}(m)\subset\mathfrak{SO}(4m))$, we see from Lemma 3.2 that G is semi-simple or $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{C}$ (direct sum), $\mathfrak{G}_1$ being a semi-simple ideal where $\mathfrak{C}$ is the center of $\mathfrak{G}$ and has the form $\mathfrak{C}=\{b\phi | b\in R\}$ $(\phi^2=-1)$. Since $\mathfrak{G}\subset\mathfrak{Sp}(m)$, there exist ϕ' and ϕ'' in $\mathfrak{SO}(4m)$ such that ϕ' and ϕ'' are commutative with any element of $\mathfrak{Sp}(m)$ and $\phi'^2=\phi''^2=-1$, $\phi\phi'=-\phi'\phi=\phi''$. Taking an arbitrary element $b\phi$ of C, we get $(b\phi)\phi'=\phi'(b\phi)=-b\phi\phi'$, because $b\phi$ belongs to $\mathfrak{Sp}(m)$. Thus $b=0$, i.e., $\mathfrak{C}=\{0\}$ which means that $\mathfrak{G}$ is semi-simple. Here, for convenience, we consider the following two cases: (a) $\mathfrak{G}$ is not simple; (b) $\mathfrak{G}$ is simple.

Case (a). Let G be not simple. Then $\mathfrak{G}$ can be written as the direct sum of two semi-simple ideals, i.e., $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2$. Putting $\dim\mathfrak{G}_i=r_i$ $(i=1, 2)$, we can assume $r_1\geqq r_2$ without loss of generality. Since $\dim\mathfrak{G}=r=r_1+r_2\geqq 2m^2-3m+4$, we get $r_1\geqq(2m^2-3m+4)/2$. If $\mathfrak{G}_i$ consists of $(m_i\times m_i)$-matrices, taking accounts of case (3) of Lemma 3.3, we have $m_1m_2=4m$ $(m_1\geqq 2, m_2\geqq 2)$. So we get $m\geqq m_1/2$ and hence

$$r_1\geqq(m_1^2-3m_1+8)/2. \tag{4.1}$$

On the other hand, taking account of Lemma 3.3 and $\mathfrak{G}\subset\mathfrak{Sp}(m)$, we get $\mathfrak{G}_1\subset\mathfrak{Sp}\left(\frac{m_1}{4}\right)$, from which $r\leqq 2\left(\frac{m_1}{4}\right)^2+\frac{m_1}{4}=(m_1^2+2m_1)/8$. It contradicts the inequality (4.1). Therefore $\mathfrak{G}$ is necessarily simple.

Case (b). Let $\mathfrak{G}$ be simple. We denote V_1 and $\bar{V}_1$ the $\hat{\mathfrak{G}}$-invariant subspaces of V^c, which appeared in Lemma 3.1. The R-irreducibility of G implies that $\tilde{\mathfrak{G}}$

acts irreducibly on V_1. If we assume that $\tilde{\mathfrak{G}}$ is not simple as a complex Lie algebra, then $\tilde{\mathfrak{G}}$ can be written in the form $\tilde{\mathfrak{G}}=\mathfrak{H}_1+\mathfrak{H}_2$, where $\mathfrak{H}_1$ is simple. Since $\mathfrak{G}\subset\mathfrak{Sp}(m)$, by the same way as in the case (a), we can conclude $\mathfrak{H}_1\subset\mathfrak{Sp}\left(\frac{m_1}{2}\right)$ and $\mathfrak{H}_2\subset\mathfrak{Sp}\left(\frac{m_2}{2}\right)$, where $\mathfrak{H}_i$ $(i=1,2)$ consists of $(m_i\times m_i)$-matrices. In our case $m_1m_2=2m$ and $m_1\geqq 2$, $m_2\geqq 2$. So, we get $m_1\leqq m$ and $m_2\leqq m$, from which

$$\dim_c\tilde{\mathfrak{G}}=2\left(\frac{m_1}{2}\right)^2+\frac{m_1}{2}+2\left(\frac{m_2}{2}\right)^2+\frac{m_2}{2}\leqq m^2+m\,. \tag{4.2}$$

This inequality contradicts the assumption

$$\dim_c\tilde{\mathfrak{G}}\geqq 2m^2-3m+4\,,\qquad m\geqq 3\,.$$

Therefore $\tilde{\mathfrak{G}}$ is necessarily simple.

Proof of Proposition 4.1. First we assume that G is R-irreducible. By means of Lemma 4.2, the Lie algebra $\tilde{\mathfrak{G}}$ is simple and acting on a $2m$-dimensional complex vector space. We now take account of a theorem due to E. Cartan, in which simple complex linear Lie algebras are classified. (See E. Cartan [3]). In Cartan's classification, we have to consider only the cases in which $\tilde{\mathfrak{G}}$ is acting on a complex vector space of even dimension $2m$. If we suppose that $\tilde{\mathfrak{G}}$ is special linear or that $\tilde{\mathfrak{G}}$ is symplectic, then $\dim_c\tilde{\mathfrak{G}}=4m^2-1$ or $\dim_c\tilde{\mathfrak{G}}=2m^2+m$, respectively. However, since G is a proper subgroup of $Sp(m)$, we have $\dim_c\tilde{\mathfrak{G}}<2m^2+m$. Therefore $\tilde{\mathfrak{G}}$ can not be special linear or sympletic. Next, we assume that $\tilde{\mathfrak{G}}$ is orthogonal. Then $\tilde{\mathfrak{G}}$ is the Lie algebra of all matrices of the type

$$\begin{pmatrix} A+iB & -C+iD \\ C-iD & A+iB \end{pmatrix}\in SO(2m, C)\,,$$

where A, B, C and D are real $(m\times m)$-matrices, from which we find

$${}^tA=-A\,,\qquad {}^tB=-B\,,\qquad {}^tC=C\,,\qquad {}^tD=D\,.$$

Thus

$$\dim_c\tilde{\mathfrak{G}}=\frac{1}{2}\times 2\times\frac{m(m+1)}{2}+\frac{1}{2}\times 2\times\frac{m(m-1)}{2}=m^2\,,$$

which contradicts the assumption that $\dim G\geqq 2m^2-3m+4$.

Among the exceptional cases, we have to consider only the case $2m=26$, the case $2m=56$ and the case $2m=248$. In these three cases, we have $\dim_c\tilde{\mathfrak{G}}=52$ for $2m=26$, $\dim_c\tilde{\mathfrak{G}}=133$ for $2m=56$ and $\dim_c\tilde{\mathfrak{G}}=248$ for $2m=248$, respectively. On the other hand, we have $f(m)=2m^2-3m+4=303$ for $2m=26$, $f(m)=1488$ for $2m=56$ and $f(m)=30384$ for $2m=248$. Therefore, because of the assumption that $\dim G>f(m)$, i. e., $\dim_c\tilde{\mathfrak{G}}>f(m)$, we can conclude that the exceptional cases can not occur in our problem. Summing up, all the cases appearing in Cartan's classification excluded for our problem. Consequently, there is no closed proper subgroup G of $Sp(m)$ which is R-irreducible in V, if $\dim G \geqq 2m^2-3m+4$ and $m \geqq 3$. Therefore, G is necessarily R-reducible. This proves Proposition 4.1.

Next, using Proposition 4.1, we can easily prove the following.

PROPOSITION 4.3. *An in Proposition 4.1, if $m \geqq 3$ and $\dim G \geqq 2m^2-3m+4$, then G is conjugate to the group of matrices of the form*

$$Sp(1)+Sp(m-1)=\{A+B \mid A \in Sp(1),\ B \in Sp(m-1)\}\ ,$$

where $+$ means the direct sum of matrices.

As a corollary to Proposition 4.3, we have

LEMMA 4.4. *Let $\bar{\mathfrak{G}}$ be a proper subalgebra of the Lie algebra $\mathfrak{Sp}(m)+\mathfrak{Sp}(1)$ (direct sum) of $Sp(m)\cdot Sp(1)$, satisfying $\dim \bar{\mathfrak{G}} > 2m^2-3m+7$. If $m \geqq 3$, then $\pi_1(\bar{G}) = \mathfrak{Sp}(m)$, where π_1 is the projection $\mathfrak{Sp}(m)+\mathfrak{Sp}(1)$ to the $\mathfrak{Sp}(m)$-part.*

Proof. We denote by π_2 the projection to the $\mathfrak{Sp}(1)$-part. Then putting $\mathfrak{G}'=\pi_1\bar{\mathfrak{G}}$ and $\mathfrak{K}'=\pi_2\bar{\mathfrak{G}}$, we obtain

$$\dim \mathfrak{G}' \geqq \dim \bar{\mathfrak{G}} - \dim \mathfrak{K}' > 2m^2-3m+7-3=2m^2-3m+4\ .$$

Thus, using Proposition 4.3, we get $\mathfrak{G}'=\mathfrak{Sp}(m)$, which proves Lemma 4.4.

§ 5. The main theorem.

First we prove the following

PROPOSITION 5.1. *Let M be a $4m$-dimensional quaternion Kähler manifold and G be a proper closed subgroup in the group of automorphisms of M satisfying $2m^2+m+7 < \dim G < 2m^2+5m+3$. Then the isotropy subgroup G_P of G at any point P is conjugate to $Sp(m)\cdot 1$ or $Sp(m)\cdot K$, K being a 1-dimensional subgroup of $Sp(1)$.*

Proof. The isotropy group G_P is a subgroup of $Sp(m)\cdot Sp(1)$. Then denoting by $\mathfrak{G}_P$ the Lie algebra of G_P, we have $\mathfrak{G}_P \subset \mathfrak{Sp}(m)+\mathfrak{Sp}(1)$. Using Lemma 4.4, we see that $\pi_1\mathfrak{G}_P=\mathfrak{Sp}(m)$ and that $\pi_2\mathfrak{G}_P=\{0\}$ or $\mathfrak{K}$ (a certain 1-dimensional subalgebra of $\mathfrak{Sp}(1)$). If we assume that $\pi_2\mathfrak{G}_P=\{0\}$, then we have obviously $G_P= Sp(m)\cdot 1$. Next, if we assume that $\pi_2\mathfrak{G}_P=\mathfrak{K}$, then we see that $\mathfrak{H}=\pi_2^{-1}(0)$ is an ideal of $\mathfrak{G}_P$ and so $\pi_1\mathfrak{H}$ is an ideal of $\mathfrak{Sp}(m)$. Thus, since $\mathfrak{Sp}(m)$ is simple, we find $\pi_1\mathfrak{H}=\{0\}$ or $\mathfrak{Sp}(m)$ when $\pi_2\mathfrak{G}_P=\mathfrak{K}$.

In the case where $\pi_2\mathfrak{G}_P=\mathfrak{K}$ and $\pi_1\mathfrak{H}=\{0\}$, we have $\mathfrak{H}=\{0\}$, because we put $\mathfrak{H}=\pi_2^{-1}(0)$, which means that π_2 is an isomorphism. Since $\mathfrak{K}$ is 1-dimensional, this fact contradicts the assumption for the dimension. In the case where $\pi_2\mathfrak{G}_P=\mathfrak{K}$ and $\pi_1\mathfrak{H}=\mathfrak{Sp}(m)$, we have $\mathfrak{G}_P=\mathfrak{Sp}(m)+\mathfrak{K}$. In fact, $\pi_1\mathfrak{H}=\mathfrak{Sp}(m)$ and $\mathfrak{H}=\pi_2^{-1}(0)$ implies $\mathfrak{H}=\{(X,0) \mid X \in \mathfrak{Sp}(m)\}$. Thus $\mathfrak{G}_P$ contains subalgebra $\{(X,0) \mid X$

$\in \mathfrak{Sp}(m)\}$, and similarly $\mathfrak{G}_P$ contains subalgebra $\{(0, X) | X \in \mathfrak{K}\}$. Consequently, $\mathfrak{G}_P = \mathfrak{Sp}(m) + \mathfrak{K}$, which means that $G_P = Sp(m) \cdot K$. Summing up, G_P is conjugate to $Sp(m) \cdot 1$ or $Sp(m) \cdot K$, which proves Proposition 5.1.

Using Proposition 5.1, we have

THEOREM. *Let M be a $4m$-dimensional ($m \geqq 3$) quaternion Kähler manifold. Then the group of automorphisms of M contains no proper closed subgroup of dimension r for $2m^2+m+7<r<2m^2+5m$.*

Proof. Let G be a closed subgroup of the group of automorphisms of M such that $\dim G = r$ and G_P denote the isotropy subgroup at $P \in M$. Then, G_P is a subgroup of $Sp(m) \cdot Sp(1)$. Suppose $r > 2m^2+m+7$, then $\dim G_P \geqq \dim G - \dim M > 2m^2+m+7-4m = 2m^2-3m+7$. Thus, by Proposition 5.1, $G_P = Sp(m) \cdot Sp(1)$, $Sp(m) \cdot K$ ($\dim K = 1$) or $Sp(m) \cdot 1$.

Now we shall show that G is transitive on M. If Q and R are two points of M which can be joined by a geodesic. Let P be the midpoint of this geodesic segment and Z be the vector tangent to this geodesic at P. Then there is a transformation f belonging to G_P such that $f^*(Z) = -Z$ for any tangent vector Z at P, because G_P is, in our case, conjugate to one of $Sp(m) \cdot Sp(1)$, $Sp(m) \cdot K$ and $Sp(m) \cdot 1$. So, we have obviously $f(Q) = R$ and $f(R) = Q$. If we take arbitrary two points A and B in M, then we can join them by a finite number of geodesic segments and apply the arguments above to each of these geodesic segments. In this way, we see that there is an element of G which sends A into B. This fact means that G is transitive.

Since G is transitive on M, we have $\dim G = \dim M + \dim G_P \geqq 4m + 2m^2 + m = 2m^2 + 5m$, which contradicts the assumption that $\dim G < 2m^2 + 5m$. Thus the theorem is completely proved.

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