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KAEHLER IMMERSIONS WITH VANISHING BOCHNER CURVATURE TENSORS

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Introduction.

In [4] Tachibana has introduced the notion of the Bochner curvature tensor in a Kaehler manifold and Yamaguchi-Sato [6] have proved that a complex hypersurface M^n with vanishing Bochner curvature tensor in a Kaehler manifold $\bar{M}^{n+1}$ with vanishing Bochner curvature tensor is totally geodesic if $n \geqq 6$. On the other hand, by Theorem 3 of O'Neill [2], we can see that a complex submanifold M^n of a Kaehler manifold $\bar{M}^{n+p}$ is totally geodesic if $p<n(n+1)/2$ under the assumption both manifolds are of constant holomorphic sectional curvature. With these connection, the purpose of this note is to prove the following:

THEOREM. *Let $\bar{M}^{n+p}$ be a Kaehler manifold of complex dimension $n+p$ with vanishing Bochner curvature tensor, and let M^n be a complex submanifold of $\bar{M}$ of complex dimension n with vanishing Bochner curvature tensor. If $p<(n+1)(n+2)/(4n+2)$, then M is totally geodesic in $\bar{M}$.*

COROLLARY. *Under the same assumption as in Theorem, if $p=1$ and $n \geqq 2$, then M is totally geodesic in $\bar{M}$.*

1. Preliminaries.

Let $\bar{M}$ be a Kaehler manifold of complex dimension $n+p$ with the structure tensor J and the Kaehler metric $\langle\ ,\ \rangle$. We denote by $\bar{R}$, $\bar{S}$ and $\bar{Q}$ the curvature tensor, the Ricci tensor and the Ricci operator of $\bar{M}$ respectively. $\bar{S}$ and $\bar{Q}$ have the relation $\langle \bar{Q}\bar{x}, \bar{y}\rangle = \bar{S}(\bar{x}, \bar{y})$ for any vectors $\bar{x}, \bar{y} \in T_m(\bar{M})$. And we can see $\bar{Q}J = J\bar{Q}$ and $\bar{S}(J\bar{x}, J\bar{y}) = \bar{S}(\bar{x}, \bar{y})$. The Bochner curvature tensor $\bar{K}$ of $\bar{M}$ is defined by setting

$$
\begin{aligned}
(1.1) \qquad \bar{K}(\bar{x}, \bar{y})\bar{z} = {} & \bar{R}(\bar{x}, \bar{y})\bar{z} \\
& -\frac{1}{(2r+4)}\{\langle \bar{y}, \bar{z}\rangle \bar{Q}\bar{x} - \langle \bar{Q}\bar{x}, \bar{z}\rangle \bar{y} + \langle J\bar{y}, \bar{z}\rangle \bar{Q}J\bar{x} - \langle \bar{Q}J\bar{x}, \bar{z}\rangle J\bar{y} \\
& + \langle \bar{Q}\bar{y}, \bar{z}\rangle \bar{x} - \langle \bar{x}, \bar{z}\rangle \bar{Q}\bar{y} + \langle \bar{Q}J\bar{y}, \bar{z}\rangle J\bar{x} - \langle J\bar{x}, \bar{z}\rangle \bar{Q}J\bar{y} \\
& -2\langle J\bar{x}, \bar{Q}\bar{y}\rangle J\bar{z} - 2\langle J\bar{x}, \bar{y}\rangle \bar{Q}J\bar{z}\}
\end{aligned}
$$

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$$+\frac{\bar{k}}{(2r+2)(2r+4)}\{\langle \bar{y}, \bar{z}\rangle \bar{x}-\langle \bar{x}, \bar{z}\rangle \bar{y}+\langle J\bar{y}, \bar{z}\rangle J\bar{x}-\langle J\bar{x}, \bar{z}\rangle J\bar{y}$$
$$-2\langle J\bar{x}, \bar{y}\rangle J\bar{z}\}, \qquad r=n+p,$$

for any $\bar{x}, \bar{y}, \bar{z}\in T_m(\bar{M})$ where $\bar{k}$ is the scalar curvature of $\bar{M}$.

Let M be an n-dimensional complex submanifold of $\bar{M}$. The Riemannian metric induced on M is a Kaehler metric, which is denoted by the same $\langle\ ,\ \rangle$, and the complex structure of M is written by the same J as in $\bar{M}$. The covariant differentiation in $\bar{M}$ (resp. M) will be denoted by $\bar{\nabla}$ (resp. ∇). Then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y), \qquad X, Y\in\mathfrak{X}(M),$$
$$\bar{\nabla}_X N=-A^N(X)+D_X N, \qquad X\in\mathfrak{X}(M), \qquad N\in\mathfrak{X}(M)^{\perp}$$

where $\langle B(X, Y), N\rangle=\langle A^N(X), Y\rangle$ and D is the linear connection in the normal bundle $T(M)^{\perp}$. Since M is minimal in $\bar{M}$, we have $\sum B(e_i, e_i)=0$ for a frame $e_1, \cdots, e_{2n}$ in $T_m(M)$. If the second fundamental form B of M is identically zero, M is called a totally geodesic submanifold of $\bar{M}$. By the Gauss-Weingarten formulas, the Gauss-equation is given by, for $x, y, z, w\in T_m(M)$,

$$\langle \bar{R}(x, y)z, w\rangle=\langle R(x, y)z, w\rangle-\langle B(x, w), B(y, z)\rangle+\langle B(y, w), B(x, z)\rangle \tag{1.2}$$

where R is the Riemannian curvature tensor of M. In the following, we denote by S, Q and k the Ricci tensor, Ricci operator and the scalar curvature of M respectively. Let $v_1, \cdots, v_{2p}$ be a frame for $T_m(M)^{\perp}$. Hereafter we write A^{v_a} by A^a to simplify the presentation.

Simons [3] has defined the following operators which are symmetric, positive semi-definite:

$$\tilde{A}={}^tA\circ A \quad \text{and} \quad \underset{\sim}{A}=\sum_{a=1}^{2p} ad A^a ad A^a.$$

And we define the operator A^* by setting

$$A^*=\sum_{a=1}^{2p}(A^a)^2,$$

which is also symmetric, positive semi-definite. Obviously we have $\mathrm{Tr}\, A^*=\|A\|^2$ where $\|A\|$ denotes the length of the second fundamental form A of M. And we have also $2\,\mathrm{Tr}\,(A^*)=\langle \underset{\sim}{A}\circ A, A\rangle$ (cf. [1], [3]).

On the other hand, the second fundamental form A has the following properties:

$$A^v J+JA^v=0 \quad \text{and} \quad A^{Jv}-JA^v=0.$$

2. Proof of Theorem.

By the Gauss-equation (1.2), we obtain

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle R(e_i, A^a e_j)e_j, A^a e_i\rangle$$
$$=\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\{\langle\bar{R}(e_i, A^a e_j)e_j, A^a e_i\rangle$$
$$+\langle B(e_i, A^a e_i), B(e_j, A^a e_j)\rangle-\langle B(A^a e_i, A^a e_j), B(e_i, e_j)\rangle\}\,.$$

Hereafter we use a frame $e_1, \cdots, e_{2n}$ for $T_m(M)$ such that $e_{n+i}=Je_i$ and a frame $v_1, \cdots, v_{2p}$ for $T_m(M)^{\perp}$ such that $v_{p+a}=Jv_a$. Then we can see

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle B(A^a e_i, A^a e_j), B(e_i, e_j)\rangle$$
$$=\sum_{a}\sum_{i,j}\langle A^{(Be_i,e_j)}A^a e_j, A^a e_i\rangle$$
$$=\sum_{a,b}\sum_{i,j}\langle A^b A^a e_j, A^a e_i\rangle\langle A^b e_i, e_j\rangle$$
$$=\sum_{a,b}\sum_{j}\langle A^a A^b A^a e_j, A^b e_j\rangle=0$$

because $A^{Ja}A^b A^{Ja}=JA^a A^b JA^a=-A^a A^b A^a$, where $Ja\equiv Jv_a$. By the definition of $\tilde{A}$, we have also

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle B(e_i, A^a e_i), B(e_j, A^a e_j)\rangle=\langle A\circ\tilde{A}, A\rangle\,.$$

Consequently we obtain

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle R(e_i, A^a e_j)e_j, A^a e_i\rangle=\sum_{a}\sum_{i,j}\langle\bar{R}(e_i, A^a e_j)e_j, A^a e_i\rangle+\langle A\circ\tilde{A}, A\rangle\,. \tag{2.1}$$

By the assumption, M has the vanishing Bochner curvature tensor and we have by using (1.1)

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle R(e_i, A^a e_j)e_j, A^a e_i\rangle=\frac{-4}{n+2}\operatorname{Tr} QA^*+\frac{k}{(n+1)(n+2)}\|A\|^2\,. \tag{2.2}$$

Similarly we obtain

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle\bar{R}(e_i, A^a e_j)e_j, A^a e_i\rangle=\frac{-4}{r+2}\operatorname{Tr}\bar{Q}A^*+\frac{\bar{k}}{(r+1)(r+2)}\|A\|^2 \tag{2.3}$$

where $r=n+p$ and we take the trace of $\bar{Q}A^*$ on $T_m(M)$. In the following we calculate $\operatorname{Tr}\bar{Q}A^*$. By (1.1) and (1.2), we get

$$S(x, y)=\frac{1}{2r+4}\{(2n+4)\bar{S}(x, y)+\operatorname{Tr}\bar{Q}\langle x, y\rangle\}$$
$$-\frac{(n+1)\bar{k}}{2(r+1)(r+2)}\langle x, y\rangle-\langle A^*x, y\rangle\,, \tag{2.4}$$

$$k=\frac{(2n+2)}{(r+2)}\operatorname{Tr}\bar{Q}-\frac{(n+1)n\bar{k}}{(r+1)(r+2)}-\|A\|^2\,. \tag{2.5}$$

Using (2.4) and (2.5), we obtain

$$\operatorname{Tr} QA^*=\frac{(n+2)}{(r+2)}\operatorname{Tr}\bar{Q}A^*+\frac{1}{(2r+4)}\operatorname{Tr}\bar{Q}\|A\|^2$$

$$-\frac{(n+1)\bar{k}}{2(r+1)(r+2)}\|A\|^2-\operatorname{Tr}(A^*)^2,$$

$$\operatorname{Tr}\bar{Q}=\frac{(r+2)k}{2(n+1)}+\frac{(r+2)}{2(n+1)}\|A\|^2+\frac{n\bar{k}}{2(r+1)}.$$

From these equations, we have

$$\operatorname{Tr}\bar{Q}A^*=\frac{(r+2)}{(n+2)}\operatorname{Tr} QA^*+\frac{\bar{k}}{4(r+1)}\|A\|^2-\frac{(r+2)k}{4(n+1)(n+2)}\|A\|^2 \tag{2.6}$$

$$-\frac{(r+2)}{4(n+1)(n+2)}\|A\|^4+\frac{(r+2)}{(n+2)}\operatorname{Tr}(A^*)^2.$$

Therefore (2.3) and (2.6) imply

$$\sum_{a=1}^{2p}\sum_{i,j=1}^{2n}\langle\bar{R}(e_i, A^a e_j)e_j, A^a e_i\rangle \tag{2.7}$$

$$=\frac{-4}{(n+2)}\operatorname{Tr} QA^*+\frac{k}{(n+1)(n+2)}\|A\|^2$$

$$+\frac{1}{(n+1)(n+2)}\|A\|^4-\frac{2}{(n+2)}\langle \underset{\sim}{A}\circ A, A\rangle.$$

Consequently, from (2.1), (2.2) and (2.7), we have

$$\frac{1}{(n+1)(n+2)}\|A\|^4+\langle A\circ\tilde{A}, A\rangle=\frac{2}{(n+2)}\langle \underset{\sim}{A}\circ A, A\rangle. \tag{2.8}$$

On the other hand, we have the following inequalities (see [1]):

$$\frac{1}{2p}\|A\|^4\leqq\langle A\circ\tilde{A}, A\rangle\leqq\frac{1}{2}\|A\|^4 \quad \text{and} \quad \frac{1}{n}\|A\|^4\leqq\langle \underset{\sim}{A}\circ A, A\rangle\leqq\|A\|^4.$$

Hence (2.8) becomes

$$\frac{1}{(n+1)(n+2)}\|A\|^4+\frac{1}{2p}\|A\|^4\leqq\frac{2}{(n+2)}\|A\|^4,$$

and hence we get

$$\left\{p-\frac{(n+1)(n+2)}{(4n+2)}\right\}\|A\|^4\geqq 0.$$

Thus if $p<(n+1)(n+2)/(4n+2)$, then M is totally geodesic in $\bar{M}$, which proves our Theorem and Corollary is verified by Theorem obviousely.

Remark. Let $\bar{M}^{n+p}$ be a Kaehler manifold with vanishing Bochner curvature tensor, and let M^n be a complex submanifold of $\bar{M}$. If M is totally geodesic in

$\bar{M}$, we can see that the Bochner curvature tensor of M vanishies, by using (1.1), (2.4) and (2.5).

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