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ON THE DEFICIENCY OF AN ENTIRE FUNCTION OF FINITE GENUS

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1. Introduction. In [1], Edrei and Fuchs established the following

THEOREM A. *Let $f(z)$ be an entire function of finite order having only negative zeros. If the order is greater than one, then $f(z)$ has zero as a Nevanlinna deficient value.*

The extension of this result to more general distributions of the arguments of the zeros of an entire function was investigated in [2], [3] and [4].

Indeed Ozawa [4] gave the following result.

THEOREM B. *Let $g(z)$ be a canonical product of genus one and with only zeros $\{a_n\}$ in the sector*

$$|\arg a_n-\pi|\leqq\frac{\pi}{4}.$$

Then

$$\delta(0, g)\geqq\frac{A}{1+A}$$

with a positive constant A.

In this paper we shall prove the following theorems.

THEOREM 1. *Let $g(z)$ be a canonical product of finite genus q $(\geqq 1)$ and having only zeros in the sector*

$$\left\{z:|\arg z-\pi|\leqq\frac{\pi}{2(q+1)}\right\}.$$

Then

$$\delta(0, g)\geqq\frac{A(q)}{1+A(q)}$$

with a positive constant $A(q)$.

THEOREM 2. *The assumptions of Theorem 1 imply*

$$q\leqq\mu\leqq\rho\leqq q+1$$

where ρ and μ indicate the order and the lower order of $g(z)$, respectively.

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COROLLARY. *Let $f(z)$ be an entire function of finite genus q ($\geqq 1$). If its zeros lie in the sector*

$$\left\{z : |\arg z| \leqq \frac{\pi}{2(q+1)}\right\},$$

then $f(z)$ has zero as a deficient value.

This Corollary is an immediate consequence of the above two theorems. Further it should be remarked that for each positive integer q and for each $S_q(>\pi/(2q+2))$, there exists an entire function $f(z)$ of genus q whose zeros lie in the sector

$$\{z : |\arg z| < S_q\}$$

but

$$\delta(0, f)=0 .$$

Therefore, the result of Corollary is no longer true when the opening of the sector is greater than $\pi/(q+1)$.

2. Lemmas. Our proofs of Theorem 1 and 2 depend heavily on the following lemmas.

LEMMA 1. *For each positive integer q, set*

$$S_q(r, x, y)=\frac{1}{2}\log L(r, x, y)+\sum_{n=1}^{q}\frac{2}{n}r^n \cos nx \cos ny$$

where

$$L(r, x, y)=1-4r\cos x\cos y+2r^2\cos 2x$$
$$+4r^2\cos^2 y-4r^3\cos x\cos y+r^4 .$$

Then

$$\frac{L(r, x, y)}{2r^{q+1}}\frac{d}{dy}S_q(r, x, y)$$
$$=-(\cos qx\sin qy)r^3+A_q^*(x, y)r^2+B_q^*(x, y)r$$
$$+\cos(q+1)x\sin(q+1)y ,$$
$$A_q^*(x, y)=4\cos x\cos y\cos qx\sin qy-\cos(q-1)x\sin(q-1)y ,$$
$$B_q^*(x, y)=\sin qx\sin qy\sin 2x-\cos qx\cos qy\sin 2y$$
$$-2\cos^2 x\cos qx\sin qy-\cos 2y\cos qx\sin qy .$$

Proof. It is sufficient to prove the result for $q\geqq 5$. By a simple calculation we have

$$\frac{1}{2}L(r, x, y)\frac{d}{dy}S_q(r, x, y)$$
$$=-(\cos qx \sin qy)r^{q+4}+A_q^*(x, y)r^{q+3}$$
$$+B_q(x, y)r^{q+2}+C_q(x, y)r^{q+1}+R_q(r, x, y)$$

where

$$B_q(x, y)=-\cos(q-2)x\sin(q-2)y-2\cos 2x\cos qx\sin qy$$
$$+4\cos x\cos y\cos(q-1)x\sin(q-1)y$$
$$-4\cos^2 y\cos qx\sin qy\,,$$
$$C_q(x, y)=-\cos(q-3)x\sin(q-3)y$$
$$+4\cos x\cos y\cos(q-2)x\sin(q-2)y$$
$$-4\cos^2 y\cos(q-1)x\sin(q-1)y$$
$$-2\cos 2x\cos(q-1)x\sin(q-1)y$$
$$+4\cos x\cos y\cos qx\sin qy\,,$$
$$R_q(r, x, y)=(\cos x\sin y)r-2(\cos y\sin y)r^2$$
$$+(\cos x\sin y)r^3-\sum_{n=1}^{q}r^n\cos nx\sin ny$$
$$+4\sum_{n=1}^{q-1}r^{n+1}\cos x\cos y\cos nx\sin ny$$
$$-2\sum_{n=1}^{q-2}r^{n+2}(\cos 2x+2\cos^2 y)\cos nx\sin ny$$
$$+4\sum_{n=1}^{q-3}r^{n+3}\cos x\cos y\cos nx\sin ny$$
$$-\sum_{n=1}^{q-4}r^{n+4}\cos nx\sin ny\,.$$

Let us consider $R_q(r, x, y)$. Then

$$R_q(r, x, y)=\sum_{n=2}^{q}D_n(x, y)r^n$$

where

$$D_n(x, y)=4\cos x\cos y\cos(n-1)x\sin(n-1)y$$
$$+4\cos x\cos y\cos(n-3)x\sin(n-3)y$$
$$-2(\cos 2x+2\cos^2 y)\cos(n-2)x\sin(n-2)y$$
$$-\cos nx\sin ny-\cos(n-4)x\sin(n-4)y\,.$$

Using elementary trigonometric relations, we have

$$D_n(x, y)=0$$

for all n. Thus $R_q(r, x, y)$ is identically equal to zero. On the other hand

$$\begin{aligned} C_q(x, y) &= \cos x \cos y \cos qx \sin qy + \cos x \sin y \cos qx \cos qy \\ &\quad - \sin x \cos y \sin qx \sin qy - \sin x \sin y \sin qx \cos qy \\ &= (\cos x \cos qx - \sin x \sin qx)(\cos y \sin qy + \sin y \cos qy) \\ &= \cos (q+1)x \sin (q+1)y \end{aligned}$$

and

$$\begin{aligned} B_q(x, y) &= (\cos 2x - 2\cos^2 x) \sin 2y \cos qx \cos qy \\ &\quad + (2\cos^2 y - \cos 2y) \sin 2x \sin qx \sin qy \\ &\quad - (2+\cos 2y) \cos 2x \cos qx \sin qy \\ &\quad + (4\cos^2 x - 4) \cos^2 y \cos qx \sin qy \\ &= B_q^*(x, y)\,. \end{aligned}$$

Thus we have the desired result.

LEMMA 2. *For each even integer q ($\geqq 2$),*

$$\frac{d}{dy} S_q(r, x, y) \leqq 0$$

for $r \geqq 0$, $\frac{\pi}{2q} \leqq x \leqq \frac{\pi}{2q-2}$ *and* $\frac{2q+1}{2q+2}\pi \leqq y \leqq \pi$.

Proof. Evidently $(\cos (q+1)x \sin (q+1)y)$, $-(\cos qx \sin qy)$, $A_q^*(x, y)$ and $B_q^*(x, y)$ are all non-positive under the given conditions. Hence Lemma 1 implies the desired fact.

LEMMA 3. *For each odd integer q ($\geqq 1$),*

$$\frac{d}{dy} S_q(r, x, y) \leqq 0$$

for $r \geqq 0$, $0 \leqq x \leqq \frac{\pi}{4q}$ *and* $\frac{2q+1}{2q+2}\pi \leqq y \leqq \pi$.

Proof. Under the given conditions

$$\begin{aligned} A_q^*(x, y) &\leqq (4\cos x \cos qx - 1) \sin qy \cos y \\ &\leqq \sin qy \cos y \\ &\leqq 0 \end{aligned}$$

and

$$\begin{aligned} B_q^*(x, y) &\leqq \sin qx \sin qy \sin 2x \\ &\quad -(\cos 2x + 2\cos^2 y)\cos qx \sin qy \\ &\leqq \left(\frac{\sqrt{2}}{2} - \cos qx\right)\sin qy \\ &\leqq 0; \end{aligned}$$

Thus from Lemma 1 we have the desired result.

3. Proof of Theorem 1. By the definition of $S_q(r, x, y)$ we have

$$S_q(r, u, v) = \log|E(re^{iu}e^{iv}, q)| + \log|E(re^{iu}e^{-iv}, q)|$$

where $E(z, q)$ is the Weierstrass primary factor. Let $g(z)$ be a canonical product of genus q and with only zeros $\{a_n\}$ in the sector

$$|\arg a_n - \pi| \leqq \frac{\pi}{2q+2}.$$

Put

$$\arg a_n = v_n \qquad (n=1, 2, 3, \cdots).$$

Then we have

$$\log|g(re^{iu})| + \log|g(re^{-iu})| = \sum_{n=1}^{\infty} S_q\left(\frac{r}{|a_n|}, u, v_n\right).$$

In the first place we assume that q is even ($\geqq 2$). From Lemma 2,

$$\log|g(re^{iu})g(re^{-iu})| \leqq \sum_{n=1}^{\infty} S_q\left(\frac{r}{|a_n|}, u, \frac{2q+1}{2q+2}\pi\right)$$

for $r \geqq 0$ and $\frac{\pi}{2q} \leqq u \leqq \frac{\pi}{2q-2}$. Let

$$F(z) = \prod E\left(-\frac{z}{|a_n|}, q\right).$$

Then

$$\log|F(re^{iu}v_*)F(re^{iu}\bar{v}_*)| = \sum_{n=1}^{\infty} S_q\left(\frac{r}{|a_n|}, u, \frac{2q+1}{2q+2}\pi\right)$$

where $v_* = \exp\left(\frac{i\pi}{2q+2}\right)$. Thus we have

$$\log|g(re^{iu})g(re^{-iu})| \leqq \log|F(re^{iu}v_*)F(re^{iu}\bar{v}_*)|$$

for $r \geqq 0$ and $\frac{\pi}{2q} \leqq u \leqq \frac{\pi}{2q-2}$. Integrating with respect to u from $\frac{\pi}{2q}$ to $\frac{(q+2)\pi}{2q(q+1)}\left(\leqq \frac{\pi}{2q-2}\right)$ gives

$$m(r, 0, g) \geqq -\frac{1}{2\pi}\int_{I_q} \log|F(re^{iu})|\,du$$

where

$$I_q=\left[\frac{2q+1}{2q(q+1)}\pi,\ \frac{\pi}{q}\right]+\left[\frac{\pi}{2q(q+1)},\ \frac{\pi}{q(q+1)}\right].$$

Now by Shea's representation [5],

$$2m(r, 0, g)\geqq\int_0^{+\infty}N(rt, 0, g)K_q^*(t)dt$$

where

$$K_q^*(t)=K_q\left(t, 1, \frac{2q+1}{2q(q+1)}\pi\right)-K_q\left(t, 1, \frac{\pi}{q}\right)$$

$$+K_q\left(t, 1, \frac{\pi}{2q(q+1)}\right)-K_q\left(t, 1, \frac{\pi}{q(q+1)}\right),$$

$$K_q(t, 1, x)=\frac{(-1)^q}{\pi}t^{-q-1}\frac{t\sin(q+1)x+\sin qx}{t^2+2t\cos x+1}.$$

Put

$$A_q=2\cos\frac{\pi}{2q(q+1)},\qquad B_q=2\cos\frac{\pi}{q(q+1)},$$

$$C_q=2\cos\frac{2q+1}{2q(q+1)}\pi,\qquad D_q=2\cos\frac{\pi}{q}.$$

Then

$$K_q^*(t)=\frac{1}{\pi}t^{-q-1}(tI_q(t)+J_q(t)),$$

$$I_q(t)=\frac{\sin\frac{\pi}{2q}}{t^2+1+A_qt}-\frac{\sin\frac{\pi}{q}}{t^2+1+B_qt}-\frac{\sin\frac{\pi}{2q}}{t^2+1+C_qt}+\frac{\sin\frac{\pi}{q}}{t^2+1+D_qt},$$

$$J_q(t)=\frac{\sin\frac{\pi}{2q+2}}{t^2+1+A_qt}-\frac{\sin\frac{\pi}{q+1}}{t^2+1+B_qt}+\frac{\sin\frac{\pi}{2q+2}}{t^2+1+C_qt}.$$

Let us consider $I_q(t)$, which is equal to

$$\frac{tB_q-tD_q}{(X+B_qt)(X+D_qt)}\sin\frac{\pi}{q}+\frac{tC_q-tA_q}{(X+A_qt)(X+C_qt)}\sin\frac{\pi}{2q}$$

where $X=t^2+1$. Evidently

$$(X+A_qt)(X+C_qt)\geqq(X+B_qt)(X+D_qt)$$

for all $t\geqq0$. Hence

$$t^{-1}(X+A_qt)(X+C_qt)I_q(t)\geqq(B_q-D_q)\sin\frac{\pi}{q}+(C_q-A_q)\sin\frac{\pi}{2q}$$

$$\geqq4\sin\frac{\pi}{2q+2}\left(\sin\frac{(q+2)\pi}{2q(q+1)}\sin\frac{\pi}{q}-\sin^2\frac{\pi}{2q}\right)$$

$$>0.$$

Thus we have

$$I_q(t)>0$$

for all $t>0$. Next, consider $J_q(t)$. Then

$$J_q(t) \geqq \frac{2}{X+A_q t} \sin \frac{\pi}{2q+2} - \frac{1}{X+B_q t} \sin \frac{\pi}{q+1}$$

where $X=t^2+1$. Therefore

$$\begin{aligned}
&(X+A_q t)(X+B_q t)J_q(t) \\
&\geqq \left(2 \sin \frac{\pi}{2q+2} - \sin \frac{\pi}{q+1}\right) X \\
&\quad + \left(4 \cos \frac{\pi}{q(q+1)} \sin \frac{\pi}{2q+2} - 2 \cos \frac{\pi}{2q(q+1)} \sin \frac{\pi}{q+1}\right) t .
\end{aligned}$$

From

$$\begin{aligned}
&4 \sin \frac{\pi}{2q+2} \left(\cos \frac{\pi}{q(q+1)} - \cos \frac{\pi}{2q(q+1)} \cos \frac{\pi}{2q+2}\right) \\
&\geqq 4 \sin \frac{\pi}{2q+2} \left(\cos \frac{\pi}{q(q+1)} - \cos \frac{\pi}{2q+2}\right) \\
&\geqq 0 ,
\end{aligned}$$

we have

$$(X+A_q t)(X+B_q t)J_q(t) \geqq \left(2 \sin \frac{\pi}{2q+2} - \sin \frac{\pi}{q+1}\right) X > 0$$

for all $t \geqq 0$. Hence we have

$$J_q(t)>0$$

for all $t>0$. Then $K_q^*(t)$ is positive for all $t>0$. Therefore

$$\begin{aligned}
2m(r, 0, g) &\geqq \int_1^{+\infty} N(rt, 0, g) K_q^*(t) dt \\
&\geqq N(r, 0, g) \int_1^{+\infty} K_q^*(t) dt
\end{aligned}$$

for each $r>0$. Put

$$A(q) = \frac{1}{2} \int_1^{+\infty} K_q^*(t) dt .$$

Then evidently $A(q)$ is positive and

$$\varlimsup_{r \to +\infty} \frac{N(r, 0, g)}{T(r, g)} \leqq \frac{1}{1+A(q)} .$$

Next, consider the case that q is odd ($q \geqq 1$). In this case from Lemma 3,

$$\log |g(re^{iu}) g(re^{-iu})| \leqq \sum_{n=1}^{+\infty} S_q\left(\frac{r}{|a_n|}, u, \frac{2q+1}{2q+2}\pi\right)$$

for $r \geqq 0$ and $0 \leqq u \leqq \frac{\pi}{4q}$. Let

$$F(z)=\Pi E\left(-\frac{z}{|a_n|}, q\right).$$

Then for $r \geqq 0$, $0 \leqq u \leqq \frac{\pi}{4q}$

$$\log|g(re^{iu})g(re^{-iu})| \leqq \log|F(re^{iu}v_*)F(re^{iu}\bar{v}_*)|$$

where $v_*=\exp\left(\frac{i\pi}{2q+2}\right)$. The integration with respect to u from 0 to $\frac{\pi}{2q(q+1)}$ $\left(\leqq \frac{\pi}{4q}\right)$ yields

$$2m(r, 0, g) \geqq \int_0^{+\infty} N(rt, 0, g)K_q^*(t)dt,$$

$$K_q^*(t)=K_q\left(t, 1, \frac{q-1}{2q(q+1)}\pi\right)-K_q\left(t, 1, \frac{1}{2q}\pi\right).$$

Put

$$A_q=2\cos\frac{\pi}{2q}, \qquad B_q=2\cos\frac{(q-1)\pi}{2q(q+1)}, \qquad X=1+t^2.$$

Then

$$K_q^*(t)=\frac{1}{\pi}t^{-q-1}(tI_q(t)+J_q(t)),$$

$$I_q(t)=\left(\frac{1}{X+A_qt}-\frac{1}{X+B_qt}\right)\cos\frac{\pi}{2q};$$

$$J_q(t)=\frac{1}{X+A_qt}-\frac{1}{X+B_qt}\sin\frac{q-1}{2q+2}\pi.$$

Evidently

$$\frac{1}{X+A_qt} \geqq \frac{1}{X+B_qt}$$

for all $t \geqq 0$. Therefore $K_q^*(t)$ is positive for all $t>0$. Thus we have

$$\overline{\lim_{r\to+\infty}}\frac{N(r, 0, g)}{T(r, g)} \leqq \frac{1}{1+A(q)},$$

$$A(q)=\frac{1}{2}\int_1^{+\infty}K_q^*(t)dt.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2. Using the same notations as in the section 3, we have

$$2T(r, g) \geqq \int_0^{+\infty} N(rt, 0, g)K_q^*(t)dt.$$

Further we have shown that

$$K_q^*(t) \geqq \frac{1}{\pi} t^{-q-1} J_q(t)$$

and

$$J_q(t) \geqq M_q > 0$$

for $0 \leqq t \leqq 1$. Hence

$$\begin{aligned} 2T(r, g) &\geqq \int_0^1 N(rt, 0, g) K_q^*(t) dt \\ &\geqq \frac{M_q}{\pi} \int_0^1 N(rt, 0, g) t^{-q-1} dt \\ &= r^q \frac{M_q}{\pi} \int_0^r N(t, 0, g) t^{-q-1} dt . \end{aligned}$$

Since $g(z)$ is of genus q,

$$\lim_{r \to +\infty} \int_0^r N(t, 0, g) t^{-q-1} dt = +\infty .$$

Therefore

$$\lim_{r \to +\infty} \frac{r^q}{T(r, g)} = 0 ,$$

which yields $q \leqq \mu$. $\mu \leqq \rho \leqq (q+1)$ is well known.

References

[1] EDREI, A. AND W. H. J. FUCHS, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc., **93** (1959), 292–328.

[2] EDREI, A., W. H. J. FUCHS AND S. HELLERSTEIN, Radial distribution and deficiencies of the values of a meromorphic function, Pacific Journ. Math., **11** (1961), 135–151.

[3] OZAWA, M., Radial distribution of zeros and deficiency of a canonical product of finite genus, Kōdai Math. Sem. Rep., **25** (1973), 506–512.

[4] OZAWA, M., Distribution of zeros and deficiency of a canonical product of genus one, Hokkaido Math. Journ., **3** (1974), 218–231.

[5] SHEA, D. F., On the Valiron deficiencies of meromorphic functions of finite order, Trans. Amer. Math. Soc., **124** (1966), 201–227.

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