C. D. MINDA
KŌDAI MATH. SEM. REP.
27 (1976), 308–312

SQUARE INTEGRABLE DIFFERENTIALS ON RIEMANN SURFACES

BY CARL DAVID MINDA

1. Introduction. Suppose R is a Riemann surface and $\Gamma(R)$ is the Hilbert space of square integrable first-order differential forms on R. A rather complete account of $\Gamma(R)$ is given in [2, Chapter V]; the notation and basic results developed in [2] will be employed freely without further mention. The subspace of semiexact differentials was generally neglected until it was employed in the study of the Hilbert space isomorphism $f^{\#}: \Gamma(R')\to\Gamma(R)$ induced by a quasiconformal homeomorphism $f: R\to R'$ of Riemann surfaces [6]. Further evidence of the usefulness of this subspace and its orthogonal complement are presented. First, the relationship between semiexact differentials and dividing cycles is explored. Then several tests for degeneracy in the classification theory of Riemann surfaces are derived. The use of the subspace of semiexact differentials and their orthogonal complement enables us to obtain a complete parallelism between the important subspaces of $\Gamma(R)$ and those of $\Gamma_h(R)$, the subspace of harmonic differentials.

2. Semiexact differentials and orthogonal decompositions. To begin with, let us recall a few basic facts from [6, § 1].

DEFINITION. Let R be a Riemann surface. (i) $\Gamma_{se}^1(R)=\left\{\omega\in\Gamma^1(R): \int_d \omega=0\right.$ for all dividing cycles d on $\left.R\right\}$. $\Gamma_{se}(R)$, the subspace of semiexact differentials, is the closure of $\Gamma_{se}^1(R)$ in the Hilbert space $\Gamma(R)$. (ii) $\Gamma_{coe}(R)$ is the closure of $\Gamma_{co}^1(R)\cap\Gamma_e^1(R)$ in $\Gamma(R)$.

The subspaces $\Gamma_{se}(R)$ and $\Gamma_{coe}^*(R)$ are orthogonal complements in $\Gamma(R)$; that is, $\Gamma(R)=\Gamma_{se}\oplus\Gamma_{coe}^*(R)=\Gamma_{se}^*(R)\oplus\Gamma_{coe}(R)$. Moreover, the following identities are valid.

(1) $\Gamma_{se}(R)=\Gamma_{co}(R)\oplus(\Gamma_{hse}(R)\cap\Gamma_{he}^*(R))$.
(2) $\Gamma_c(R)=\Gamma_{se}(R)\oplus\Gamma_{hm}^*(R)$.
(3) $\Gamma_e(R)=\Gamma_{coe}(R)\oplus(\Gamma_{hse}^*(R)\cap\Gamma_{he}(R))$.
(4) $\Gamma_{coe}(R)=\Gamma_{eo}(R)\oplus\Gamma_{hm}(R)$.

All of these results will be employed later.

Received May 27, 1974.

The definition of semiexact differentials reveals their close connection with dividing cycles. The elements of $\Gamma^1_{se}(R)$ are characterized by their periods over dividing cycles; an analogous description of the differentials in $\Gamma_{se}(R)$ is given below. For a similar characterization of other classes of differential forms by their periods see ([3], [4]). Recall that a statement is said to hold for almost all curves in a family F if the subfamily F' for which the statement is false has infinite extremal length; that is, $\lambda(F')=\infty$.

THEOREM 1. *Let $\omega\in\Gamma(R)$. A necessary and sufficient condition for $\omega\in\Gamma_{se}(R)$ is that $\int_d\omega=0$ for almost all dividing cycles d.*

Proof. First, suppose that $\omega\in\Gamma_{se}$ and let $(\omega_n)_{n=0}^\infty$ be a sequence in Γ^1_{se} with $\|\omega-\omega_n\|\to 0$. Set $\rho_n(z)|dz|=|\omega_n-\omega|$, then $\rho_n(z)|dz|$ is a linear density on R and $A(\rho_n)=\iint_R\rho_n^2(z)dxdy=\|\omega_n-\omega\|\to 0$. By Fuglede's lemma ([3], [8, Lemma 2B, p. 128]) these is a subsequence $(\omega_{n_j})_{j=0}^\infty$ such that for almost all dividing cycles d

$$0\leqq\left|\int_d(\omega_{n_j}-\omega)\right|\leqq\int_d\rho_{n_j}(z)|dz|\longrightarrow 0\,.$$

Thus, $\int_d\omega=\lim\int_d\omega_{n_j}=0$ for almost all dividing cycles d because $\omega_{n_j}\in\Gamma^1_{se}$ implies that $\int_d\omega_{n_j}=0$.

Conversely, suppose that $\int_d\omega=0$ for almost all dividing cycles d. Since every bounding cycle is a dividing cycle, $\int_d\omega=0$ for almost all cycles d homologous to zero. Therefore, $\omega\in\Gamma_c(R)$ [3]. Now, from equation (2), $\Gamma_c=\Gamma_{se}\oplus\Gamma^*_{hm}$ so $\omega=\omega_{se}+{}^*\omega_{hm}$ where $\omega_{se}\in\Gamma_{se}$, $\omega_{hm}\in\Gamma_{hm}$. The first portion of the proof shows that $\int_d\omega_{se}=0$ for almost all dividing cycles d. Consequently, $\int_d{}^*\omega_{hm}=0$ for almost all dividing cycles d. But this means that $\int_d{}^*\omega_{hm}=0$ for all dividing cycles d because each homology class has finite extremal length and ${}^*\omega_{hm}\in\Gamma_h$. Hence, ${}^*\omega_{hm}\in\Gamma_{hse}$. Since $\Gamma_h=\Gamma_{hse}\oplus\Gamma^*_{hm}$, it follows that ${}^*\omega_{hm}=0$ and $\omega=\omega_{se}\in\Gamma_{se}$.

This shows that semiexact differentials are characterized by their periods over dividing cycles. On the other hand, dividing cycles are typified by the fact that every $\omega\in\Gamma^1_{se}$ has zero period over each of them.

THEOREM 2. *Let d denote a cycle. d is a dividing cycle if and only if $\int_d\omega=0$ for all $\omega\in\Gamma^1_{se}(R)$.*

Proof. The necessity follows from the definition of Γ^1_{se}. Conversely, suppose that $\int_d\omega=0$ for all $\omega\in\Gamma^1_{se}$. This also holds for all $\omega\in\Gamma^1_{co}\subset\Gamma^1_{se}$ and this implies that d is weakly homologous to zero [7, Satz III. 14, p. 92]. A finite cycle is weakly homologous to zero if and only if it is dividing [2, Theorem 32C, p. 73].

3. Classification results. The following inclusion diagrams will be helpful in this section.

$$
\begin{array}{ccccc}
 & & \Gamma_e(R) & & \\
 & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\supset$} & \\
\Gamma_c(R)\supset\Gamma_{se}(R) & & & & \Gamma_{co}(R)\cap\Gamma_e(R)\supset\Gamma_{coe}(R)\supset\Gamma_{eo}(R)\\
 & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\supset$} & \\
 & & \Gamma_{co}(R) & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \Gamma_{he}(R) & & \\
 & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\supset$} & \\
\Gamma_h(R)\supset\Gamma_{hse}(R) & & & & \Gamma_{ho}(R)\cap\Gamma_{he}(R)\supset\Gamma_{hm}(R)\supset(0)\\
 & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\supset$} & \\
 & & \Gamma_{ho}(R) & &
\end{array}
$$

Observe the similarity between these two inclusion relationships. The second is obtained by intersecting each subspace in the first with $\Gamma_h(R)$; on the other hand, the first may be derived by taking the direct sum of each term in the second with $\Gamma_{eo}(R)$. Without the subspaces $\Gamma_{se}(R)$ and $\Gamma_{coe}(R)$ this parallelism is absent.

Let O_{HD}, O_{KD}, O_{HM} denote the class of Riemann surfaces R for which $\Gamma_{he}(R)=(0)$, $\Gamma^*_{hse}(R)\cap\Gamma_{he}(R)=(0)$, $\Gamma_{hm}(R)=(0)$; respectively. Riemann surfaces in these classes exhibit certain types of degeneracy. Knowledge that certain of the inclusions in the above diagrams are not proper will enable us to assert that the surface belongs to one of these classes. The following classification theorem is well known [2, Theorem 11G, p. 287]; it is stated for completeness because it is the prototype for the other theorems in this section.

THEOREM 3. *The following conditions on a Riemann surface R are equivalent:*

(i) $R\in O_{HD}$.
(i′) $\Gamma_e(R)=\Gamma_{eo}(R)$.
(ii) $\Gamma_h(R)=\Gamma_{ho}(R)$.
(ii′) $\Gamma_c(R)=\Gamma_{co}(R)$.

It is possible to derive analogous theorems for O_{KD} and O_{HM}. The class O_{HM} has not been studied extensively.

Observe that the conditions (ii) and (ii′) above are clearly equivalent because the subspaces in each are corresponding terms in the two inclusion diagrams. A similar comment applies to (i) and (i′). In other words, whenever two subspaces in one diagram coincide, then the corresponding subspaces in the other are identical. We shall make use of this observation by omitting the proof of obvious equivalences in the following.

THEOREM 4. *The following conditions on a Riemann surface R are equivalent:*

(i) $R \in O_{KD}$.
(ii) $\Gamma_{he}(R) = \Gamma_{hm}(R)$.
(ii′) $\Gamma_{e}(R) = \Gamma_{coe}(R)$.
(iii) $\Gamma_{ho}(R) = \Gamma_{hse}(R)$.
(iii′) $\Gamma_{co}(R) = \Gamma_{se}(R)$.

Proof. (i)$\Leftrightarrow$(ii). This is a simple consequence of the orthogonal decomposition $\Gamma_{he} = \Gamma_{hm} \oplus (\Gamma^*_{hse} \cap \Gamma_{he})$.

(ii)$\Leftrightarrow$(iii). This is evident from $\Gamma_{he} \oplus \Gamma^*_{ho} = \Gamma_{h} = \Gamma_{hm} \oplus \Gamma^*_{hse}$.

The equivalence of (i) and (ii′) is essentially due to Royden [9, p. 54].

THEOREM 5. *The following conditions on a Riemann surface R are equivalent:*

(i) $R \in O_{HM}$.
(i′) $\Gamma_{eo}(R) = \Gamma_{coe}(R)$.
(ii) $\Gamma_{h}(R) = \Gamma_{hse}(R)$.
(ii′) $\Gamma_{c}(R) = \Gamma_{se}(R)$.
(iii) $\Gamma_{he}(R) = \Gamma^*_{hse}(R) \cap \Gamma_{he}(R)$.
(iii′) $\Gamma_{e}(R) = \Gamma^*_{se}(R) \cap \Gamma_{e}(R)$.

Proof. (i)$\Leftrightarrow$(ii). This is due to the decomposition $\Gamma_{h} = \Gamma^*_{hm} \oplus \Gamma_{hse}$.

(i)$\Leftrightarrow$(iii). Observe that $\Gamma_{he} = \Gamma_{hm} \oplus (\Gamma^*_{hse} \cap \Gamma_{he})$.

The decomposition $\Gamma_{he}(R) = \Gamma_{hm}(R) \oplus (\Gamma^*_{hse}(R) \cap \Gamma_{he}(R))$ shows that $R \in O_{HD}$ if and only if $R \in O_{HM} \cap O_{KD}$. Therefore, Theorem 3 may be derived easily from Theorems 4 and 5. For other characterizations of O_{HM} see ([5], [8, Theorem 1C, p. 202]). There is a similar result to determine whether a Riemann surface is planar.

THEOREM 6. *The following conditions on a Riemann surface R are equivalent:*

(i) *R is planar.*
(ii) $\Gamma_{hse}(R) = \Gamma_{he}(R)$.
(ii′) $\Gamma_{se}(R) = \Gamma_{e}(R)$.
(iii) $\Gamma_{ho}(R) = \Gamma_{hm}(R)$.
(iii′) $\Gamma_{co}(R) = \Gamma_{coe}(R)$.
(iv) $\Gamma^*_{hse}(R) \cap \Gamma_{ho}(R) = (0)$.
(iv′) $\Gamma^*_{se}(R) \cap \Gamma_{co}(R) = (0)$.

Proof. (i)$\Leftrightarrow$(ii). If R is planar, then every cycle is dividing so $\Gamma_{he} = \Gamma_{hse}$. Conversely, suppose that $\Gamma_{he} = \Gamma_{hse}$, then $\Gamma^1_{e} = \Gamma^1_{se}$. Therefore, $\int_d \omega = 0$ for all $\omega \in \Gamma^1_{e} = \Gamma^1_{se}$, where d is any cycle. Theorem 2 guarantees that every cycle on R is dividing so that R is planar.

(ii)⇔(iii). This is evident from $\Gamma_{hse}\oplus\Gamma^*_{hm}=\Gamma_h=\Gamma_{he}\oplus\Gamma^*_{ho}$.

(ii)⇔(iv). Notice that $\Gamma_{hse}=\Gamma_{he}\oplus(\Gamma_{hse}\cap\Gamma^*_{ho})$ follows immediately from $\Gamma_h=\Gamma_{he}\oplus\Gamma^*_{ho}$.

As the final result of this type we determine a condition under which $\Gamma_{hm}(R)=\Gamma_{ho}(R)\cap\Gamma_{he}(R)$. In general, $\Gamma_{hm}(R)$ is a proper subspace of $\Gamma_{ho}(R)\cap\Gamma_{he}(R)$; the equality of the two subspaces is equivalent to the validity of the special bilinear relation on R [1]. Clearly, this holds if and only if $\Gamma_{coe}(R)=\Gamma_{co}(R)\cap\Gamma_e(R)$ and $\Gamma_{coe}(R)$ is typically a proper subspace of $\Gamma_{co}(R)\cap\Gamma_e(R)$.

REFERENCES

[1] R. D. M. ACCOLA, The bilinear relation on open Riemann surfaces, Trans. Amer. Math. Soc., **96** (1960), 143–161.

[2] L. V. AHLFORS AND L. SARIO, Riemann surfaces, Princeton Math. Series, no. 26, Princeton Univ. Press, Princeton, N. J., 1960.

[3] B. FUGLEDE, Extremal length and functional completion, Acta Math., **98** (1957), 171–219.

[4] A. MARDEN AND B. RODIN, Periods of differentials on open Riemann surfaces, Duke Math. J., **33** (1966), 103–108.

[5] C. D. MINDA, Extremal length and harmonic functions on Riemann surfaces, Trans. Amer. Math. Soc., **171** (1972), 1–22.

[6] C. D. MINDA, Square integrable differentials on Riemann surfaces and quasiconformal mappings, Trans. Amer. Math. Soc., **195** (1974), 365–381.

[7] A. PFLUGER, Theorie der Riemannschen Flächen, Die Grundlehren der math. Wissenschaften, Band 89, Springer-Verlag, Berlin, 1957.

[8] B. RODIN AND L. SARIO, Principal functions, Van Nostrand, Princeton, N. J., 1968.

[9] H. L. ROYDEN, On a class of null-bounded Riemann surfaces, Comment. Math. Helv., **34** (1960), 52–66.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CINCINNATI
CINCINNATI, OHIO 45221