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CONFORMAL AUTOMORPHISMS OF A COMPACT BORDERED RIEMANN SURFACE OF GENUS 3

Dedicated to Professor Y. Komatu

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1. Introduction. Let R be a finite Riemann surface of genus g and with k boundary components, and G be the group of all conformal mappings of R onto itself. For given non-negative integers g and k, we put

$$N(g, k)=\max(\text{ord.}\, G)$$

where ord. G means the order of G, and the maximum is taken with respect to all R having the genus g and k boundary components.

For compact surfaces, that is, $k=0$, Hurwitz [5] proved that $N(g, 0)=84(g-1)$. After it many results have been obtained. For special g, the accurate values of $N(g, 0)$ have been known. However, the problem is still open, for infinitely many values of g.

On the other hand, for $k\geqq 1$, Heins [4], Oikawa [9] and the author [11] determined $N(0, k)$, $N(1, k)$ and $N(2, k)$ respectively. And, Kato [6] determined $N(g, k)$ for $k=1, 2, 3$.

In this paper, we shall determine $N(3, k)$ as follows.

THEOREM. *The value of* $N(3, k)$ *is*

168 *for* $84n+0, 24, 56, 80$
96 *for* $48n+0, 12, 32, 44$ *except above cases*
48 *for* $4n$ *except above cases*
24 *for* $24n+2, 6, 14, 18$
16 *for* $24n+10$
14 *for* $14n+1, 3, 7, 9$ *and* $168n+22, 70, 94, 142$
12 *for* $168n+46, 118, 166$
and $84n+5, 13, 19, 25, 27, 39, 41, 53, 55, 61, 67, 75$
9 *for* $252n+11, 47, 81, 83, 95, 117, 131, 153, 165, 167, 201, 237$
8 *for* $504n+33, 179, 249, 251, 321, 467$
6 *for* $504n+69, 215, 285, 431, 501, 503$

2. Method of research. Let $N'(g, k)$ be the order of the largest group of automorphisms of a k-times punctured compact Riemann surface of genus g.

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Oikawa [10] proved

LEMMA 1.

$$N(g, k)=N'(g, k).$$

Therefore, it is sufficient to prove the theorems for $N'(g, k)$.

Let R be a compact Riemann surface of genus g, and G be the group of automorphisms. For any subgroup H of G, we can regard R/H as a Riemann surface having a conformal structure which is induced from the conformal structure of R[5]. R can be regarded as a covering surface of $R/H=R_0$. Let π denote the natural projection of R onto R_0. We project all the branch points of R with respect to π into R_0 and denote them by $\hat{p}_1, \cdots, \hat{p}_r$. Noting that the ramification indices of all the points over $\hat{p}_i$ $i=1, 2, \cdots, r$, are the same respectively, we denote the corresponding indices by $\nu_1-1, \cdots, \nu_r-1$. Let the genus of R_0 be g_0, and the order of H be n, then the following Riemann-Hurwitz relation [5] holds.

LEMMA 2.

$$\frac{2g-2}{n}=2g_0-2+\sum_{i=1}^{r}\left(1-\frac{1}{\nu_i}\right).$$

If a Riemann surface of genus g and with k punctured points has an automorphisms group H, then the compactified surface has also same automorphisms group H. Therefore, our task is to determine all the automorphisms groups and their subgroups. A necessary information on the group is given by values of g, g_0, ν_i, $i=1, \cdots, r$. From them, we can count n by Lemma 2. Without breaking the structure of this group, we puncture the points, by taking all equivalent points in regard to this group to be punctured or not. Unramified points must be punctured by n, and the branch points of (ν_i-1) order must be punctured by n/ν_i. Thus, we can conclude

LEMMA 3. *If R has an automorphism group H of order n,*

$$N(g, k)\geqq n \qquad \textit{for} \quad k=ln+\sum_{i=1}\delta_i\frac{n}{\nu_i}$$

where l is a non-negative integer, $\delta_i=0$ or 1, and ν_i-1 is the branch order of R at $\hat{p}_i$ on $R/H=R_0$. We denote this group $(\nu_1, \nu_2, \cdots, \nu_r)$ group.

3. Exceptional Cases. For given g, g_0, ν_i, $i=1, 2, \cdots, r$, we must study the existence of R having automorphism group H which produces above values. But, our problem is to determine the maximal order of the groups. So, it is useful to remove the exceptional cases having small orders.

As shown later, for a certain surface of genus 3, $k=6l+\sum_{i=1}^{r}6\delta_i/\nu_i$ runs over all the positive integers. By Lemma 3, $N(3, k)\geqq 6$. Therefore we have no interest the group of order$\leqq 6$.

By Lemma 2, for $g_0 \geqq 2$, $2g-2=n(2g_0-2)+n\sum(1-1/\nu_i) \geqq 2n$ or $n \leqq g-1$ $(=2)$. These cases need not be considered. For $g_0=1$, $2g-2=n\sum(1-1/\nu_i)$. But as Accola [1] shows, there must be at least two branch points. Because $\nu_i \geqq 2$, $2g-2=n\sum(1-1/\nu_i) \geqq n\{(1-1/2)+(1-1/2)\}$, or $n \leqq 2g-2$ $(=4)$. We have also no interest for these cases.

Thus, for the surface R having the group G of order>4, we know the genus of $R_0=R/G$ is 0.

Let R be the Riemann surface having the group G of order $n>6$, and H be a cyclic subgroup of G having maximal order m. If $m>4$, the above reason shows that the genus of R/H is 0. In this case, we can represent the surface as $y^m=f(x)$.

By Lemma 2, for $g_0=0$, $4/n=r-2-1/\nu_1-\cdots-1/\nu_r$. From this, $r \geqq 3$. (If $m>4$, this result can be used to $y^m=f(x)$.)

If $r \geqq 5$, $4/n \geqq r-2-(1/2+\cdots+1/2)=r/2-2 \geqq 1/2$, or $n \leqq 8$. Equality occurs only for $(2, 2, 2, 2, 2)$ group.

If $r=4$, $4/n=2-1/\nu_1-1/\nu_2-1/\nu_3-1/\nu_4$. For $\nu_i=2, 3, 4$ $n>6$, only possible combinations are $(2, 2, 2, 3)$, $(2, 2, 3, 3)$, $(2, 2, 2, 4)$ and $(2, 2, 4, 4)$.

If $r=3$, $4/n=1-1/\nu_1-1/\nu_2-1/\nu_3$. Possible combinations are $(3, 3, 4)$, $(3, 4, 4)$ and $(4, 4, 4)$. Thus we have,

LEMMA 4. *If the surface of genus* 3 *have the group of order* $n>6$,
(i) *either it can be represented as* $y^m=f(x)$ $(m>4)$,
(ii) *or its group is one of the next eight types:*

$$(2, 2, 2, 2, 2),\quad (2, 2, 2, 3),\quad (2, 2, 3, 3),\quad (2, 2, 2, 4),\quad (2, 2, 4, 4),$$
$$(3, 3, 4),\quad (3, 4, 4),\quad (4, 4, 4)\,.$$

4. Genus of $y^m=f(x)$. Our main task is to study the surface $y^m=f(x)$. This surface is an m-sheeted covering of Riemann sphere. Of course each power of the factors of $f(x)$ may be less than m. Zeros of $f(x)$ are branch points, and the number of branch points is more than 2 by the consideration of previous section.

By factorzation of $f(x)$, the surface can be represented as follows

$$y^m=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r} \qquad 0 \leqq \beta_i < m\,.$$

If $\sum\beta_i \equiv 0 \pmod m$, $x=\infty$ is not a branch point. In this case, by a simple transformation we can remove the one factor, and can make $x=\infty$ a branch point. After now, we always assume $\sum\beta_i \not\equiv 0 \pmod m$, and consider $x=\infty$ as a branch point.

Let G.C.M. $(m, \beta_i)=l_i$, G.C.M. $(m, \sum\beta_i)=l_\infty$. The ramification index at $x=\alpha_i$ is m/l_i-1. We put $m/l_i=\nu_i$. Then, considering that branch points are $\alpha_1, \cdots, \alpha_r$, ∞

$$\frac{2g-2}{m}=(r+1)-2-\frac{1}{\nu_1}-\cdots-\frac{1}{\nu_r}-\frac{1}{\nu_\infty}=r-1-\frac{l_1}{m}-\cdots-\frac{l_r}{m}-\frac{l_\infty}{m}\,,$$

$$2g-2=m(r-1)-\sum^{\infty} l_i\,, \qquad \text{where} \quad \sum^{\infty} l_i=l_1+l_2+\cdots+l_r+l_\infty$$

by Lemma 2.

Thus

LEMMA 5. *By the notations above, the genus of the Riemann surface*

$$y^m=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r} \quad \textit{is} \quad g=\frac{m(r-1)-\sum^{\infty} l_i+2}{2}\,.$$

In our cases, $g=3$, and remembering $m>4$, and $1\leqq l_i<m/2$,

$$4=m(r-1)-\sum^{\infty} l_i \geqq m(r-1)-\frac{m}{2}(r+1)=\frac{m(r-3)}{2}>2(r-3)\,.$$

From this, $5>r$. Thus we have

LEMMA 6. *In the case* (i) *of Lemma* 4, $f(x)$ *must be one of the next three types:*

(a) $(x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}$,

(b) $(x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}$, $\sum\beta_i\neq 0 \pmod m$

(c) $(x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}(x-\alpha_4)^{\beta_4}$.

Our surface $y^m=f(x)$ may be assumed to be irreducible.

5. Further restriction. Further investigations in what follow together with Lemmas 5, 6 make us possible to determine the surfaces. First of all, the value of m is not arbitrary. Concerning it, we use the next two lemmas.

LEMMA 7 (Wiman [12]). $4g+2$ *is the order of the largest cyclic group of automorphisms which a compact Riemann surface of genus g can admit.*

LEMMA 8 (Accola [1]). *Let G be a group of automorphisms. Then the prime factors of order of G occur among the prime factors of* $(2g+1)$ $((g+1)\,!)$.

From Lemma 7, $m\leqq 14$. We have assumed $4<m$ already, so by Lemma 8, necessary numbers to be considered are, $m=6, 7, 8, 9, 12, 14$.

We shall show that the case (c) of Lemma 6 does not occur. If $m>8$, the same reason explained before shows $4>r$. Therefore, we must examine the cases $m=8, 7, 6$.

For $r=4$, from Lemma 5, $3m-4=l_1+l_2+l_3+l_4+l_\infty$.

For $m=8$, $20=l_1+l_2+l_3+l_4+l_\infty$, where l_i is a divisor of 8.

Only possibility is, $l_1=l_2=l_3=l_4=l_\infty=4$, but this contradicts irreducibility of our surface.

For $m=7$, $17=l_1+l_2+l_3+l_4+l_\infty$.

Divisor of 7 less than itself is only 1. So, this case does not occur.

For $m=6$, $14=l_1+l_2+l_3+l_4+l_\infty$.

This equation seems to have a solution 2, 3, 3, 3, 3. But in this case, $\nu_i=2, 2, 2, 2, 3$. So we can put it into exceptional cases.

For $r=3$, $2m-4=l_1+l_2+l_3+l_\infty$.

For $m=14, 12, 9, 8, 7$, we can show as above that these cases do not occur. Therefore, only the case $m=6$ occurs.

For $r=2$, $m-4=l_1+l_2+l_\infty$.

For $m=14, 12, 9, 8, 7$, all cases can occur. But, for $m=6$, this equation has no solution obviously.

Thus we have an improved form of Lemma 6.

LEMMA 9. *In the case* (i) *of Lemma* 4, *only the next cases occur.*
(a) $y^m=(x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}$, *where* $m=14, 12, 9, 8, 7$.
(b) $y^6=(x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}$.

6. Graphic Representation. We have considered Riemann surfaces in regard to a cyclic subgroup H. But our main purpose is to determine whole group G containing H. For the purpose, we use the graphic representation of the group.

R can be regarded as an m sheeted covering surface of a Riemann sphere. By Lemma 9, the number of branch points is 3 or 4. Without loss of generality, three of them may be $0, 1, \infty$. If the fourth branch point occurs, let it be located on the real nxis, because we want the maximum order, and the symmetry condition serve to increase the order of the group.

We have assumed that the largest order of the cyclic subgroup be m. As we shall see soon, in every case the branch point of order $m-1$ exists. Let one of them be p_0, and without loss of generality, p_0 may lie over $x=\infty$.

As we are treating hyperbolic Riemann surfaces, their universal covering surface R can be considered as the unit disc. In R, with the center p_0, we consider normal polygon Π, that is $\Pi=\{w\,;\,\rho(0, w)\leqq\rho$ (equivalent point of $0, w)\}$, where ρ is the non-euclidean metric of the unit disc. As p_0 is a branch point of order $m-1$, Π is unchanged by the rotation $w\to e^{\frac{2\pi i}{m}}w$. A $1/m$ part of Π corresponds to one x plane. From the symmetry for real axis, the part of real axis which connects $x=\infty$ and the neighboring branch point corresponds to the straight line which connects $w=0$ with one vertex of Π.

The part of real axis other than the above, corresponds to the sides of Π, and by the symmetry principle, Π is symmetric with respect to all non-euclidean lines which correspond to a part of real axis having no branch points. The fundamental regions of H is divided in two (let them be shaded and unshaded) half fundamental regions, according to the upper or lower half x-plane.

This is the graphic representation of a cyclic group of order m. Generally, every group has the graphic representation. In detail, see Burnside [3].

If the cyclic group H is a proper subgroup of a large group G, we can divide the half fundamental region F of H into half fundamental regions S of G. Conversely, if F is divided to S, and S can be the half fundamental regions of some group G containing H, we can extend H to G on the same Riemann surface.

Simple geometrical division is dangerous, because we can not distinguish them to be the half fundamental region of a conformal group or not. In this respect, we must research more carefully.

7. Local property. We want to know the relations between the graph and the equation which produces the group. For finer research, we must study local properties to begin with.

Let $f(x)=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r}$, $\sum\beta_r\neq 0 \pmod m$, and l_i, ν_i be the same as before. Let $m=l_i\nu_i$, $\beta_i=l_i\mu_i$, where ν_i and μ_i are mutually prime. Therefore $\mu_i b_i=\nu_i a_i+1$ has non-negative integral solutions. We take the least pair (a_i, b_i). At the points $x=\alpha_i$, we can take $t=(x-\alpha_i)^{1/\nu_i}$ as the local parameter. Then, locally

$$y^m=c(x-\alpha_i)^{\beta_i}=c(x-\alpha_i)^{l_i\mu_i}=ct^{l_i\mu_i\nu_i}=ct^{\mu_i m}.$$

From this, $y=c_1t^{\mu_i}$,

$$y^{b_i}=c_1^{b_i}t^{b_i\mu_i}=c_1't^{1+\nu_i a_i}=c_1't(x-\alpha_\nu)^{a_i}.$$

$$\therefore\quad t=\frac{y^{b_i}}{c_1'(x-\alpha_i)^{a_i}}.$$

Other solutions have the form $(a_i+k\mu_i, b_i+k\nu_i)$. Corresponding them, local parameters are

$$t=\frac{y^{b_i+k\mu_i}}{c_1'(x-\alpha_i)^{a_i+k\mu_i}}\qquad (k=0, 1, 2, \cdots, l_i-1).$$

Different values correspond to different branch points.

As for the points at infinity, let $m=l_\infty\nu_\infty$, $\sum\beta_i=l_\infty\mu_\infty'$. We consider the equation $\mu_\infty' b_\infty+1=\nu_\infty a_\infty$, which has non-negative integral solutions. Taking the least solution (a_∞, b_∞), and considering the local parameter $t=(1/x)^{1/\nu_\infty}$, as before we can get

$$t=\frac{y^{b_\infty+k\mu_\infty}}{c_2'x^{a_\infty+k\mu_\infty}}\qquad (k=0, 1, 2, \cdots, l_\infty-1).$$

By cyclic automorphism $y\to\varepsilon y$ $(\varepsilon=e^{\frac{2\pi i}{m}})$, t changes $\varepsilon^{b_i+k\mu_i}t$ or $\varepsilon^{b_\infty+k\mu_\infty}t$ respectively. But, this does not mean the rotation around the poi . Generally, the points lying over the same α_i are changed each other. When this a.to-morphisms repeated l_i-times, the rotation around the branch points occur. For the symmetry, we use $\mu_\infty\equiv-\mu_\infty' \pmod{\nu_\infty}$ instead of μ_∞'. Then we have,

LEMMA 10. *By cyclic rotation $y\to\varepsilon^{l_i}y$, the local parameter of the branch points lying over $x=\alpha_i$ changes $t\to\varepsilon^{l_ib_i}t$, where $\varepsilon=e^{\frac{2\pi i}{m}}$, and* $b_i\mu_i\equiv 1 \pmod{\nu_i}$, $i=1, 2, \cdots, r, \infty$.

8. Side-Correspondence. Let R be one of a Riemann surface of Lemma 9, and Π be its normal polygon. The sides of Π are pairly equivalent. And if we know the side-correspondence, by attaching them, we can get the original surface. Thus, Π and its side-correspondence determines R completely.

Let R be given by $y^m=x^{\beta_1}(x-1)^{\beta_2}$, and $x=\infty$ be a branch point of $(m-1)$-th order. Let $\beta_\infty\equiv-(\beta_1+\beta_2) \pmod m$ satisfy $0<\beta_\infty<m$.

Let H be the cyclic group corresponding to $y\to e^{\frac{2\pi i}{m}}y$, and F be one of the unshaded half fundamental region. F is a (non-euclidean) triangle, and $x=\infty$ corresponds to 0 ($w=0$). Let the vertices corresponding to $x=0$ and $x=1$ be A_1 and B_1 respectively. Here, we use Lemma 10. Notations are all same. Figures are all non-euclidean, but we use euclidean language.

By the rotation around 0, the side $\overline{A_1B_1}$ changes to $\overline{A_2B_2}, \cdots, \overline{A_mB_m}$. From the assumption, $l_\infty=1$. Therefore, by the cyclic automorphisms $y\to\varepsilon^{\beta_2}y$, the local parameter of the origin changes $t\to\varepsilon^{\beta_2b_\infty}t$ by Lemma 10. So, $\overline{A_1B_1}$ changes to $\overline{A_{\beta_2b_\infty}B_{\beta_2b_\infty}}$. On the other hand, this automorphism gives arise to a rotation around B_1. Again by Lemma 10, the local parameter of B_1 changes $t\to\varepsilon^{\beta_2b_2}t=\varepsilon^{l_2}t$. This means that F changes to F' neighboring the shaded triangle which has the side $\overline{A_1B_1}$, and their common vertex is B_1.

F' and $\triangle OA_{\beta_2b_\infty}B_{\beta_2b_\infty}$ are equivalent in regard to R, so the shaded triangles neighboring them along the side which correspond to $x=\infty$ and $x=1$ are also equivalent.

Side-correspondence determines R, so the number β_2b_∞ determines R. If we exchange the rolle of $x=0$ and $x=1$, we get the number β_1b_∞. But

$$\beta_1b_\infty+\beta_2b_\infty\equiv(\beta_1+\beta_2)b\equiv-\beta_\infty b_\infty\equiv1 \pmod m.$$

If we consider $y^m=x^{\beta_1}(x-c)^{\beta_3}(x-1)^{\beta_2}$, $0<c<1$, the condition does not change. Conversely, if two surfaces are conformally equivalent, clearly their side-correspondences in respect to Π are equal. Thus we have

LEMMA 11. *For two surfaces,*

$$\begin{aligned} y^m&=x^{\beta_1}(x-c)^{\beta_3}(x-1)^{\beta_2} \\ & \qquad\qquad\qquad\qquad (0<c<1) \\ y^m&=x^{\beta'_1}(x-c)^{\beta'_3}(x-1)^{\beta'_2} \end{aligned}$$

let them have common $l_\infty=1$, l_1, l_2, l_3. Then they are conformally equivalent if and only if

$$\beta_1b_\infty\equiv\beta'_1b'_\infty \quad or \quad \equiv1-\beta'_1b'_\infty \pmod m.$$

9. Possible formulas. We want to restrict the cases in Lemma 9 further. For the case $y^{14}=(x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}$, by Lemma 5 the genus condition is

$$l_1+l_2+l_\infty=10\,.$$

l_i must be a divisor of 14, so the only possible combination is 1, 2, 7. From this, we can get the possible combinations of β_1, β_2, β_∞. Their sum must be a multiple of 14, therefore, possible combinations are, 5, 2, 7; 3, 4, 7; 1, 6, 7. Other combinations for example 9, 12, 7 exist, but they can be changed to one of these three by a simple birational transformation. So we need not consider them.

Above three types are conformally equivalent by Lemma 11. For the convenience the next table is useful.

	β_1	β_2	β_∞	μ_1	μ_2	μ_∞	b_1	b_2	b_∞	$\beta_1 b_\infty$	$\beta_2 b_\infty$
(1)	7	2	5	1	1	5	1	1	3	21	6
(2)	7	4	3	1	2	3	1	4	5	35	20
(3)	7	6	1	1	3	1	1	5	1	7	6

Same procedures are applied to the cases, $m=12, 9, 8, 7, 6$. But we describe only the results.

LEMMA 12. *The surfaces appearing in Lemma* 9 *are conformally equivalent to one of the next ten surfaces:*

(i)	$y^{14}=x^7(x-1)^2$	(vi)	$y^8=x(x-1)$
(ii)	$y^{12}=x^6(x-1)^5$	(vii)	$y^7=x(x-1)$
(iii)	$y^{12}=x^3(x-1)$	(viii)	$y^7=x^2(x-1)$
(iv)	$y^9=x^3(x-1)$	(ix)	$y^6=x^3(x-1)^2(x-\alpha)^2$
(v)	$y^8=x^2(x-1)$	(x)	$y^6=x^3(x-1)(x-\alpha)$.

10. Extended group. For a given surface $y^m=x^{\beta_1}(x-1)^{\beta_2}$, the branch order ν_i-1 is determined, and the cyclic automorphism $y\to e^{\frac{2\pi i}{m}}y$ generates $(\nu_1, \nu_2, \nu_\infty)$ group whose order is m. Representing as a normal polygon Π, Π is a $2m$-sided polygon, and the half fundamental region F is a triangle whose angle is $(2\pi/\nu_1, 2\pi/\nu_2, 2\pi/\nu_\infty)$. If $l_\infty=1$, then $\nu_\infty=m$.

We have assumed that this cyclic group H has the maximal order m among all cyclic subgroups of automorphism group G. So, the half fundamental region S of G must have a vertex of angle $2\pi/m$. Let the angles of S be $2\pi/\nu_1'$, $2\pi/\nu_2'$, $2\pi/m$. From Lemma 2, the order n of G is given by

$$\frac{4}{n}=1-\frac{1}{\nu_1'}-\frac{1}{\nu_2'}-\frac{1}{m} .$$

From the known result in the non-euclidean geometry, the area of k-sided polygon whose vertex is $2\pi/\nu_i$ $(i=1, 2, \cdots, k)$ is given $k-2-\sum 1/\nu_i$. So, for our cases $4/n$ represents the area of S. As the area of F is a multiple of that of S, for given positive integers ν_1, ν_2, allowed pair (ν_1', ν_2') is restricted.

To know whether (ν_1, ν_2, m) group is a subgroup of (ν_1', ν_2', m) group, we write their graphical representation, and see whether several "S" fill up F. For small ν_1, ν_2, it is an easy combination problem.

As we noted in § 6, geometrical extension is not sufficient a (ν_1', ν_2', m) group. G may be a topological extension of H, but may not be a conformal extension of H, because we know that a conformal automorphisms preserve the positions of Weierstrass points, and we have not concerned with it. Conversely if the positions of Weierstrass points are preserved, we can connect Π side by side in the former correspondence, and the constructed surface may be regarded as the same Riemann surface as R, because allowed algebraic function field on it is not changed for their Weierstrass points. In this respect, see Bliss [2].

Our problem has been reduced to divide F into several S not disturbing the position of Weierstrass points. We denote the set of Weierstrass points simply by W.

11. Divisors. To study the distribution of W, we must investigate the differentials of the first kind.

Let R be given by $y^m=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r}$, $\sum\beta_i\not\equiv 0 \pmod m$.

For prime m, Kuribayashi [7] has studied the differentials of the first kind of this type. We want to extend his result to general m.

Let $A_\iota^1, \cdots, A_i^l$ be points lying over $x=\alpha_\iota$ $(\iota=1, 2, \cdots, r, \infty)$. We take one of them, say A_i^1. At A_ι^1, the local parameter is $t=(x-\alpha_\iota)^{1/\nu_i}$. We denote the divisor of $\mathfrak{a}$ by $(\mathfrak{a})$, $\mathfrak{a}$ may be a function or may be a differential.

$x-\alpha_i=t^{\nu_i}$ has zeros of order ν_i at $A_i^1, \cdots, A_i^{l_\iota}$, and has pole of order ν_∞ at $A_\infty^1, \cdots, A_\infty^{l_\infty}$. So the divisor is

$$(x-\alpha_\iota)=\nu_i\sum_k^{l_\iota}A_i^k-\nu_\infty\sum_k^{l_\infty}A_\infty^k$$

$dx=\nu_i t^{\nu_i-1}$ has zeros of order ν_i-1 at $A_i^1, \cdots, A_i^{l_\iota}$, and at $x=\infty$, the local parameter is $t=(1/x)^{1/\nu_\infty}$, $x=t^{-\nu_\infty}$. So, $dx=-\nu_\infty t^{-\nu_\infty-1}$ has poles of order ν_∞ at $A_\infty^1, \cdots, A_\infty^{l_\infty}$. So the divisor is

$$(dx)=\sum_\iota^r(\nu_i-1)\sum_k^{l_i}A_i^k-(\nu_\infty+1)\sum_k^{l_\infty}A_\infty^k\,.$$

The function $y=\sqrt[m]{(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r}}$ is locally $(x-\alpha_\iota)^{\beta_i/m}=t^{\nu_i\beta_i/m}$, and at $x=\infty$, we can get $t^{-\nu_\infty\Sigma\beta_i/m}$. So the divisor is

$$(y)=\sum_\iota^r\frac{\nu_i\beta_i}{m}\sum_k^{l_\iota}A_i^k-\frac{\nu_\infty\sum\beta_i}{m}\sum_k^{l_\infty}A_\infty^k\,.$$

For simplicity, we write $\sum_k^{l_i}\frac{A_i^k}{l_\iota}=A_\iota$, $\sum_k^{l_\infty}\frac{A_\infty^k}{l_\infty}=A_\infty$. Using the relations, $\nu_i=m/l_\iota$, $\nu_\infty=m/l_\infty$, we have

LEMMA 13. *For*

$$y^m=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r}\,,\qquad \sum\beta_i\neq 0 \pmod m\,,$$

divisors are given by

$$(x-\alpha_\iota)=m(A_i-A_\infty)\,,$$

$$(dx)=\sum_\iota(m-l_\iota)A_i-(m+l_\infty)A_\infty\,,$$

$$(y)=\sum_\iota\beta_i(A_i-A_\infty)\,.$$

12. Differentials of the first kind. We consider the divisor by Lemma 13

$$(y^{m-1}/dx)=(m-1)\sum_i \beta_i(A_i-A_\infty)-\sum_i(m-l_i)A_i+(m+l_\infty)A_\infty$$
$$=\sum_i\{(m-1)\beta_i-(m-l_i)\}A_i+\{(m+l_\infty)-(m-1)\sum_i\beta_i\}A_\infty .$$

At A_i, $(m-1)\beta_i-(m-l_i)=(\beta_i-1)(m-1)+l_i-1\geqq 0$. If we consider an arbitrary differential of the first kind ω on R, $(\omega/dx)y^{m-1}$ is a function on R. As ω is the first kind, this function is finite in $|x|<\infty$, so must be polynomial of x and y. We put it $R(x,y)=(\omega/dx)y^{m-1}$, $\omega=\{R(x,y)dx\}/y^{m-1}$. $R(x,y)$ is a polynomial of degree$\leqq m-2$ in y. Otherwise, at A_∞, the poles of dx remains, and contradicts ω to be of the first kind. Thus we have gotten the form ω. The uniqueness of the representation is obvious.

LEMMA 14. *The differentials of the first kind of the surface $y^m=f(x)$ is uniquely given by $R(x, y)dx/y^{m-1}$, where $R(x, y)$ is a polynomial of x and y, and the degree of y is less than $m-1$.*

In order to get a simpler form of $R(x, y)$, we must take a suitable basis of differentials. Corresponding to the cyclic automorphisms $y\to e^{\frac{2\pi i}{m}}y$, the differentials of the first kind changes by a linear transformation. As Hurwitz [5] shows, the corresponding matrix can be diagonal, whose diagonal elements are $\zeta_1, \zeta_2, \zeta_3$, where ζ_i is an m-th root of unity.

We take the base differentials corresponding them. We consider the differential ω_1 corresponding to $\zeta_i=\varepsilon^{e_i}(\varepsilon=e^{2\pi i/m})$. By Lemma 14, we can put

$$\omega_1=\frac{R(x, y)}{y^{m-1}}dx=\frac{P(x, y)y^c+Q(x)}{y^k}dx\,, \qquad \text{where} \quad P(x, 0)\neq 0,\ 0<c<k<m\,.$$

By the transformation $y\to\varepsilon y$, $\omega_1\to\varepsilon^{e_1}\omega_1$. So,

$$\frac{P(x, \varepsilon y)\varepsilon^c y^c+Q(x)}{\varepsilon^k y^k}=\varepsilon^{e_1}\frac{P(x, y)y^c+Q(x)}{y^k}\,,$$
$$P(x, \varepsilon y)\varepsilon^c y^c+Q(x)=\varepsilon^{e_1+k}\{P(x, y)y^c+Q(x)\}\,.$$

Putting $y=0$, $Q(x)\,(1-\varepsilon^{e_1+k})=0$. So either $Q(x)=0$ or $e_1+k\equiv 0 \pmod m$.

If $e_1+k\equiv 0 \pmod m$ occurs, $P(x, \varepsilon y)=\varepsilon^{-c}P(x, y)$. Putting $y=0$, $P(x, 0)=\varepsilon^{-c}P(x, 0)$. As $P(x, 0)\neq 0$, this implies $\varepsilon^{-c}=1$, contradicting $0<c<m$.

Thus $Q(x)=0$, that is $\omega_1=(P(x, y)dx)/y^{k-c}$. If $P(x, y)$ contains y, the same method can be used. By repeating this process, we know that $R(x, y)$ does not contain y. Thus we have

LEMMA 15. *The basis of the differentials of the first kind of the surface $y^m=f(x)$ is given by $P(x)dx/y^t$, where $P(x)$ is a polynomial.*

Let R be given by $y^m=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r}$, one of the base differential is, by Lemma 15, $\omega=\{P(x)dx\}/y^t$. Let the degree of $P(x)$ be s. We consider the divisor of ω at A_∞. By Lemma 13, the coefficient of A_∞ is

$$t\sum\beta_i-m-l_\infty-ms\geqq 0 \qquad (\because\ \omega \text{ is the first kind}).$$

If $m-l_i-t\beta_i\geqq 0$ for all i, $(i=1,2,\cdots,r)$, dx/y^t, $(x-\alpha_1)dx/y^t,\cdots,(x-\alpha_1)^s dx/y^t$ are all first kind, and ω is generated by these differentials.

If for some i, $m-l_i-t\beta_i<0$, $P(x)$ must contain the factor $(x-\alpha_i)^{k_i}$, because ω is the first kind. And $m-l_i-t\beta_i+mk\geqq 0$ holds. Considering each i respectively, we can get $f(x)=(x-\alpha_1)^{k_1}\cdots(x-\alpha_r)^{k_r}g(x)$, where order $g(x)=s-\sum^r k_i$. In this case

$$\frac{(x-\alpha_1)^{k_1+d}(x-\alpha_2)^{k_2}\cdots(x-\alpha_r)^{k_r}}{y^t}dx \qquad (d=1,2,\cdots,s-\sum^r k_i)$$

are of the first kind and generate ω. Thus we have.

LEMMA 16. *The basis of the differentials of the first kind of the surface* $y^m=(x-\alpha_1)^{\beta_1}\cdots(x-\alpha_r)^{\beta_r}$ *is given by* $\dfrac{(x-\alpha_1)^{k_1}\cdots(x-\alpha_r)^{k_r}}{y^t}dt$.

As an important result, we can get a method of calculating differentials. The principle lies in the proof of Lemma 15. We summalize it as follows.

LEMMA 17. *The differential* $(x-\alpha_1)^{k_1}\cdots(x-\alpha_r)^{k_r}dx/y^t$ *is the first kind if and only if*

$$\begin{cases} t\sum\beta_i-m-l_\infty-m\sum k_i\geqq 0 \\ m-l_i-t\beta_i+mk_i\geqq 0 \end{cases} \qquad (i=1,2,\cdots,r).$$

We call the basis corresponding to $y\to\varepsilon y$ the normalized basis.

13. Simple example. For an application of Lemma 17, we consider the simple case;

$$y^m=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_r).$$

Here, $\beta_i=l_i=1$, $\sum\beta_i=r$, $l_\infty=$G.C.M.(m,r).

The second conditions of Lemma 17, that is, $m-l_i-t\beta_i\geqq 0$ are always satisfied, without using k_i. So we may consider only $t\sum\beta_i-m-l_\infty-m\sum k_i\geqq 0$. As $\sum k_i$ is the degree of $P(x)$ used in Lemma 15, we put it s.

$tr-m-l_\infty-ms\geqq 0$ has solutions for each $t=1,2,\cdots,m-1$. We count the number of the solutions. For fixed t, allowed values of s are

$$s=0,1,2,\cdots,\left[\frac{tr-m-l_\infty}{m}\right], \qquad ([\ \ ] \text{ means Gaussian symbol}).$$

The number of solution is $\left[\dfrac{tr-l_\infty}{m}\right]$.

Summing them for $t=1,2,\cdots,m-1$, we have

$$\left[\frac{r-l_\infty}{m}\right]+\left[\frac{2r-l_\infty}{m}\right]+\cdots+\left[\frac{(m-1)r-l_\infty}{m}\right].$$

But the dimension of the space which is constructed by differentials of the first kind is clearly g. Therefore, above sum must be g, and we have got the basis

$$\frac{dx}{y},\ \frac{x}{y}dx, \cdots, \frac{x^{s_1}}{y}dx;\ \frac{dx}{y^2},\ \frac{x}{y^2}dx, \cdots, \frac{x^{s_2}}{y^2}dx;\ \cdots;\ \frac{dx}{y^{m-1}},\ \frac{x}{y^{m-1}}, \cdots, \frac{x^{s_{m-1}}}{y^{m-1}}dx\,.$$

where $s_i=\left[\frac{ir-l_\infty}{m}-1\right]$.

The genus of this surface can be calculated directly by Lemma 5, that is,

$$g=\frac{m(r-1)-\sum l_i+2}{2}=\frac{(r-1)(m-1)+1-l_\infty}{2}\,.$$

Comparing above results, we have a formula in elementary number theory.

LEMMA 18. *For given positive integer m, r, let* G.C.M. $(m, r)=d$. *Then*

$$\left[\frac{r-d}{m}\right]+\left[\frac{2r-d}{m}\right]+\cdots+\left[\frac{(m-1)r-d}{m}\right]=\frac{(r-1)(m-1)+1-d}{2}\,.$$

Of course this can be proved elementarily. But it is worth remarking that there is a function-theoretic proof.

14. points at vertex. Let R be a Riemann surface given by $y^m=f(x)$, and the base differentials of the first kind be $P_1(x)dx/y^{t_1}, \cdots, P_g(x)dx/y^{t_g}$. By the cyclic automorphism $y\to\varepsilon y$, these differentials are multiplied by $\varepsilon^{m-t_1}, \cdots, \varepsilon^{m-t_g}$. Concerning it, Lewittes has proved as follows.

LEMMA 19. *Let $\gamma_1, \cdots, \gamma_g$ be the gap sequence at P, and P be the fixed point of a cyclic automorphisms of order m. If this automorphism caused the rotation for the local parameter $t\to\zeta t$, then the diagonal elements with respect to the normalized bases are $\zeta^{\gamma_1}, \zeta^{\gamma_2}, \cdots, \zeta^{\gamma_g}$.*

We know already the value ζ by Lemma 10. If $y\to\varepsilon^{l_i}y$, $t\to\zeta^{l_i}t=\varepsilon^{l_ib_i}t$. By Lemma 19, the diagonal elements are $\varepsilon^{l_ib_i\gamma_1}, \varepsilon^{l_ib_i\gamma_2}, \cdots, \varepsilon^{l_ib_it_g}$. On the other hand, as shown above, the diagonal elements are $\varepsilon^{l_i(m-t_1)}, \cdots, \varepsilon^{l_i(m-t_g)}$. The order of permutation is not same, so for suitable pair,

$$l_ib_i\gamma\equiv l_i(m-t) \pmod{m}, \qquad b_i\gamma+t\equiv 0 \pmod{\nu_i}\,.$$

If $\nu_i=m$, that is $l_i=1$, at A_i the gap sequence $\gamma_1, \cdots, \gamma_g$ is completely determined, and we can determine whether $A_i\in W$ or not.

LEMMA 20. *If the normalized differentials of $y^m=f(x)$ are $\{p_1(x)dx\}/y^{t_1}, \cdots, \{p_g(x)dx\}/y^{t_g}$, the Weierstrass gap sequences of A_i are given by $b_i\gamma+t_k\equiv 0$ (mod ν_i), $k=1, 2, \cdots, g$.*

If $\nu_i \neq m$, γ sequences are not always determined uniquely, but give us some information. For the equations in Lemma 12, each equation has $b_i=1$ for at least one i $(=1, 2, \infty)$. This shows $\gamma+t_k \equiv 0 \pmod{\nu_i}$. As every gap sequence contains 1 always, and contains 2 if and only if R is not hyperelliptic, we have

LEMMA 21. *As for the equations in Lemma* 12, *the inequallities of Lemma* 17 *have always solutions for $t=m-1$. And the surface is hyperelliptic if and only if the inequallities has no solution for $t=m-2$.*

15. Distribution of W. If $\nu_i \neq m$, the gap sequence is not determined uniquely. At an arbitrary point P on R, we consider the gap sequence $\gamma_1, \cdots, \gamma_g$. We call $m=\sum \gamma_k - g(g+1)/2$ the rest number at P. Of course, $m>0$ if and only if $P \in W$. As Hurwitz [5] shows, the snm of the rest number over R is $(g-1)g(g+1)$. When we cannot see wheather some vertex belongs to W, or when the total sum of the rest numbers at vertices is not equal to $(g-1)g(g+1)$, we must study W points in detail.

Let the normalized basis be $P_1(x)dx/y^{t_1}, \cdots, P_g(x)dx/y^{t_g}$. At an arbitrary point on R, we take a local parameter $t=x-c$. Substituting it in $y^m=f(x)$ we can get the normalized differentials in term of t. At an ordinary points, the normalized property is of no use. In order to get diagonal basis, we must consider linear substitutions of them, and bring them to a diagonal form.

As we are concerned with the case $g=3$, we proceed in this case. Let the differentials of the first kind be given in terms of t, as follows:

$$(a_0+a_1t+a_2t^2+O(t^3))dt\,,$$

$$(a_0'+a_1't+a_2't^2+O(t^3))dt\,,$$

$$(a_0''+a_1''t+a_2''t^2+O(t^3))dt$$

This point belongs to W if and only if

$$\Delta=\begin{vmatrix} a_0 & a_1 & a_2 \\ a_0' & a_1' & a_2' \\ a_0'' & a_1'' & a_2'' \end{vmatrix}=0\,,$$

that is, at least one of the eigen-values of Δ must be zero. If we need to know the rest number, we must consider further terms of higher order, and practise the elimination processes.

In the case of branch points, the calculation is easier, but in the case of an ordinary point, the coefficients a_i is a complex form containing c. So, it is a laborious work. But, at any rate, we have a way to proceed.

16. Calculations. We have already preliminary knowledges. We shall investigate the distribution of W about ten equations defined in Lemma 12.

(i) $y^{14}=x^7(x-1)^2$, $m=14$, $\beta_1=7$, $\beta_2=2$, $\sum\beta_i=9$, $l_1=7$, $l_2=2$, $l_\infty=1$.

The inequallityes in Lemma 17 are

$$\begin{cases} 9t-14-1-14(k_1+k_2)\geqq 0 \\ 14-7-7t+14k_1\geqq 0 \\ 14-2-2t+14k_2\geqq 0\,. \end{cases}$$

These have solutions $t=5$, $k_1=2$, $k_2=0$; $t=11$, $k_1=5$, $k_2=1$; $t=13$, $k_1=6$, $k_6=1$. This means that the normalized differentials are x^2dx/y^5, $x^5(x-1)dx/y^{11}$, $x^6(x-1)dx/y^{13}$.

We can see that this surface is hyperelliptic by Lemma 21. But the next table is useful for our understanding.

$$\begin{array}{cccccc|ccc}
 & l & \beta & (\mathrm{mod}\,\nu) & \mu & b & m-t_1=9 & m-t_2=3 & m-t_3=1 \\
0 & 7 & 7 & 2 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 7 & 1 & 1 & 2 & 3 & 1 \\
\infty & 1 & 5 & 14 & 5 & 3 & 3 & 1 & 5
\end{array}.$$

Outside the bracket, every data has been known. In the bracket, by Lemma 20, possible value of γ is added. From this, $x=\infty$ is a W point of the rest number 3, nnd $x=1$ does not belong to W. For $x=0$, in case of (mod 2) some ambiguity arises, but original sequence is clearly 1, 3, 5. So, seven points lying over $x=0$ must be W points of the rest number 3.

We have gained 8 points which have rest number 3 respectively. Considering $(g+1)g(g-1)=24=8\cdot 3$, there is no W point other than these.

(ii) $y^{12}=x^6(x-1)^5$. By Lemma 17, we get as normalized differentials

$$\frac{x^3(x-1)^2}{y^7}dx\,,\quad \frac{x^4(x-1)^3}{y^9}dx\,,\quad \frac{x^5(x-1)^4}{y^{11}}dx\,.$$

This is also hyperelliptic. The table is

$$\begin{array}{cccccc|ccc}
l & \beta & \nu & \mu & b & & (5) & (3) & (1) \\
0 & 6 & 6 & 2 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 5 & 12 & 5 & 5 & 1 & 3 & 5 \\
\infty & 1 & 1 & 12 & 1 & 1 & 5 & 3 & 1
\end{array}.$$

As before, we can see all the vertices are W points with rest number 3. Over $x=0$, six points lie, so all W points are known.

(iii) $y^{12}=x^3(x-1)$. By Lemma 17, we have normalized differentials

$$\frac{x}{y^7}dx\,,\quad \frac{x^2}{y^{10}}dx\,,\quad \frac{x^2}{y^{11}}dx\,.$$

This is not hyperelliptic. The table is

l	β	ν	μ	b		(5)	(2)	(1)
0	3	3	4	1	1	1	2	1
1	1	1	12	1	1	5	2	1
∞	4	8	3	2	2	1	1	2

This shows that gap sequence of $x=0$ and $x=\infty$ are $1, 2, 5$ and $1, 2, 4$ respectively. So, four points over $x=0$ or $x=1$ has the rest number 2, and four point over $x=\infty$ has the rest number 1. As their sum is 12, there must be one $x=c$, over which each of twelve W points has the rest number 1.

We take local parameter as $t=x-c$. Multiplying a common factor to the normalized differentials, we may consider y^4, xy, x. Substituting $x=t+c$ in $y^{12}=x^3(x-1)$, and removing a multiplying constant, we get

$$y^4=1+\frac{4c-3}{3c(c-1)}t+\frac{2c-3}{9c(c-1)^2}t^2+0(t^3)\,,$$

$$xy=1+\frac{16c-15}{12c(c-1)}t+\frac{64c^2-120c+45}{288c^2(c-1)^2}t^2+0(t^3)\,,$$

$$x=c+t\,.$$

The determinant stated in § 15 must be zero, so we get $c=9/8$. This shows that on each side of the twelve sides corresponding to $\overline{1\infty}$, there exists a W point, which has the rest number 1.

For (iv)$\sim$(x), the similar considerations are neede. We summalize them, omit the calculations, and only list the results.

17. The table of W-distribution.

			Rest Number			
(i)	$y^{14}=x^7(x-1)^2$ H.E.	x^2dx/y^5, $x^5(x-1)dx/y^{11}$, $x^6(x-1)dx/y^{13}$	$0(7)$ 3	$\infty(1)$ 3		
(ii)	$y^{12}=x^6(x-1)^5$ H.E.	$x^3(x-1)^2dx/y^7$, $x^4(x-1)^3dx/y^9$, $x^5(x-1)^4dx/y^{11}$	$0(6)$ 3	$1(1)$ 3	$\infty(1)$ 3	
(iii)	$y^{12}=x^3(x-1)$	xdx/y^7, x^2dx/y^{10}, x^2dx/y^{11}	$0(3)$ 2	$1(1)$ 2	$\infty(4)$ 1	$9/8(12)$ 1
(iv)	$y^9=x^3(x-1)$	xdx/y^5, x^2dx/y^7, x^2dx/y^8	$0(3)$ 1	$1(1)$ 1	$\infty(1)$ 2	$(-3\pm\sqrt{3})/2(9)$ 1
(v)	$y^8=x^2(x-1)$	dx/y^3, xdx/y^6, xdx/y^7	$0(2)$ 2	$1(1)$ 2	$\infty(1)$ 2	$2(8)$ 2

(vi)	$y^8=x(x-1)$	H.E.	$dx/y^5,\ dx/y^6,\ dx/y^7$	$\underset{3}{1/2(8)}$
(vii)	$y^7=x(x-1)$	H.E.	$dx/y^4,\ dx/y^5,\ dx/y^6$	$\underset{3}{\infty(1)}\ \underset{3}{1/2(7)}$
(viii)	$y^7=x^2(x-1)$		$dx/y^3,\ xdx/y^5,\ xdx/y^6$	$\underset{1}{0(1)}\ \underset{1}{1(1)}\ \underset{1}{\infty(1)}\ \underset{1}{c_1(7)}\ \underset{1}{c_2(7)}\ \underset{1}{c_3(7)}$
(ix)	$y^6=x^3(x^2-1)^2$		$dx/y,\ xdx/y^2,\ x^2(x^2-1)dx/y^5$	$\underset{2}{0(3)}\ \underset{1}{\pm 1(2)}\ \underset{2}{\infty(1)}\ \underset{1}{\pm 3(6)}$
(x)	$y^6=x^3(x-1)^3(x-\alpha)$	H.E.	$dx/y,\ x(x-1)dx/y^3,\ x^2(x-1)^2dx/y^5$	$\underset{3}{0(3)}\ \underset{3}{1(3)}\ \underset{3}{\alpha(1)}\ \underset{3}{\infty(1)}$

The meaning of this table may be seen by the examples already mentioned. For (viii), c_1, c_2, c_3 are three zeros of c^3+5c^2-8c+1, so they belongs $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$ respectively. For (ix), in order to increase symmetricity, we put $\alpha=-1$.

18. Extension of the group. The last work to do is, as shown in § 10, to extend the cyclic group without disturbing the distribution of W.

(i) $y^{14}=x^7(x-1)^2$. Original cyclic group H is (14, 7, 2) group. If this is a subgroup of some G, G must be $(14, \nu_1', \nu_2')$ group. As $1-1/14-1/7-1/2=2/7$, the value $(7/2)(1-1/14-1/\nu_1'-1/\nu_2')$ must be an integer >1.

The only group which satisfy this condition is (14, 3, 2) group. Above number is three, so if this group exists, a half fundamental region F of H contains three half fundamental region S of G.

This division is possible. Let F be $\triangle OAB$, where O, A and B correspond to $x=\infty$, 0 and 1 respectively. Let the intersection of $\overline{OA}$ with the bisecting line of $\angle ABO$ be C, and the intersection of $\overline{OB}$ with the perpendicular line through C be D. BC and CD divide F into congruent three triangles whose angles are $2\pi/14$, $2\pi/3$, $2\pi/2$ respectively.

But this extension cannot be allowed, because W points is not equivalent for this division. If O and B were W points instead of O and A, this divison would be allowed.

Thus H cannot be extended. So, we know that the automorphism group of R is H itself.

We can also know that there is no (14, 3, 2) group for $g=3$ from the work of Accola(1).

(ii) $y^{12}=x^6(x-1)^5$. The cyclic group H is (12, 12, 2) group. Three vertices are all W points. This group has an extension. Using same notions, let the midpoint of OB be C. AC divide F into two congruent triangles, which has W points at congruent point. Further division is not possible. This extended group G is (12, 4, 2) group, and the order of G is 24.

We can study the cases (iii)$\sim$(x) in the same way. If we treat non-hyperelliptic cases, we must distinguish, the rest numbers. Again we omit the intermediate explanations, and show the list of the results.

19. The list of extended group.

		original group	extended group	order
(i)	$y^{14}=x^7(x-1)^2$	(14, 7, 2)	(14, 7, 2)	14
	H.E.	3 3	3 3	
(ii)	$y^{12}=x^6(x-1)^5$	(12, 12, 12)	(12, 4, 2)	24
	H.E.	3 3 3	3 3	
(iii)	$y^{12}=x^3(x-1)$	(12, 4, 3) 9/8	(12, 3, 2)	48
		2 2 1 1	2 1	
(iv)	$y^9=x^3(x-1)$	(9, 9, 3) $(-3\pm\sqrt{3})/2$	(9, 9, 3) $(-3\pm\sqrt{3})/2$	9
		2 1 1 1	2 1 1 1	
(v)	$y^8=x^2(x-1)$	(8, 8, 4) 2	(8, 3, 2)	96
		2 2 2 2	2	
(vi)	$y^8=x(x-1)$	(8, 8, 4) 1/2	(8, 4, 2)	32
	H.E.	3	3	
(vii)	$y^7=x(x-1)$	(7, 7, 7) 1/2	(7, 7, 7) 1/2	7
	H.E.	3 3	3 3	
(viii)	$y^7=x^2(x-1)$	(7, 7, 7) c_1 c_2 c_3	(7, 3, 2)	168
		1 1 1 1 1 1	1	
(ix)	$y^6=x^3(x^2-1)^2$	(6, 3, 3, 2) ± 3	(6, 3, 3)	24
		2 1 1 2 1	2 1 1	
(x)	$y^6=x^3(x-1)^3(x-c)$	(6, 6, 2, 2)	(6, 4, 2)	48
	H.E.	3 3 3 3	3	

The meaning of this list may be understood the examples already explained. For (x), c must be suitably chosen in order to extend maximally.

20. Subgroups of the extended group. We want to know not only the extended group, but also their subgroups. To make subgroups from their extended group is easier. In the case of extension, we must care the position of W, and not to break the maximal branch order $m-1$. But to make subgroups, we collect the half-fundamental regions S of G, and only need to examine if the order of the new group can be got by this polygon.

As stated in § 2, for given subgroups, the possibility of k (the number of boundary components) is determined.

(i) (14, 7, 2) group. For simplicity, we denote the vertices A_{14}, A_7, A_2. Here the suffices correspond to their branch orders. Reflecting S against the side

A_{14}, A_2, we get a new triangle.

Each vertices have $2\pi/7$ angle, so this is the half fundamental region of (7, 7, 7) group. From the nature of the figure, there is no other subgroup.

For (7, 7, 7) group, allowed values of k are $14n+\delta_1+2\delta_2+7\delta_3$, by Lemma 3. That is, $14n+\{0, 1, 2, 3, 7, 8, 9, 10\}$. They are equivalent to $7n+\{0, 1, 2, 3\}$. We know that for this value of k, $N(3, k)\geqq 14$.

For (7, 7, 7) group, allowed values of k are $7n+\delta_1+\delta_2+\delta_3=7n+\{0, 1, 2, 3\}$. From this, we get $N(3, k)\geqq 7$. But comparing the above bounds, this result is meaningless, because the set of allowed k is the same.

(ii) (12, 4, 2) group. We use the same notations as before. Reflecting S with respect to side A_2A_4, $A_{12}A_2$ and $A_{12}A_4$, we get the half fundamental regions of (12, 12, 2), (6, 4, 4) and (6, 2, 2, 2) group respectively. Each order of these groups is 12. If we collect three S around A_{12}, we get the half fundamental regions of (4, 4, 2, 2) group of order 8.

The area of the half fundamental region and the order of the group is reversely proportional. So, by further reflecting the half fundamental regions of order 12, we get groups of order 6. The type is different according to the way of reflections. Their types are, (6, 6, 2, 2), and (3, 2, 2, 2, 2).

If we reflect the half fundamental region of order 8, new groups may be got. But we need not them, because the orders of them are only 4.

For (12, 4, 2) group, allowed values of k are $24n'+2\delta_1+6\delta_2+12\delta_3=6n+\{0, 2\}$. Values of k for the group of order 12 are as follows

(12, 12, 2)	group	$12n'+\delta_1+\delta_2+6\delta_3=6n+\{0, 1, 2\}$
(6, 4, 4)	group	$12n+2\delta_1+3\delta_2+3\delta_3=12n+\{0, 2, 3, 5, 6, 8\}$
(6, 2, 2, 2)	group	$12n'+2\delta_1+6\delta_2+6\delta_3+6\delta_4=6n+\{0, 2\}$.

The third set is contained in the first set, so this group is useless. Likewise, allowed values of k for (4, 4, 2, 2) group are $8n'+2\delta_1+2\delta_2+4\delta_3+4\delta_4=2n$.

We have got two types of the group of order 6. (6, 6, 2, 2) group is of interest, because the values of k are $6n+\delta_1+\delta_2+2\delta_3+2\delta_4$. These mean that k can run over all integers. Thus we have $N(3, k)\geqq 6$ for all k. We need not examine the group of order 6 no more.

For (iii)$\sim$(x), similar consideration is needed. Again we omit them, and summalize the result.

21. The list of the groups and allowed values of k.

	order	group		allowed k	
(i)	14	(14, 7, 2)	◎	$14n+\delta_1+2\delta_2+7\delta_3$	$7n+\{0, 1, 2, 3\}$
	7	(7, 7, 7)		$7n+\delta_1+\delta_2+\delta_3$	$7n+\{0, 1, 2, 3\}$
(ii)	24	(12, 4, 2)	◎	$24n+2\delta_1+6\delta_2+12\delta_3$	$6n+\{0, 2\}$
	12	(12, 12, 2)	◎	$12n+\delta_1+\delta_2+6\delta_3$	$6n+\{0, 1, 2\}$

	12	(6, 4, 4)		$12n+2\delta_1+3\delta_2+3\delta_3$	$12n+\{0, 2, 3, 5, 6, 8\}$
	12	(6, 2, 2, 2)		$12n+2\delta_1+6\delta_2+6\delta_3+6\delta_4$	$6n+\{0, 2\}$
	8	(4, 4, 2, 2)		$8n+2\delta_1+2\delta_2+4\delta_3+4\delta_4$	$2n$
	6	(6, 6, 2, 2)	◎	$6n+\delta_1+\delta_2+2\delta_3+2\delta_4$	n
(iii)	48	(12, 3, 2)	◎	$48n+4\delta_1+16\delta_2+24\delta_3$	$24n+\{0, 4, 16, 20\}$
	24	(6, 3, 3)		$24n+4\delta_1+8\delta_2+8\delta_3$	$4n$
	12	(12, 4, 3)	◎	$12n+\delta_1+3\delta_2+4\delta_3$	$12n+\{0, 1, 3, 4, 5, 7, 8\}$
(iv)	9	(9, 9, 3)	◎	$9n+\delta_1+\delta_2+3\delta_3$	$9n+\{0, 1, 2, 3, 4, 5\}$
(v)	96	(8, 3, 2)	◎	$96n+12\delta_1+32\delta_2+48\delta_3$	$48n+\{0, 12, 32, 44\}$
	48	(4, 3, 3)		$48n+12\delta_1+16\delta_2+16\delta_3$	$16n+\{0, 12\}$
	32	(8, 4, 2)		$32n+4\delta_1+8\delta_2+16\delta_3$	$4n$
	16	(4, 4, 4)		$16n+4\delta_1+4\delta_2+4\delta_4$	$4n$
	16	(4, 2, 2, 2)		$16n+4\delta_1+8\delta_2+8\delta_3+8\delta_4$	$4n$
	16	(8, 8, 2)	◎	$16n+2\delta_1+2\delta_2+8\delta_3$	$8n+\{0, 2, 4\}$
	8	(8, 8, 4)	◎	$8n+\delta_1+\delta_2+2\delta_3$	$8n+\{0, 1, 2, 3, 4\}$
	8	(4, 4, 2, 2)		$8n+2\delta_1+2\delta_2+4\delta_3+4\delta_4$	$2n$
	8	(2, 2, 2, 2, 2)		$8n+4\delta_1+4\delta_2+4\delta_3+4\delta_4+4\delta_5$	$4n$
(vi)	32	(8, 4, 2)		appeared already	
(vii)	7	(7, 7, 7)			
(viii)	168	(7, 3, 2)	◎	$168n+24\delta_1+56\delta_2+84\delta_3$	$84n+\{0, 24, 56, 80\}$
(ix)	24	(6, 3, 3)		appeared already	
(x)	48	(6, 4, 2)	◎	$48n+8\delta_1+12\delta_2+24\delta_3$	$12n+\{0, 8\}$
	24	(6, 6, 2)		$24n+4\delta_1+4\delta_2+12\delta_3$	$4n$
	24	(4, 4, 3)		$24n+6\delta_1+6\delta_2+8\delta_3$	$24n+\{0, 6, 8, 12, 14, 20\}$
	24	(3, 2, 2, 2)		$24n+8\delta_1+12\delta_2+12\delta_3+12\delta_4$	$12n+\{0, 8\}$
	16	(4, 2, 2, 2)		$16n+4\delta_1+8\delta_2+8\delta_3+8\delta_4$	$4n$
	12	(3, 3, 2, 2)		$12n+4\delta_1+4\delta_2+6\delta_3+6\delta_4$	$2n$ (except 2)
	12	(6, 6, 3)	◎	$12n+2\delta_1+2\delta_2+4\delta_3$	$2n+\{0, 2, 4, 6, 8\}$

The subgroups which has the type appeared already are all omitted, and the group of order 6 is shown for only one case for the reason stated in the previous section. We had eight exceptional cases in Lemma 4, but these types are all appeared in this list. So, we know the existence of the groups.

In this list, the groups with the symbol ◎ are needed to determine the values $N(3, k)$. Summing up these results, we can obtain the theorem.

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