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ON THE SPECTRUM OF LENS SPACES

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Let (S^{2n-1}, g_0) be a $(2n-1)$-dimensional sphere of constant curvature 1, and be imbedded in $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$. Let T be an element of $SO(2n)$ which is defined by

$$T:\ (z_1, \cdots, z_n) \longrightarrow (e^{\frac{2\pi}{p}\sqrt{-1}}z_1, \cdots, e^{\frac{2\pi}{p}\sqrt{-1}}z_n),$$

and G be a cyclic group of order p generated by T.

Then G acts on (S^{2n-1}, g_0) as a deck transformation group and we have the lens space $M=S^{2n-1}/G$ which has a homogeneous riemannain metric of constant curvature 1 (See J. Wolf [2]).

In M. Berger [1], spectrum of spheres, real and complex projective spaces are given. In the present note we shall give the spectrum of homogeneous lens space of constant curvature explicitely.

§ **1.** *Spectrum of* (S^{2n-1}, g_0). (For the proof, see [1], pp. 172). Let $S^{2n-1}\subset\boldsymbol{C}^n$ be a shere of constant curvature 1. Let $(z_j, \bar{z}_j)$ $(j=1, \cdots, n)$ be complex coordinates of $\boldsymbol{C}^n$ and put

$$\partial/\partial z_j=1/2(\partial/\partial x_j-\sqrt{-1}\,\partial/\partial y_j), \qquad \partial/\partial \bar{z}_j=1/2(\partial/\partial x_j+\sqrt{-1}\,\partial/\partial y_j),$$

where $(x_1, \cdots, x_n;\ y_1, \cdots, y_n)$ be the coordinates of $\boldsymbol{R}^{2n}=\boldsymbol{C}^n$, i.e. $z_j=x_j+\sqrt{-1}\,y_j$ $(j=1, \cdots, n)$.

Now we define the Laplacian acting on $C_c^\infty(C^n)$—space of complex-valued C^∞-functions on C^n—by

$$-\Delta f=\sum_{j=1}^n\left(\frac{\partial^2}{\partial x_j^2}f+\frac{\partial^2}{\partial y_j^2}f\right)=4\sum_{j=1}^n \frac{\partial^2 f}{\partial \bar{z}^j\partial z^j}\ .$$

Let P be a bihomogeneous polinomial of bidegree (k, l), i.e., degree k on z and degree l on $\bar{z}$, then P is harmonic if and only if $\Delta P=0$ holds.

$\mathcal{P}_{k,l}$ (resp. $\mathcal{H}_{k,l}$) denotes a ring of all bihomogeneous polynomials (resp. harmonic bihomogeneous polynomials) of bidegree (k, l). Then we have

$$\mathcal{P}_{k,l}=\mathcal{H}_{k,l}\oplus r^2\mathcal{P}_{k-1,l-1}\,.$$

If we define $\bar{\mathcal{P}}_{k,l}=\mathcal{P}_{k,l}|(S^{2n-1})$, and $\bar{\mathcal{H}}_{k,l}=\mathcal{H}_{k,l}\cap\bar{\mathcal{P}}_{k,l}$, we have

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PROPRSITION 1.1. *$\bar{\mathscr{P}}_{k,l}=\bar{\mathscr{H}}_{k,l}\oplus\bar{\mathscr{P}}_{k-1,l-1}$, that is, $\bigoplus_{k,l}\bar{\mathscr{H}}_{k,l}=\bigoplus_{k,l}\bar{\mathscr{P}}_{k,l}$ holds., and this $\bigoplus_{k,l}\bar{\mathscr{H}}_{k,l}$ is dense in $C_c^\infty(S^{2n-1})$ in the sense of uniform convergence.*

If $P\in\mathscr{H}_{k,l}$, then we have

$$\Delta^{S^{2n-1}}\bar{P}=\Big(\Delta^{C^n}P+(2n-1)\frac{\partial P}{\partial r}+\frac{\partial^2 P}{\partial r^2}\Big)_{|S^{2n-1}}$$
$$=(k+l)(2n+k+l-2)\bar{P}$$

and $\dim\bar{\mathscr{H}}_{k,l}=\dim\bar{\mathscr{P}}_{k,l}-\dim\bar{\mathscr{P}}_{k-1,l-1}$.

From this we have

PROPOSITION 1.2. *The spectrum of (S^{2n-1}, g_0) is the set $\lambda_p=p(2n+p-2)$ (p; non-negative integer) with multiplicity $\binom{2n+p-1}{p}-\binom{2n+p-3}{p-2}$.*

2. Spectrum of homogeneous lens space. In this section we shall consider the spectrum of homogeneous lens space $M=S^{2n-1}/G$ with constant curvature 1, where $G=\{T^k\}_{k=0}^{k=p-1}$ with $T:(z_1,\cdots,z_n)\to(e^{\frac{2\pi}{p}\sqrt{-1}}z_1,\cdots,e^{\frac{2\pi}{p}\sqrt{-1}}z_n)$. First we consider the bihomogeneous polynomials of bidegree (k, l) which is invariant under the action of G. Since the monomial $z_1^{i_1}\cdots z_n^{i_n}\bar{z}_1^{j_1}\cdots\bar{z}_n^{j_n}$ $(i_1+\cdots+i_n=k,\ j_1+\cdots+j_n=l)$ is taken into $e^{\frac{2\pi}{p}\sqrt{-1}(k-l)}z_1^{i_1}\cdots z_n^{i_n}\bar{z}_1^{j_1}\cdots\bar{z}_n^{j_n}$ via the action of $T\in G$, bi-homogeneous polynomial of bi-degree (k, l) is G-invariant if and only if $k\equiv l$ $(\mathrm{mod}\,p)$ holds.

Let $\tilde{\mathscr{P}}_{k,l}$ (resp. $\tilde{\mathscr{H}}_{k,l}$) denotes the space of functions on M, which is deduced from $\mathscr{P}_{k,l}$ (resp. $\mathscr{H}_{k,l}$) by first restricting on S^{2n-1} and next passing to quotient by the covering map $\varphi: S^{2n-1}\to M$. By proposition 1.1, $\tilde{\mathscr{H}}_{k,l}$ $(k=l\ (\mathrm{mod}\,p))$ is a subspace of proper subspace relative to the eigenvalue $(k+1)(2n+k+1-2)$.

Next we shall show that $\bigoplus_{k\equiv l\,(\mathrm{mod}\,p)}\tilde{\mathscr{H}}_{k,l}$ is dense in $C_c^\infty(M)$ in the sense of uniform convergence. This implies that $\bigoplus_{k\equiv l\,(\mathrm{mod}\,p)}\tilde{\mathscr{H}}_{k,l}=\bigoplus_{k\equiv l\,(\mathrm{mod}\,p)}\tilde{\mathscr{P}}_{k,l}$ gives the decomposition of $C_c^\infty(M)$ by the proper subspaces of Laplacian ([1], pp. 143). Since $\bigoplus_{k\equiv l\,(\mathrm{mod}\,p)}\tilde{\mathscr{P}}_{k,l}$ is a subalgebra of $C_c^\infty(M)$ which is self-conjugate and contains the constants, it suffices to show that $\bigoplus_{k\equiv l\,(\mathrm{mod}\,p)}\tilde{\mathscr{P}}_{k,l}$ separates the points of $M=S^{2n-1}/G$ (Stone-Weierstrass theorem [1] pp. 144).

Let $x, y\in S^{2n-1}$ be points with $\varphi(x)\neq\varphi(y)$. We put $x=(z_1,\cdots,z_n)$, $y=(z_1',\cdots,z_n')$. Since $x\neq y$, there exists i such that $z_i\neq z_i'$ holds.

Case I. $z_i\bar{z}_i\neq z_i'\bar{z}_i'$ (not summed up). In this case $\varphi(x)$, $\varphi(y)$ are separated by $z_i\bar{z}_i$.

Case II. $z_i\bar{z}_i=z_i'\bar{z}_i'$, but $z_j=z_j'$ for some $j\neq i$. In this case $\varphi(x)$, $\varphi(y)$ are separated by $z_i\bar{z}_j$.

Case III. $z_k\neq z_k'$ $(k=1,\cdots,n)$, but $z_k\bar{z}_k=z_k'\bar{z}_k'$. In this case we may write

$z_k'=e^{2\pi\theta_k\sqrt{-1}}z_k$ with $\theta_k\not\equiv 0 \pmod 1$, and we have $z_j'\bar{z}_i'=e^{2\pi(\theta_j-\theta_i)\sqrt{-1}}z_j\bar{z}_i$. If $\theta_j\not\equiv\theta_i \pmod 1$ for some distinct j, i, $\varphi(x)$ and $\varphi(y)$ are separated by the $z_j\bar{z}_i$. If $z_k'=e^{2\pi\theta\sqrt{-1}}z_k$ $(k=1,\cdots,n)$ holds for some $\theta\not\equiv 0 \pmod 1$, we have $(z_k')^{p+1}\bar{z}_k'=e^{2\pi p\theta\sqrt{-1}}(z_k)^{p+1}\bar{z}_k$. In the case $e^{2\pi p\theta\sqrt{-1}}\neq 1$, $\varphi(x)$ and $\varphi(y)$ are separated by $(z_k)^{p+1}\bar{z}_k$. In the case $e^{2\pi p\theta\sqrt{-1}}=1$, we have $z_k'=e^{2\pi\frac{l}{p}\sqrt{-1}}z_k$ $(1\leqq l\leqq p-1\,;\,k=1,\cdots,n)$ and we have $\varphi(x)=\varphi(y)$.

So the eigenvalues of M is $\lambda_{k,m}=(2k+mp)\times(2n-2+2k+mp)$ $(k=0,1,2,\cdots;\ m=1,2,\cdots)$ and the multiplicity of $\lambda_{k,m}$ is equal to $\dim\tilde{\mathscr{P}}_{k,k+mp}+\dim\tilde{\mathscr{P}}_{k+mp,k}-\dim\tilde{\mathscr{P}}_{k-1,k-1+mp}-\dim\tilde{\mathscr{P}}_{k-1+mp,k-1}$.

But there is the possibility that $2k+mp=2k'+m'p$ holds for distinct values k, k' and m, m'.

A) p : *odd*. The different values of $2k+mp$ $(k=0,1,2,\cdots:\ m=0,1,2,\cdots)$ are the following;

(i) $2(s-1)p+2t$ $(s=1,2,\cdots:\ 0\leqq t\leqq(p-1)/2)$.

(ii) $(2s-1)p+2t$ $(s=1,2,\cdots:\ 0\leqq t\leqq(p-1)/2)$.

(iii) $2(s-1)p+2\left(t+\dfrac{p+1}{2}\right)$ $(s=1,2,\cdots:\ 0\leqq t\leqq(p-3)/2)$.

(iv) $(2s-1)p+2\left(t+\dfrac{p+1}{2}\right)$ $(s=1,2,\cdots:\ 0\leqq t\leqq(p-3)/2)$.

Case (i). For given s, t $(s=1,2,\cdots:\ 0\leqq t\leqq(p-1)/2)$, the k and m's which satisfy $2k+mp=2(s-1)p+2t$ are

$$\begin{cases} k=t\quad, p+t\quad, \cdots, (s-1)p+t \\ m=2(s-1), 2(s-2), \cdots, 0\,. \end{cases}$$

So the corresponding eigenvalue is $4\{(s-1)p+t\}\{(n-1)+(s-1)p+t\}$ with multiplicity

$$\sum_{\substack{a+b=\\2(s-1)\\a,b\geqq 0}}\binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}-\sum_{\substack{a+b=\\2(s-1)\\a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}.$$

Case (ii). For given s, t $(s=1,2,\cdots:\ 0\leqq t\leqq(p-1)/2)$, the k and m's which satisfy $2k+mp=(2s-1)p+2t$ are

$$\begin{cases} k=t\quad, p+t\quad, (s-1)p+t \\ m=2s-1, 2s-3, \cdots, 1\,. \end{cases}$$

So the corresponding eigenvalue of Laplacian is $\{(2s-1)p+2t\}\{2n-2+(2s-1)p+2t\}$ with multiplicity

$$\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}}\binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}-\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}.$$

Case (*iii*). For given s, t ($s=1, 2, \cdots : 0 \leqq t \leqq (p-3)/2$), the k and m's which satisfy $2k+mp=2(s-1)p+2\left(t+\frac{p+1}{2}\right)$ are

$$\begin{cases} k=t+\frac{p+1}{2},\ t+\frac{3p+1}{2},\ \cdots,\ t+(2s-1)p+1 \\ m=2(s-1)\ ,\ 2(s-2)\ ,\ \cdots,\ 0\,. \end{cases}$$

So the corresponding eigenvalue of Laplacian is $4\{(2s-1)p/2+t+1/2\}\{n+(2s-1)p/2+t-1/2\}$ with multiplicity

$$\sum_{\substack{a+b=\\2(s-1)\\a,b\geqq 0}} \binom{n+t+\frac{p-1}{2}+ap}{t+\frac{p+1}{2}+ap}\binom{n+t+\frac{p-1}{2}+bp}{t+\frac{p+1}{2}+bp} - \sum_{\substack{a+b=\\2(s-1)\\a,b\geqq 0}} \binom{n+t+\frac{p-3}{2}+ap}{t+\frac{p-1}{2}+ap}\binom{n+t+\frac{p-3}{2}+bp}{t+\frac{p-1}{2}+bp}.$$

Case (*iv*). For a given s, t ($s=1, 2, \cdots : 0 \leqq t \leqq (p-3)/2$), the k and m's which satisfy $2k+mp=2(s-1)+2(t+(p+1)/2)$ are

$$\begin{cases} k=t+(p+1)/2,\ t+(3p+1)/2,\ \cdots,\ t+\frac{(2s-1)p+1}{2} \\ m=2s-1\ ,\ 2s-3\ ,\ 1\,. \end{cases}$$

So the corresponding eigenvalue is $\{2sp+2t+1\}\{(2n-2)+2sp+2t+1\}$ with multiplicity

$$\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}} \binom{n+t+\frac{p-1}{2}+ap}{t+\frac{p+1}{2}+ap}\binom{n+t+\frac{p-1}{2}+bp}{t+\frac{p+1}{2}+bp} - \sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}} \binom{n+t+\frac{p-3}{2}+ap}{t+\frac{p-1}{2}+ap}\binom{n+t+\frac{p-3}{2}+bp}{t+\frac{p-1}{2}+bp}.$$

A) p: *even*. In this case the different values of $2k+mp$ ($k=0, 1, 2, \cdots : m=0, 1, 2, \cdots$) are the following:

(i) $(2s-1)p+2t$ ($s=1, 2, \cdots ; 0 \leqq t \leqq (p-2)/2$).
(ii) $2sp+2t$ ($s=0, 1, 2, \cdots ; 0 \leqq t \leqq (p-2)/2$).

Case (*i*) For a given s, t ($s=1, 2, \cdots ; 0 \leqq t \leqq (p-2)/2$), the k and m's which satisfy $2k+mp=(2s-1)p+2t$ are

$$\begin{cases} k=t \quad , t+p, \cdots, t+(s-1)p \\ m=2s-1, 2s-3, \cdots, 1, \end{cases}$$

or

$$\begin{cases} k=t+p/2 \ , t+\frac{3}{2}p, \cdots, t+(2s-1)p/2 \\ m=2(s-1), 2(s-2), \cdots, 0. \end{cases}$$

So the corresponding eigenvalue is $\{(2s-1)p+2t\}\{2n-2+(2s-1)p+2t\}$ with multiplicity

$$\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}} \binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}$$

$$+\sum_{\substack{a+b=\\2(s-1)\\a,b\geqq 0}} \binom{n+t+\left(a+\frac{1}{2}\right)p-1}{t+\left(a+\frac{1}{2}\right)p}\binom{n+t+\left(b+\frac{1}{2}\right)p-1}{t+\left(b+\frac{1}{2}\right)p}$$

$$-\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}} \binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}$$

$$-\sum_{\substack{a+b=\\2(s-1)\\a,b\geqq 0}} \binom{n+t+\left(a+\frac{1}{2}\right)p-2}{t+\left(a+\frac{1}{2}\right)p-1}\binom{n+t+\left(b+\frac{1}{2}\right)p-2}{t+\left(b+\frac{1}{2}\right)p-1}.$$

Case (ii) For a given s, t ($s=0, 1, 2, \cdots$; $0\leqq t\leqq (p-2)/2$), the k and m's which satisfy $2k+mp=2sp+2t$ are

$$\begin{cases} k=t \ , t+p \quad , \cdots, t+sp \\ m=2s, 2(s-1), \cdots, 0 \end{cases}$$

or

$$\begin{cases} k=t+p/2, t+\frac{3}{2}p, \cdots, t+(2s-1)p/2 \\ m=2s-1, 2s-3 \quad , \cdots, 1. \end{cases}$$

So the corresponding eigenvalue of Laplacian is $4(t+sp)(n-1+t+sp)$ with multiplicity

$$\sum_{\substack{a+b=\\2s\\a,b\geqq 0}} \binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}$$

$$+\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}} \binom{n+t+\left(a+\frac{1}{2}\right)p-1}{t+\left(a+\frac{1}{2}\right)p}\binom{n+t+\left(b+\frac{1}{2}\right)p-1}{t+\left(b+\frac{1}{2}\right)p}$$

(i) p: *odd.*

eigenvalues of Δ	range of s, t
$4\{(s-1)p+t\}\{(n-1)+(s-1)p+t\}$	$s=1, 2, \cdots$ $t=0, 1, \cdots, (p-1)/2$
$\{(2s-1)p+2t\}\{2n-2+(2s-1)p+2t\}$	$s=1, 2, \cdots$ $t=0, 1, \cdots, (p-1)/2$
$4\{(2s-1)p/2+t+1/2\}\{n+(2s-1)p/2+t-1/2\}$	$s=1, 2, \cdots$ $t=0, 1, \cdots, (p-3)/2$
$\{2sp+2t+1\}\{(2n-2)+2sp+2t+1\}$	$s=1, 2, \cdots$ $t=0, 1, \cdots, (p-3)/2$

(ii) p: *even*

$\{(2s-1)+2t\}\{2n-2+(2s-1)p+2t\}$	$s=1, 2, \cdots$ $t=0, 1, \cdots, (p-2)/2$
$4(sp+t)(n+t+sp-1)$	$s=0, 1, 2, \cdots$ $t=0, 1, \cdots, (p-p)/2$

Table 1.

multiplicity
$\sum_{\substack{a+b=2(s-1)\\ a,b\geqq 0}}\binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}-\sum_{\substack{a+b=2(s-1)\\ a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}$
$\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}-\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}$
$\sum_{\substack{a+b=2(s-1)\\ a,b\geqq 0}}\binom{n+t+\frac{p-1}{2}+ap}{t+\frac{p+1}{2}+ap}\binom{n+t+\frac{p-1}{2}+bp}{t+\frac{p+1}{2}+bp}-\sum_{\substack{a+b=2(s-1)\\ a,b\geqq 0}}\binom{n+t+\frac{p-3}{2}+ap}{t+\frac{p-1}{2}+ap}\binom{n+t+\frac{p-3}{2}+bp}{t+\frac{p-1}{2}+bp}$
$\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+\frac{p-1}{2}+ap}{t+\frac{p+1}{2}+ap}\binom{n+t+\frac{p-1}{2}+bp}{t+\frac{p+1}{2}+bp}-\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+\frac{p-3}{2}+ap}{t+\frac{p-1}{2}+ap}\binom{n+t+\frac{p-3}{2}+bp}{t+\frac{p-1}{2}+bp}$

$\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+ap-1}{t+ap}\binom{n+t+bp-1}{t+bp}+\sum_{\substack{a+b=2(s-1)\\ a,b\geqq 0}}\binom{n+t+\left(a+\frac{1}{2}\right)p-1}{t+\left(a+\frac{1}{2}\right)p}\binom{n+t+\left(b+\frac{1}{2}\right)p-1}{t+\left(b+\frac{1}{2}\right)p}$ $-\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}-\sum_{\substack{a+b=2(s-1)\\ a,b\geqq 0}}\binom{n+t+\left(a+\frac{1}{2}\right)p-2}{t+\left(a+\frac{1}{2}\right)p-1}\binom{n+t+\left(b+\frac{1}{2}\right)p-2}{t+\left(b+\frac{1}{2}\right)p-1}$
$\sum_{\substack{a+b=2s\\ a,b\geqq 0}}\binom{n+t+ap-1}{t+ap-1}\binom{n+t+bp-1}{t+bp}+\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+\left(a+\frac{1}{2}\right)p-1}{t+\left(a+\frac{1}{2}\right)p}\binom{n+t+\left(b+\frac{1}{2}\right)p-1}{t+\left(b+\frac{1}{2}\right)p}$ $-\sum_{\substack{a+b=2s\\ a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}-\sum_{\substack{a+b=2s-1\\ a,b\geqq 0}}\binom{n+t+\left(a+\frac{1}{2}\right)p-2}{t+\left(a+\frac{1}{2}\right)p-1}\binom{n+t+\left(b+\frac{1}{2}\right)p-2}{t+\left(b+\frac{1}{2}\right)p-1}$

Table 2.

$$-\sum_{\substack{a+b=\\2s\\a,b\geqq 0}}\binom{n+t+ap-2}{t+ap-1}\binom{n+t+bp-2}{t+bp-1}$$

$$-\sum_{\substack{a+b=\\2s-1\\a,b\geqq 0}}\binom{n+t+\left(a+\frac{1}{2}\right)p-2}{t+\left(a+\frac{1}{2}\right)p-1}\binom{n+t+\left(b+\frac{1}{2}\right)p-2}{t+\left(b+\frac{1}{2}\right)p-1}.$$

Summing up the above, we have the following:

THEOREM. *The spectrum of homogeneous lens space* $M=S^{2n-1}/G$ *of constant curvature* 1 *with cyclic fundamental group of order* p *is given in the following table.*

Remark. If $p=2$, the spectrum in the table coincides with the result in ([1], pp. 166).

REFERENCES

[1] BERGER, M., Le spectre d'une variétés riemannienne, Lecture Notes ın Math., vol. 194. Springer-Verlag, 1971.
[2] WOLF, J. A., Spaces of constant curvature, New York: McGraw-Hill 1967.

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