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IMMERSIONS OF CODIMENSION TWO WITH TRIVIAL NORMAL CONNEXION INTO ELLIPTIC SPACES

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Introduction.

The actual article is concerned with the study of isometric immersions of *codimension two with trivial normal connexion* of arbitrary dimensional C^∞-manifolds into elliptic spaces.

Its main purpose is to generalize a number of results concerning pseudo-umbilical immersions to immersions having an *umbilical normal direction* which is not necessarily determined by the mean curvature point. In particular such immersions for which the umbilical normal direction is *parallel in the normal bundle* are investigated. The latter immersions can be considered as a generalization of the pseudo-umbilical immersions of codimension two with constant mean curvature.

The results we'll generalize now are mostly due to R. Rosca and ourselves [16], [18], [19], [21], and are closely related to the work done by several other authors who are cited in the text.

§ 1. Preliminaries.

Let P_e^{n+2} be an $(n+2)$-dimensional real elliptic space of curvature 1, and $x: M^n \to P_e^{n+2}$ an isometric immersion of an orientable n-dimensional C^∞-manifold M^n into P_e^{n+2}. With the general point $\boldsymbol{X}_0(u^i)$, $(i, j, k, l \in \{1, 2, \cdots, n\})$, of M^n we associate an *orthonormal simplex* $\mathcal{S}_{X_0} \equiv \{\boldsymbol{X}_A\}$, $(A, B, C \in \{0, 1, \cdots, n+2\})$, such that the *dual tangent space* $T_{X_0}(M^n)$ of M^n at $\boldsymbol{X}_0$ is determined by the points $\boldsymbol{X}_i$. Then $N_0 = [\boldsymbol{X}_{n+1}, \boldsymbol{X}_{n+2}]$ is the *principal quasi-normal* of M^n at $\boldsymbol{X}_0$ [15]. In the following we'll say that each point of N_0 defines a normal direction of M^n at $\boldsymbol{X}_0$.

M^n is structured by the *connexion*

$$(1) \qquad d\boldsymbol{X}_A = \omega_A^B \boldsymbol{X}_B\,, \qquad (\omega_A^B + \omega_B^A = 0)\,,$$

where ω_A^B are the connexion 1-forms(**). The *structure equations* are given by

$$(2) \qquad d \wedge \omega_A^B = \omega_A^C \wedge \omega_C^B\,.$$

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(*) Aspirant H. F. W. O. (**) ω_0^A will also be denoted as ω^A.

We have the relations

(3) $$\omega_A^B=\gamma_{Ai}^B\omega^i ,$$

where $\omega^i(u^j|du^j)$ is the dual base and γ_{Ai}^B are the connexion coefficients. Since $x(M^n)$ is an integral manifold of

(4) $$\omega^r=0 , \qquad (r, s\in\{n+1, n+2\}) ,$$

we find by exterior differentiation using E. Cartan's lemma

(5) $$\gamma_{ij}^r=\gamma_{ji}^r .$$

From the above formulae we obtain

(6) $$d\wedge\omega_i^j=\omega_i^k\wedge\omega_k^j+\Omega_i^j, \qquad \Omega_i^j=\frac{1}{2}R_i{}^j{}_{kl}\omega^k\wedge\omega^l ;$$
$$d\wedge\omega_r^s=\Omega_r^s , \qquad \Omega_r^s=\frac{1}{2}R_r{}^s{}_{kl}\omega^k\wedge\omega^l ;$$
$$R_i{}^j{}_{kl}+R_i{}^j{}_{lk}=0 , \qquad R_r{}^s{}_{kl}+R_r{}^s{}_{lk}=0 ;$$

(7) $$R_i{}^j{}_{kl}=\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}+\sum_r(\gamma_{il}^r\gamma_{jk}^r-\gamma_{ik}^r\gamma_{jl}^r) ;$$
$$R_r{}^s{}_{kl}=\sum_i(\gamma_{il}^r\gamma_{ik}^s-\gamma_{ik}^r\gamma_{il}^s) .$$

By definition the *normal connexion* of x is *trivial* if $R_r{}^s{}_{kl}=0$ [9]. The following assertions concerning x are equivalent:

(i) the normal connexion is trivial;
(ii) the *scalar normal curvature* $K_N=\sum(R_r{}^s{}_{kl})^2$ [2] is zero;
(iii) the *Gaussian torsion* τ_g [1] vanishes;
(iv) the *second fundamental* forms $\varphi_r=-\langle d\boldsymbol{X}_0, d\boldsymbol{X}_r\rangle=\omega_i^r\omega^i$ can be diagonalized simultaneously.

Observe that if a point on N_0 defines an *umbilical normal direction* of M^n at $\boldsymbol{X}_0$ (i.e. if the corresponding second fundamental tensor is proportional to the identity transformation [3]), then φ_{n+1} and φ_{n+2} can be diagonalized simultaneously. In the following all simplices $\mathcal{S}_{X_0}$ under consideration will be chosen, if possible, to be such that both φ_r's are diagonal, i.e. to be *principal.*

For an immersion x with trivial normal connexion it is always possible to determine rectangular points $\bar{\boldsymbol{X}}_r$ on N_0 such that the *unique normal connexion form* (*or torsion form*) $\bar{\omega}_{n+2}^{n+1}$ of x with respect to the orthonormal simplex $\mathcal{S}_{X_0}\equiv\{\boldsymbol{X}_0, \boldsymbol{X}_i, \bar{\boldsymbol{X}}_r\}$ *vanishes.* Such a simplex will be said to be of *type* $\mathcal{S}_{vtf}$. It follows at once that in this case also the sommets $\bar{\boldsymbol{X}}_r$ of $\mathcal{S}_{vtf}$ generate manifolds of codimension two in P_e^{n+2} with vanishing torsion form with respect to the same simplex, and for which moreover we have

(8) $$T_{X_0}(M^n)\equiv T_{\bar{X}_r}((\bar{\boldsymbol{X}}_r)) ,$$

so that actually $(\boldsymbol{X}_0)$, $(\bar{\boldsymbol{X}}_{n+1})$ and $(\bar{\boldsymbol{X}}_{n+2})$ have the same tangential connexion forms (or rotation forms) ω_i^j.

Finally we mention that all manifolds which will be discussed in this paper are supposed to be *not totally umbilical.*

PART I. Determination of orthonormal simplices of type $\mathcal{S}_{vtf}$ associated with x.

§2. A characterization for $R_r{}^s{}_{kl}=0$.

Consider the manifolds $(\boldsymbol{P})$ with general point

$$\boldsymbol{P}=\boldsymbol{X}_0\cos\varphi+\boldsymbol{X}_{n+1}\sin\varphi\,,\qquad (\varphi\in\mathcal{D}(M^n))\,. \tag{9}$$

Putting $\boldsymbol{N}=\boldsymbol{X}_{n+1}\cos t+\boldsymbol{X}_{n+2}\sin t$, $(t\in\mathcal{D}(M^n))$, we find using (1) that

$$\langle d\boldsymbol{P}, \boldsymbol{N}\rangle=0 \tag{10}$$

if and only if

$$\omega_{n+1}^{n+2}=-\operatorname{cotg} t\, dln \sin\varphi\,. \tag{11}$$

$(\boldsymbol{P})$ will be called *general* if $\boldsymbol{P}\neq\boldsymbol{X}_0$ and $\boldsymbol{N}$ will be called a *general* point *invariantly situated* on N_0 if t=constant and $\boldsymbol{N}\neq\boldsymbol{X}_{n+1}$. From (10) and (11) then follows

THEOREM 1. *There exist general manifolds $(\boldsymbol{P})$ for which the principal quasi-normal contains a general point invariantly situated on the principal quasi-normal of M^n if and only if the normal connexion of x is trivial.*

Moreover it is clear that in particular we have

THEOREM 2. *If the principal quasi-normal of a general manifold $(\boldsymbol{P})$ contains $\boldsymbol{X}_{n+2}$ then $\mathcal{S}_{\boldsymbol{X}_0}$ is of type $\mathcal{S}_{vtf}$. If conversely $\mathcal{S}_{\boldsymbol{X}_0}$ is of type $\mathcal{S}_{vtf}$ then the principal quasi-normal of each manifold $(\boldsymbol{P})$ contains $\boldsymbol{X}_{n+2}$.*

§3. Parallellism in the normal bundle. Manifolds M^n.

The *mean curvature point* of x is defined as [4], [18]

$$\boldsymbol{H}=f\gamma^r\boldsymbol{X}_r\,,\qquad (\gamma^r=\operatorname{tr}[\gamma_{ij}^r]\,;\ f=\text{factor of normalization})\,. \tag{12}$$

If x is not minimal ($H\neq0$), then it is always possible to choose $\mathcal{S}_{\boldsymbol{X}_0}$ such that $\boldsymbol{H}=\boldsymbol{X}_{n+1}$ ($\gamma^{n+2}=0$). In this case we'll denote $\boldsymbol{X}_{n+2}$ as $\boldsymbol{H}^\perp$. Then, the *scalar mean curvature* α of M^n [4] is defined by

$$\gamma^{n+1}=n\alpha\,, \tag{13}$$

and according to [11] x is *pseudo-umbilical* if and only if

$$\gamma_{ij}^{n+1}=\alpha\delta_{ij}\,. \tag{14}$$

Since φ_{n+2} can always be diagonalized we can formulate

THEOREM 3. *A pseudo-umbilical immersion x has a trivial normal connexion.*

Next we remind the following known result [18]:

(*) *If x is pseudo-umbilical then the following assertions are equivalent:*

(i) $\mathcal{S}_{X_0} \equiv \{X_0, X_i, H, H^{\perp}\}$ *is of type* $\mathcal{S}_{vtf}$;

(ii) *the scalar mean curvature of* M^n *is constant;*

(iii) *the mean curvature field of x is parallel in the normal bundle.*

As is well known $N \in N_0$ is said to determine a normal field on M^n which is *parallel in the normal bundle* [25] if

$$dN \equiv 0 \quad (\mathrm{mod}\, X_0, X_i)\,. \tag{15}$$

Hence it follows at once from (1) that (*) can be partially generalized in the following way:

THEOREM 4. *An orthonormal simplex* $\mathcal{S}_{X_0}$ *associated with x is of type* $\mathcal{S}_{vtf}$ *if and only if* X_{n+1}, *or equivalently* X_{n+2}, *determines a field on* M^n *which is parallel in the normal bundle.*

Aiming for a generalization of the other part of (*) we make the following considerations. First suppose that $\mathcal{S}_{X_0}$ is such that $\omega_{n+1}^{n+2}=0$ and that X_{n+1} determines an *umbilical* normal direction on M^n. Then exterior differentiation of

$$\omega_i^{n+1} = \lambda \omega^i\,, \tag{16}$$

where λ is the (unique) principal curvature of M^n at X_0 corresponding to X_{n+1}, yields

$$d\lambda \wedge \omega^i = 0\,. \tag{17}$$

This shows that λ is constant. Conversely suppose now that X_{n+1} determines an *umbilical* direction of M^n with *constant corresponding principal curvature.* Then exterior differentiation of (16) yields

$$\lambda_i^{n+2} \omega_i^0 \wedge \omega_{n+1}^{n+2} = 0\,, \tag{18}$$

where λ_i^{n+2} are the principal curvatures of M^n at X_0 corresponding to X_{n+2}. Consequently, if zero is not a principal curvature with multiplicity $n-1$ of M^n corresponding to X_{n+2}, or equivalently if $\lambda \cos t$ is not a principal curvature with multiplicity $n-1$ of M^n corresponding to $N \in N_0$, (18) implies that $\omega_{n+1}^{n+2}=0$. A manifold M^n having an *umbilical normal direction* with *constant corresponding principal curvature* and for which *zero is not a principal curvature with multiplicity* $n-1$ *corresponding to the normal direction* which is *rectangular to the umbilical one* will further on be denoted as $\bar{M}^n$.

THEOREM 5. (i) *The principal curvature of* M^n *corresponding to an umbilical normal field which is parallel in the normal bundle is constant;*

(ii) *the orthonormal simplices* $\mathcal{S}_{X_0}$, $X_0 \in \bar{M}^n$, *for which* X_{n+1} *determines the umbilical normal direction of* $\bar{M}^n$ *are of type* $\mathcal{S}_{vtf}$.

§ 4. Concurrent normal fields.

$N \in N_0$ determines a *concurrent normal field* on M^n in the sense of K. Yano [24] if

$$\exists f \in \mathcal{D}(M^n) \ni dX_0 + d(fN) = 0 . \tag{19}$$

After diagonalizing φ_{n+1} we have for $N = X_{n+1}$ and using (1):

$$\omega^i(1 - f\lambda_i^{n+1})X_i + f\omega_{n+1}^{n+2}X_{n+2} + (df)X_{n+1} = 0 , \tag{20}$$

where λ_i^{n+1} are the principal curvatures of M^n at X_0 corresponding to X_{n+1}.

THEOREM 6. *If* $\mathcal{S}_{X_0}$ *is an orthonormal simplex associated with* x *then the following assertions are equivalent:*

(i) $\mathcal{S}_{X_0}$ *is of type* $\mathcal{S}_{vtf}$ *and* X_{n+1} *determines an umbilical direction of* M^n *with constant corresponding principal curvature;*

(ii) X_{n+1} *determines a normal field on* M^n *which is concurrent in the sense of K. Yano.*

Based on Theorem 6 and (*) we obtain the following generalization of a result of [21]:

THEOREM 7. *A manifold* M^n *is pseudo-umbilically immersed into* P_e^{n+2} *with constant scalar mean curvature if and only if its mean curvature field is concurrent in the sense of K. Yano.*

§ 5. On the focal manifolds of the rectilinear system $\mathcal{L}_{0,n+1}$.

Consider the rectilinear system (depending on n parameters) with general element $R = [X_0, X_{n+1}]$. Putting $Q = X_0 \cos\zeta + X_{n+1} \sin\zeta$, $(\zeta \in \mathcal{D}(M^n))$, the *developpables* and corresponding *focal manifolds* $\mathcal{L}_{0,n+1}$ are determined by the condition

$$dQ \equiv 0 \qquad (\mathrm{mod}\ X_0, X_{n+1}) . \tag{21}$$

Using (1) we find (21) to be equivalent with

$$\omega_i^0(\cos\zeta - \lambda_i^{n+1}\sin\zeta) = 0 , \qquad \omega_{n+1}^{n+2}\sin\zeta = 0 . \tag{22}$$

Hence $\mathcal{L}_{0,n+1}$ admits focal manifolds if and only if $\omega_{n+1}^{n+2} = 0$.

In this case the focal manifold corresponding to

$$\omega^1 = \cdots = \hat{\omega}^i = \cdots = \omega^n = 0 , \tag{23}$$

(where ^ denotes omission), is generated by $Q(\zeta)$ where

(24) $$\operatorname{tg}\zeta=\frac{1}{\lambda_i^{n+1}}\,.$$

THEOREM 8. *$\mathcal{S}_{X_0}$ is of type $\mathcal{S}_{vtf}$ if and only if the rectilinear system $\mathcal{L}_{0,n+1}$ admits focal manifolds. In this case the n focal manifolds of $\mathcal{L}_{0,n+1}$ coincide if and only if X_{n+1} determines an umbilical direction of M^n.*

§ 6. The product manifolds $\mathcal{M}_{0,n+1}$.

A last determination of simplices $\mathcal{S}_{vtf}$ is given in the following terms. By definition [23] the *indecomposable cartesian product* $\mathcal{M}_{0,n+1}$ of M^n and its normal $[X_0, X_{n+1}]$ is the hypersurface of P_e^{n+2} with general point

(25) $$W=X_0\cos u^{n+1}+X_{n+1}\sin u^{n+1}\,,$$

where u^{n+1} is a new local coordinate. Using (1) it follows that

(26) $$dW=(\omega^i\cos u^{n+1}+\omega_{n+1}^i\sin u^{n+1})X_i$$
$$+(-\sin u^{n+1}X_0+\cos u^{n+1}X_{n+1})du^{n+1}+\omega_{n+1}^{n+2}\sin u^{n+1}X_{n+2}\,.$$

Hence

(27) $$\langle dW, X_{n+2}\rangle=0\Leftrightarrow\omega_{n+1}^{n+2}=0\,.$$

If $\omega_{n+1}^{n+2}=0$ then the second fundamental form of $\mathcal{M}_{0,n+1}$ is found to be

(28) $$\varphi(\mathcal{M}_{0,n+1})=-\langle dW, dX_{n+2}\rangle=(\cos u^{n+1}-\lambda_i^{n+1}\sin u^{n+1})\lambda_i^{n+2}(\omega^i)^2\,.$$

Denoting the *type number* of $\mathcal{M}_{0,n+1}$ [17] by $\operatorname{tn}(\mathcal{M}_{0,n+1})$, (27) and (28) prove

THEOREM 9. *$\mathcal{S}_{X_0}$ is of type $\mathcal{S}_{vtf}$ if and only if X_{n+2} determines the normal direction of the hypersurface $\mathcal{M}_{0,n+1}$. In this case*

$$\operatorname{tn}(\mathcal{M}_{0,n+1})\leqq n\,.$$

PART II. On manifolds M^n with an umbilical normal direction.

§ 7. Quadratic mean form, third fundamental form and Ricci form of M^n.

Following M. Obata [10] the *quadratic mean form* II_H, the *third fundamental form* III and the *Ricci form* ϕ associated with x are respectively given by the formulae

(29) $$\mathrm{II}_H=\sum_r\gamma^r\gamma_{ij}^r\omega^i\omega^j\,,$$

(30) $$\mathrm{III}=\sum_{i,r}(\omega_i^r)^2\,,$$

(31) $$\phi=(n-1)ds^2-\mathrm{III}+\mathrm{II}_H\,.$$

Supposing X_{n+1} determines an *umbilical direction* of M^n with corresponding principal curvature λ, we find

(32) $$\mathrm{II}_H=(n\lambda^2+\gamma^{n+2}\lambda_i^{n+2})(\omega^i)^2\,,$$

(33) $$\mathrm{III}=[\lambda^2+(\lambda_i^{n+2})^2](\omega^i)^2\,,$$

(34) $$\phi=[(n-1)(1+\lambda^2)-(\lambda_i^{n+2})^2+\gamma^{n+2}\lambda_i^{n+2}](\omega^i)^2\,,$$

where λ_i^{n+2} are the principal curvatures of M^n at X_0 corresponding to X_{n+2}.

§ 8. Einstein manifolds M^n.

From (34) we derive that M^n is *Einsteinian* ($\phi\sim ds^2$) if and only if

(35) $$(\lambda_i^{n+2})^2-\gamma^{n+2}\lambda_i^{n+2}=\rho\,,\qquad (\rho\in\mathcal{D}(M^n))\,.$$

Then

(36) $$\lambda_i^{n+2}=\frac{1}{2}\{\gamma^{n+2}+\varepsilon_i[(\gamma^{n+2})^2+4\rho]^{1/2}\}\,,\qquad (\varepsilon_i=\pm1)\,;$$

hence M^n has two different principal curvatures corresponding to X_{n+2}, say β_1 and β_2. Suppose now that β_1 and β_2 have the same multiplicity m, ($n=2m$). In this case (35) implies

(37) $$(\beta_1^2-\beta_2^2)(1-m)=0\,.$$

Thus if $n>2$ we have $\beta_1+\beta_2=0$, and so $\gamma^{n+2}=0$.

THEOREM 10. *If M^n is an Einstein manifold having an umbilical normal direction then it has exactly two different principal curvatures corresponding to any normal direction different from the umbilical one. If these curvatures are of equal multiplicity and $n>2$, then M^n is pseudo-umbilical.*

§ 9. Manifolds M with vanishing Ricci form.

Next let's consider manifolds M^n having a *vanishing Ricci form*, i. e. suppose that

(38) $$(n-1)(1+\lambda^2)-(\lambda_i^{n+2})^2+\gamma^{n+2}\lambda_i^{n+2}=0\,.$$

Summation over i yields

(39) $$n(n-1)(1+\lambda^2)-\sum_j(\lambda_i^{n+2})^2+(\gamma^{n+2})^2=0\,.$$

We recall that the norm σ of the second fundamental form, which in general is defined as [9]

(40) $$\sigma=\sum_{r,i,j}(\gamma_{ij}^r)^2\,,$$

actually is found to be

(41) $$\sigma=n\lambda^2+\sum_i(\lambda_i^{n+2})^2 .$$

Combining (39) and (41) it follows that

(42) $$n(n-1)+n^2\lambda^2-\sigma+(\gamma^{n+2})^2=0 .$$

THEOREM 11. *A manifold M^n with vanishing Ricci from is pseudo-umbilical if and only if there exists an umbilical normal direction with corresponding principal curvature λ such that*

$$\sigma=n(n-1)+n^2\lambda^2 ,$$

where σ denotes the norm of the second fundamental form.

In the previous Theorem clearly $\lambda=h$, the scalar mean curvature of M^n. In general h is defined as [19]

(43) $$h=\frac{1}{n}[(\gamma^{n+1})^2+(\gamma^{n+2})^2]^{1/2} .$$

Hence if X_{n+1} determines an umbilical direction then

(44) $$n^2h^2=n^2\lambda^2+(\gamma^{n+2})^2 .$$

Consequently supposing moreover that $\psi=0$, we have

(45) $$n^2h^2=\sigma-n(n-1) .$$

In particular (45) implies the following

THEOREM 12. *Let M^n be a manifold with vanishing Ricci form and having an umbilical normal direction. Then the scalar mean curvature is constant if and only if the norm of the second fundamental form is constant.*

§ 10. Manifolds M^n with conformal Gauss map.

As follows from (33), M^n has a *conformal Gauss map* (III$\sim ds^2$) if and only if

(46) $$\lambda_i^{n+2}=\varepsilon_i'\beta , \qquad (\beta\in\mathcal{D}(M^n)\, ;\ \varepsilon_i'=\pm 1) .$$

Hence if moreover M^n is pseudo-umbilical then the two (opposite) principal curvatures corresponding to X_{n+2} do have the same multiplicity (and conversely), and so in this case M^n is essentially even-dimensional.

On the other hand if M^n is pseudo-umbilical and has two principal curvatures corresponding to X_{n+2} (and X_{n+1} determines an umbilical direction), then clearly (46) is satisfied.

THEOREM 13. *A manifold M^n having an umbilical normal direction has the following properties:*

(i) *its Gauss map is conform if and only if M^n has but two principal curvatures corresponding to any normal direction different from the umbilical one* *and these curvatures are opposite for the normal direction which is rectangular to the umbilical one;*

(ii) *if M^n has a conformal Gauss map and is pseudo-umbilical then M^n is essentially even-dimensional;*

(iii) *if M^n has two principal curvatures of equal multiplicity corresponding to any normal direction different from the umbilical one, then its Gauss map is conform if and only if M^n is pseudo-umbilical.*

PART III. On manifolds M^n having an umbilical normal direction which is parallel in the normal bundle.

§ 11. A condition for a normal direction which is parallel in the normal bundle to be umbilical.

Let $\mathcal{S}_{X_0}$ be a simplex of type $\mathcal{S}_{vtf}$ associated with x. Then the metrical fundamental form of the manifold with general point $\boldsymbol{X}_{n+1}$ is found to be

$$\langle d\boldsymbol{X}_{n+1}, d\boldsymbol{X}_{n+1}\rangle=(\lambda_i^{n+1})^2(\omega^i)^2 . \tag{47}$$

Consequently the manifolds M^n and $(\boldsymbol{X}_{n+1})$ are *conform* to each other if and only if

$$\lambda_i^{n+1}=\varepsilon_i''\bar{\lambda} , \qquad (\bar{\lambda}\in\mathcal{D}(M^n)\,;\ \varepsilon_i''=\pm 1) , \tag{48}$$

i.e. if either $\boldsymbol{X}_{n+1}$ determines an umbilical direction or M^n has two principal curvatures corresponding to $\boldsymbol{X}_{n+1}$ and these are opposite.

Reminding that the r-th *mean curvature* $K_r(\boldsymbol{N})$ of M^n at $\boldsymbol{X}_0$ and corresponding to $\boldsymbol{N}\in N_0$ is defined by the formula [5]

$$\binom{n}{r}K_r(\boldsymbol{N})=\sum k_1(\boldsymbol{N})\cdots k_r(\boldsymbol{N}) , \qquad (1\leqq r\leqq n) , \tag{49}$$

where $k_i(\boldsymbol{N})$ are the principal curvatures of M^n at $\boldsymbol{X}_0$ corresponding to $\boldsymbol{N}$, we have the

THEOREM 14. *Let $\mathcal{S}_{X_0}$ be of type $\mathcal{S}_{vtf}$. Then for each positively valued function on M^n there exists a class of manifolds M^n which are conform to the manifolds $(\boldsymbol{X}_{n+1})$ with the given function as factor of conformality; in each such class the two manifolds for which the first mean curvature corresponding to $\boldsymbol{X}_{n+1}$ attains an extreme value are exactly those for which $\boldsymbol{X}_{n+1}$ determines an umbilical direction.*

§ 12. Manifolds $\tilde{M}^n$, $\tilde{M}'^n$, $\tilde{M}''^n$.

In the following we will always (except if explicitely mentioned otherwise) consider manifolds M^n having an *umbilical normal direction* which is *parallel in the normal bundle*. Such manifolds will be denoted as $\tilde{M}^n$. Furthermore $\mathcal{S}_{X_0}$

will be choosen such that X_{n+1} determines the normal direction with the above properties, and then we'll denote the manifolds (X_{n+1}) and (X_{n+2}) respectively as $\tilde{M}'^n$ and $\tilde{M}''^n$. As we known from Theorem 5 the principal curvature λ of M^n at X_0 corresponding to X_{n+1} is actually *constant.*

First we remark that in this case clearly $\frac{\lambda X_0 + X_{n+1}}{\sqrt{\lambda^2+1}}$ is a fix point, and consequently $\tilde{M}^n$ belongs to a hypersphere $S^{n+1}\left(bg \cos \frac{\lambda}{\sqrt{\lambda^2+1}}\right)$ of P_e^{n+2}. It is well known [5], [18] that *$\tilde{M}^n$ is a minimal hypersurface* of $S^{n+1}\left(bg \cos \frac{\lambda}{\sqrt{\lambda^2+1}}\right)$ *if and only if $\tilde{M}^n$ is pseudo-umbilical.*

Using (1) the principal curvatures of $\tilde{M}'^n$ at X_{n+1} corresponding to X_0 and X_{n+2} are respectively found to be

$$\frac{1}{\lambda}, \ -\frac{\lambda_i^{n+2}}{\lambda}. \tag{50}$$

From (50), (29), (30) and (31) then follow respectively the traces corresponding to X_0 and X_{n+2}, the quadratic mean form, the third fundamental form and the Ricci form of $\tilde{M}'^n$ as

$$\frac{n}{\lambda}, \ -\frac{\gamma^{n+2}}{\lambda}, \tag{51}$$

$$\mathrm{II}'_H = \frac{1}{\lambda^2}[n + \gamma^{n+2}\lambda_i^{n+2}](\alpha^i)^2, \tag{52}$$

$$\mathrm{III}' = \frac{1}{\lambda^2}[1 + (\lambda_i^{n+2})^2](\alpha^i)^2, \tag{53}$$

$$\psi' = \frac{1}{\lambda^2}[(n-1)(1+\lambda^2) - (\lambda_i^{n+2})^2 + \gamma^{n+2}\lambda_i^{n+2}](\alpha^i)^2, \tag{54}$$

where α^i are the dual base forms of $\tilde{M}'^n$. (51), (32), (33), (34), (52), (53) and (54) imply

THEOREM 15. (i) *$\tilde{M}^n$ is pseudo-umbilical if and only if M^n is the mean curvature manifold of $\tilde{M}'^n$ or equivalently if $\tilde{M}'^n$ is pseudo-umbilical;*

(ii) *$\tilde{M}^n$ has a conformal Gauss map if and only if $\tilde{M}'^n$ has a conformal Gauss map;*

(iii) *$\tilde{M}^n$ is Einsteinian if and only if $\tilde{M}'^n$ is Einsteinian.*

Using (1) again the principal curvatures $\tilde{M}''^n$ at X_{n+2} corresponding to X_0 and X_{n+1} are respectively found to be

$$\gamma''^0_{ii} = \frac{1}{\lambda_i^{n+2}}, \quad \gamma''^{n+1}_{ii} = -\frac{\lambda}{\lambda_i^{n+2}}, \qquad (\gamma''^0_{ij} = \gamma''^{n+1}_{ij} = 0 \text{ for } i \neq j). \tag{55}$$

Consequently

$$\gamma''^0 = \sum_i \frac{1}{\lambda_i^{n+2}}, \quad \gamma''^{n+1} = -\lambda \sum_i \frac{1}{\lambda_i^{n+2}}, \tag{56}$$

and so $\tilde{M}''^n$ clearly is minimal if and only if $K_{n-1}(X_{n+2})=0$. Moreover the quadratic mean form of $\tilde{M}''^n$ being

$$\mathrm{II}''_H=(1+\lambda^2)\Big(\sum_j \frac{1}{\lambda_j^{n+2}}\Big)\sum_i \frac{(\beta^i)^2}{\lambda_i^{n+2}}, \tag{57}$$

where β^i are the dual base forms of $\tilde{M}''^n$, it follows that $\tilde{M}''^n$ is essentially not pseudo-umbilical.

Next consider the following two linear mappings:

$$\bar{m}: T^{\perp}_{X_{n+2}}(\tilde{M}''^n) \longrightarrow \boldsymbol{R}: \xi \longrightarrow \frac{1}{n}(\gamma''^0 \cos \xi+\gamma''^{n+1} \sin \xi), \tag{58}$$

$$\psi: T^{\perp}_{X_{n+2}}(\tilde{M}''^n) \longrightarrow \boldsymbol{S}_n: \xi \longrightarrow \cos \xi[\gamma''^0_{ij}]+\sin \xi[\gamma''^{n+1}_{ij}], \tag{59}$$

where $\xi=\boldsymbol{X}_0 \cos \xi+\boldsymbol{X}_{n+1} \sin \xi$, $(\xi\in\mathcal{D}(\tilde{M}^n))$, $T^{\perp}_{X_{n+2}}(\tilde{M}''^n)$ is the totally normal space of $\tilde{M}''^n$ at $\boldsymbol{X}_{n+2}$ and $\boldsymbol{S}_n$ the space of all real symmetric $n\times n$ matrices. Then *M-index*$_{X_{n+2}}$ $\tilde{M}''^n=\dim \operatorname{Im} \psi \operatorname{Ker} \bar{m}$ [12]. From (56) and (58) it follows that

$$\operatorname{Ker} \bar{m}=\Big\{\xi \,\Big\|\, (\cos \xi-\lambda \sin \xi)\sum_i \frac{1}{\lambda_i^{n+2}}=0\Big\}, \tag{60}$$

$$\psi(\xi)=(\cos \xi-\lambda \sin \xi)\begin{bmatrix} \ddots & & 0 \\ & \frac{1}{\lambda_i^{n+2}} & \\ 0 & & \ddots \end{bmatrix}. \tag{61}$$

Hence M-index$_{X_{n+2}}$ $\tilde{M}''^n=1$ if and only if $K_{n-1}(X_{n+2})=0$, and in all other cases M-index$_{X_{n+2}}$ $\tilde{M}''^n=0$. Based on (56), (60), (61) and a Theorem of T. Ōtsuki [13] we can formulate

THEOREM 16. *The following assertions are equivalent:*
(i) *the manifold $\tilde{M}''^n$ associated with a manifold $\tilde{M}^n$ is minimal;*
(ii) *M-index$_{X_{n+2}}$ $\tilde{M}''^n=1$;*
(iii) *the $(n-1)$-th mean curvature of $\tilde{M}^n$ at $\boldsymbol{X}_0$ and corresponding to X_{n+2} is zero.*

A minimal manifold $\tilde{M}''^n$ is a minimal hypersurface of an $(n+1)$-dimensional linear subspace of P_e^{n+2}.

$\tilde{M}''^n$ is essentially not pseudo-umbilical.

§ 13. Manifolds $\tilde{M}^n$ with homothetic Gauss map.

As follows from § 10 a manifold $\tilde{M}^n$ with homothetic Gauss map has two principal curvatures corresponding to $\boldsymbol{X}_{n+2}$, and these curvatures are opposite. Suppose that

$$\lambda_{\bar{i}}^{n+2}=-\lambda_{\bar{i}'}^{n+2}=\mu, \quad (\mu=\text{constant};\ \bar{i}\in\{1, 2, \cdots, p\}, \tag{62}$$

$$\bar{i}'\in\{p+1, p+2, \cdots, n\}, \qquad (0<p<n).$$

Then by exterior differentiation of

$$\omega_{\bar{i}}^{n+2}=\mu\omega^{\bar{i}}, \qquad \omega_{\bar{i}}^{n+2}=-\mu\omega^{\bar{i}'}, \tag{63}$$

we obtain

$$\omega_{\bar{i}}^{\bar{i}'}=0. \tag{64}$$

Hence the two distributions defined on $\tilde{M}^n$ by the Pfaffian systems

$$\omega^{\bar{i}}=0, \tag{65}$$

$$\omega^{\bar{i}'}=0, \tag{66}$$

are both *completely integrable.* Moreover by exterior differentiation of (24) we find that

$$\mu^2=1+\lambda^2. \tag{67}$$

Then, using (1), it follows that the integral submanifolds of $\tilde{M}^n$ defined by (65) and (66) are respectively $V^{n-p}\cong S^{n-p}\left(bg\cos\sqrt{\frac{2\lambda^2+1}{2\lambda^2+2}}\right)$ and $V^p\cong S^p\left(bg\cos\sqrt{\frac{2\lambda^2+1}{2\lambda^2+2}}\right)$.

We remark that in view of § 10 $\tilde{M}^n$ is *pseudo-umbilical* if and only if $2p=n$. In this case, $\tilde{M}^n$ being pseudo-umbilical and having a homothetic Gauss map, $\tilde{M}^n$ is *Einsteinian.* According to (34) $\tilde{M}^n$ has a *vanishing Ricci form* if and only if

$$(n-1)(1+\lambda^2)-\mu^2=0. \tag{68}$$

It follows that the only such manifolds are 2-dimensional.

THEOREM 17. *A manifold $\tilde{M}^n$ with homothetic Gauss map has the following properties:*

(i) *it is locally a Riemannian direct product*

$$S^p\left(bg\cos\sqrt{\frac{2\lambda^2+1}{2\lambda^2+2}}\right)\times S^{n-p}\left(bg\cos\sqrt{\frac{2\lambda^2+1}{2\lambda^2+2}}\right),$$

where $0<p<n$ and λ is the principal curvature corresponding to the umbilical normal direction of $\tilde{M}^n$;

(ii) *it is pseudo-umbilical if and only if $2p=n$, and the only such manifolds with vanishing Ricci form are standard flat tori*

$$S^1\left(bg\cos\sqrt{\frac{2\alpha^2+1}{2\alpha^2+2}}\right)\times S^1\left(bg\cos\sqrt{\frac{2\alpha^2+1}{2\alpha^2+2}}\right),$$

where α is the constant scalar mean curvature.

§ 14. Pseudo-umbilical manifolds with constant scalar mean curvature.

In this paragraph we will be concerned with manifolds $\tilde{M}^n$ for which X_{n+1} *is the mean curvature point* $\boldsymbol{H}$. In this case we'll denote $\tilde{M}'^n$, X_{n+2} and $\tilde{M}''^n$ respectively as $\boldsymbol{H}^n$, $\boldsymbol{H}^\perp$ and $\boldsymbol{H}^{\perp n}$.

We remark that the results formulated in § 12 are generalizations of explicitely or implicitely stated results of [18] concerning *pseudo-umbilical manifolds with constant scalar mean curvature.* Also from [18] we know the following Theorem:

(**) *If M^n is a pseudo-umbilical manifold with constant scalar mean curvature then M^n is homothetic with its mean curvature manifold $\boldsymbol{H}^n$.*

With respect to (**) we observe that (53) implies that *a pseudo-umbilical manifold M^n with constant scalar mean curvature is Einsteinian if and only if it is conform with its associated manifold $\boldsymbol{H}^{\perp n}$* (note that this conformality becomes homothetic if and only if the *Ōtsuki curvature* of M^n at X_0 corresponding to $\boldsymbol{H}^\perp$ [14], [20] is constant). Hence *the rectangular triad $\tau \equiv \{M^n, \boldsymbol{H}^n, \boldsymbol{H}^{\perp n}\}$ build upon a pseudo-umbilical Einstein manifold M^n with constant scalar mean curvature consists of conformal components.* We remark however that τ *cannot consist of isometric components* [22].

§ 15. Minimal product manifolds $\mathcal{M}_{0,n+1}$.

From (28) it follows that the *second fundamental form* of the indecomposable cartesian product of a manifold $\tilde{M}^n$ and its normal $[X_0, X_{n+1}]$ is given by

$$\varphi(\mathcal{M}_{0,n+1}) = (\cos u^{n+1} - \lambda \sin u^{n+1}) \lambda_i^{n+2} (\omega^i)^2 , \tag{69}$$

while on the other hand the *metrical fundamental form* is, based on (26), found to be

$$ds^2(\mathcal{M}_{0,n+1}) = (\cos u^{n+1} - \lambda \sin u^{n+1})^2 \sum_i (\omega^i)^2 + (du^{n+1})^2 . \tag{70}$$

Consequently the principal curvatures of $\mathcal{M}_{0,n+1}$ are

$$\kappa_i = \frac{\lambda_i^{n+2}}{\cos u^{n+1} - \lambda \sin u^{n+1}} , \qquad \kappa_{n+1} = 0 . \tag{71}$$

THEOREM 18. *A manifold $\tilde{M}^n$ is pseudo-umbilical if and only if the associated product manifold $\mathcal{M}_{0,n+1}$ is minimal.*

§ 16. Manifolds $\tilde{M}^n$ with constant Riemannian curvature.

Inspired by [19] we now consider a manifold $\tilde{M}^n$ (see § 3) with *constant Riemannian curvature,* i. e.

(72) $$R_i{}^j{}_{kl}=-K(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}), \qquad K=\text{constant}.$$

Here K is the *Gauss curvature* of $\bar{M}^n$. Let $\mathcal{S}_{x_0}$ be chosen such that $\boldsymbol{X}_{n+1}$ determines the umbilical normal direction (with corresponding principal curvature λ) of $\bar{M}^n$. Then from (7) and (72) it follows that

(73) $$\gamma_{il}^{n+2}\gamma_{jk}^{n+2}-\gamma_{ik}^{n+2}\gamma_{jl}^{n+2}=-(K+1+\lambda^2)(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}).$$

Hence we have

(74) $$\gamma_{ii}^{n+2}\gamma_{jj}^{n+2}=-(K+1+\lambda^2), \qquad i\neq j;$$

(75) $$\gamma_{ii}^{n+2}\sum_{j\neq i}\gamma_{jj}^{n+2}=\gamma_{ii}^{n+2}(\gamma^{n+2}-\gamma_{ii}^{n+2})=-(n-1)(K+1+\lambda^2).$$

Exterior differentiation of

(76) $$\omega_i^{n+2}=-\lambda_i^{n+2}\omega_i^0, \qquad (\lambda_i^{n+2}=\gamma_{ii}^{n+2};\ \gamma_{ij}^{n+2}=0 \quad \text{if } i\neq j)$$

yields

(77) $$\sum_j \gamma_{jj}^{n+2}\omega_i{}^j\wedge\omega^j=-d\gamma_{ii}^{n+2}\wedge\omega_i{}^0-\gamma_{ii}^{n+2}\sum_j\omega_i{}^j\wedge\omega_j{}^0.$$

After multiplication by γ_{ii}^{n+2}, (77) becomes

(78) $$[K+1+\lambda^2+(\gamma_{ii}^{n+2})^2]d\wedge\omega_i{}^0+\gamma_{ii}^{n+2}d\gamma_{ii}^{n+2}\wedge\omega_i{}^0=0,$$

or equivalently

(79) $$(K+1+\lambda^2)d\wedge\omega_i{}^0+\gamma_{ii}^{n+2}d(\gamma_{ii}^{n+2}\omega_i{}^0)=0.$$

Next consider the *rectilinear system* $\mathcal{L}_{i,n+2}$ (depending on n parameters) with general element $[\boldsymbol{X}_i, \boldsymbol{X}_{n+2}]$. Putting

(80) $$\boldsymbol{T}=\boldsymbol{X}_i\cos\nu+\boldsymbol{X}_{n+2}\sin\nu, \qquad (\nu\in\mathcal{D}(\bar{M}^n)),$$

$\mathcal{L}_{i,n+2}$ is a *normal* system if and only if

(81) $$\langle d\boldsymbol{T}, \boldsymbol{X}_{n+2}\cos\nu-\boldsymbol{X}_i\sin\nu\rangle=0,$$

i. e. if and only if

(82) $$\omega_i^{n+2}+d\nu=0.$$

In this case (79) reduces to

(83) $$(K+1+\lambda^2)d\wedge\omega^i=0.$$

As follows from (74) manifolds $\bar{M}^n$ are such that essentially

(84) $$K+1+\lambda^2\neq 0,$$

and so we have the

THEOREM 19. *A manifold $\bar{M}^n$ with constant Riemannian curvature and for* *which all rectilinear systems $\mathcal{L}_{i,n+2}$ are normal is locally isometric with an n-dimensional Euclidean space E^n.*

If on the other hand the principal curvatures of $\bar{M}^n$ corresponding to X_{n+2} are *constant*, then (78) reduces to

$$[K+1+\lambda^2+(\gamma_{ii}^{n+2})^2]d\wedge\omega^i=0\,. \tag{85}$$

Then if

$$K>-(1+\lambda^2) \tag{86}$$

we have

$$d\wedge\omega^i=0\,,\qquad (\forall i)\,; \tag{87}$$

and if

$$K<-(1+\lambda^2) \tag{88}$$

we have either (87) or

$$\begin{aligned} d\wedge\omega^{\tilde{i}}&=0\,, & \tilde{i}&\in I\,,\\ K+1+\lambda^2+(\gamma_{\tilde{i}'\tilde{i}'}^{n+2})^2&=0\,, & \tilde{i}'&\in\{1,2,\cdots,n\}\backslash I\,, \end{aligned} \tag{89}$$

where $I=\emptyset$ or a real subset of $\{1, 2, \cdots, n\}$. In the latter situation it follows from (75) that

$$n(K+1+\lambda^2)+\gamma^{n+2}\gamma_{\tilde{i}'\tilde{i}'}^{n+2}=0\,, \tag{90}$$

and so

$$\gamma_{\tilde{i}'\tilde{i}'}^{n+2}=\varepsilon[-(K+1+\lambda^2)]^{1/2}\,,\qquad \varepsilon=\pm1\,. \tag{91}$$

Then however (74) implies that $\bar{M}^n$ is totally umbilical.

THEOREM 20. *A manifold $\bar{M}^n$ with constant Riemannian curvature and for which the principal curvatures corresponding to the normal direction rectangular to the umbilical one are constant is locally isometric with E^n.*

§ 17. Compact surfaces of genus 0 and having an isoperimetric normal direction which is parallel in the normal bundle.

Finally we consider an isometric immersion $x: M^2\to P_e^{2+N}$ of a C^2-manifold into P_e^{2+N}. With $X_0\in M^2$ we associate an orthonormal simplex $\mathcal{S}_{X_0}\equiv\{X_A\}$, $(A, B, C\in\{0, 1, \cdots, 2+N\})$, such that $T_{X_0}(M^2)=[X_i]$, $(i, j, k, l\in\{1, 2\})$. Then the formulae given in § 1 keep being valid with $r\in\{3, 4, \cdots, 2+N\}$.

Suppose that M^2 has an *isoperimetric normal direction* [3] which is *parallel in the normal bundle*. Choosing $\mathcal{S}_{X_0}$ such that X_3 determines this normal direction and that φ_3 is diagonal, we thus have

(92) $$\gamma^3=\gamma^3_{11}+\gamma^3_{22}=\text{constant}, \qquad (\gamma^3_{12}=\gamma^3_{21}=0),$$

(93) $$\omega^r_3=0.$$

Putting

(94) $$L_{ij}=\gamma^3_{ij}-\delta_{ij}L, \qquad (L\in\mathcal{D}(M^2);\ \delta_{ij}=\text{Kronecker delta}),$$

we define a symmetric tensor $L_{ij}\in C^1$ on M^2. Clearly

(95) $$g^{ij}L_{ij}=\gamma^3-2L.$$

We choose L such that $g^{ij}L_{ij}=0$:

(96) $$L=\frac{1}{2}\gamma^3.$$

Then we have

(97) $$L_{11}=-L_{22}=\frac{1}{2}(\gamma^3_{11}-\gamma^3_{22}), \qquad L_{12}=L_{21}=0.$$

Since the covariant derivatives of L_{ij} are given by

(98) $$\nabla_k L_{ij}=\partial_k L_{ij}+\gamma^l_{ki}L_{lj}+\gamma^l_{kj}L_{li},$$

(∂=Pfaffian derivative), we find

(99) $$\nabla_2 L_{11}=\frac{1}{2}\partial_2(\gamma^3_{11}-\gamma^3_{22})-\gamma^2_{11}(\gamma^3_{11}-\gamma^3_{22}),$$
$$\nabla_1 L_{22}=-\frac{1}{2}\partial_1(\gamma^3_{11}-\gamma^3_{22})-\gamma^2_{12}(\gamma^3_{11}-\gamma^3_{22}).$$

Exterior differentiation of

(100) $$\omega^3_1=\gamma^3_{11}\omega^1, \qquad \omega^3_2=\gamma^3_{22}\omega^2$$

yields

(101) $$\partial_2\gamma^3_{11}=\gamma^2_{11}(\gamma^3_{11}-\gamma^3_{22}),$$
$$\partial_1\gamma^3_{22}=\gamma^2_{12}(\gamma^3_{11}-\gamma^3_{22}).$$

Moreover (92) implies that

(102) $$\partial_i\gamma^3_{11}+\partial_i\gamma^3_{22}=0,$$

and consequently

(103) $$\nabla_2 L_{11}=\nabla_1 L_{22}=0.$$

Supposing that M^2 is compact and of genus 0, it then follows from a well know result of H. Hopf [8] that

(104) $$L_{ij}=0.$$

Hence as a generalization both of Theorem 2.2 in [7] and Lemma 2 in [6], we obtain

THEOREM 21. *Consider a compact C^2-manifold of genus 0 which is isometrically immersed into an elliptic space. Let η be a normal direction of this manifold which is isoperimetric and parallel in the normal bundle. Then η is umbilical.*

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