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KŌDAI MATH. SEM. REP.
27 (1976), 181—187

POSITIVE IDEMPOTENTS ON A LOCALLY COMPACT ABELIAN GROUP

Dedicated to Professor Yûsaku Komatu on his sixtieth birthday

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1. Introduction.

It will be shown by a potential theoretical method that a positive Radon measure σ on a locally compact abelian group G is a unit Haar measure on a compact subgroup of G, if the convolution $\sigma*\sigma$ is well defined and verifies the relation $\sigma*\sigma=\sigma$. This is an easy consequence of $(\hat{\sigma})^2=\hat{\sigma}$, $\hat{\sigma}$ being the Fourier transform of σ, if σ is bounded or of finite total mass. In order to show the compactness of the support of σ, we shall make use of potential theoretical properties of σ.

2. Resolvent.

Let κ be a positive (Radon) measure on a locally compact abelian group G. We shall denote by $D^+(\kappa)$ the totality of positive measures μ for which the convolution $\kappa*\mu$ is well defined. Here $\kappa*\mu$ is the positive measure defined by

$$\kappa*\mu(\varphi)=\iint\varphi(x+y)d\kappa(x)d\mu(y)$$

for every non-negative continuous function φ with compact support. This is well defined if and only if the above integral converges for every φ.

A family $\{\kappa_p\,;p>0\}$ of positive measures is called a *resolvent* associated with κ, if for every $p>0$, κ_p belongs to $D^+(\kappa)$ and

(1) $$\kappa-\kappa_p=p\kappa*\kappa_p$$

and if $\kappa=\lim_{p\downarrow 0}\kappa_p$ (vague limit). In this case we shall say that κ has a resolvent $\{\kappa_p\}$.

It is known that if $\{\kappa_p\}$ is a resolvent of κ, then for every $p>0$

(2) $$p\Big(\kappa+\frac{1}{p}\varepsilon\Big)=\sum_{n=0}^{\infty}(p\kappa_p)^n\,,$$

Received March 30, 1974.

where ε is the unit point measure on the origin 0 of G and $(p\kappa_p)^0=\varepsilon$, $(p\kappa_p)^n=(p\kappa_p)*(p\kappa_p)^{n-1}$. The positive measure of the right hand side of (2) is called an *elementary kernel*. From (2) it follows that the resolvent is uniquely determined by κ.

Let ω be an open subset of G and μ be a positive measure in $D^+(\kappa)$. A positive measure $\mu_\omega\in D^+(\kappa)$ supported by $\bar{\omega}$ (the closure of ω) will be called a *balayaged measure* of μ to ω with respect to κ, if the following three conditions are satisfied:

i) $\kappa*\mu_\omega\leqq\kappa*\mu$ in G,

ii) $\kappa*\mu_\omega=\kappa*\mu$ in ω,

iii) if ν is a positive measure in $D^+(\kappa)$ and if $\kappa*\nu$ dominates $\kappa*\mu$ in ω, then $\kappa*\nu$ dominates $\kappa*\mu_\omega$ in G.

PROPOSITION 1. *If κ has a resolvent, there exists uniquely the balayaged measure for every $\mu\in D^+(\kappa)$ and for every open set ω.*

This is proved by using the existence of balayaged measures with respect to elementary kernels and by the relation (2). For the details we refer to [2], [3]. The following proposition is also known.

PROPOSITION 2. *If κ has a resolvent, it satisfies the domination principle, that is, if μ and ν are positive measures in $D^+(\kappa)$ and $\kappa*\mu\leqq\kappa*\nu$ in a neighborhood of the support supp (μ) of μ, then $\kappa*\mu\leqq\kappa*\nu$ in G.*

3. Two lemmas.

We shall need the following convergence lemma.

LEMMA 1. *Suppose that κ has a resolvent. If a net $\{\mu_\alpha\}$ of positive measures in $D^+(\kappa)$ converges vaguely to μ and if there exists a positive measure $\nu\in D^+(\kappa)$ such that $\kappa*\nu$ dominates every $\kappa*\mu_\alpha$ in G, then $\{\kappa*\mu_\alpha\}$ converges vaguely to $\kappa*\mu$.*

Proof. Let φ be a non-negative continuous function with compact support. Assuming that $\kappa\neq0$, we shall prove

$$\kappa*\mu(\varphi)\geqq\limsup_\alpha \kappa*\mu_\alpha(\varphi)\,. \tag{3}$$

This gives the required vague convergence, since it will be immediately seen that

$$\kappa*\mu(\varphi)\leqq\liminf_\alpha \kappa*\mu_\alpha(\varphi)\,.$$

Let k be a positive integer. Then we have by (2)

$$(p\kappa_p)^k*\kappa*\mu_\alpha\leqq(p\kappa_p)^k*\left(\kappa+\frac{1}{p}\varepsilon\right)*\nu$$

$$=\frac{1}{p}\sum_{n=k}^{\infty}(p\kappa_p)^n*\nu\leqq\kappa*\nu\,.$$

Hence by a convergence theorem in [3]

$$(4) \qquad (p\kappa_p)^k * \mu(\varphi) = \lim_{\alpha} (p\kappa_p)^k * \mu_\alpha(\varphi) .$$

On the other hand, we have

$$\begin{aligned} \frac{1}{p}\sum_{n=k+1}^{\infty}(p\kappa_p)^n * \mu_\alpha &= (p\kappa_p)^k * \frac{1}{p}\sum_{n=1}^{\infty}(p\kappa_p)^n * \mu_\alpha \\ &= (p\kappa_p)^k * \kappa * \mu_\alpha \leqq (p\kappa_p)^k * \kappa * \nu \\ &= \frac{1}{p}\sum_{n=k+1}^{\infty}(p\kappa_p)^n * \nu . \end{aligned}$$

Therefore for a given positive number η, there exists a positive integer k such that

$$\frac{1}{p}\sum_{n=k+1}^{\infty}(p\kappa_p)^n * \mu_\alpha(\varphi) \leqq \frac{1}{p}\sum_{n=k+1}^{\infty}(p\kappa_p)^n * \nu(\varphi) < \eta ,$$

so that by (4)

$$\begin{aligned} \left(\kappa + \frac{1}{p}\varepsilon\right) * \mu(\varphi) &\geqq \frac{1}{p}\sum_{n=0}^{k}(p\kappa_p)^n * \mu(\varphi) \\ &= \lim_{\alpha} \frac{1}{p}\sum_{n=0}^{k}(p\kappa_p)^n * \mu_\alpha(\varphi) \\ &= \lim_{\alpha}\left\{\frac{1}{p}\sum_{n=0}^{\infty}(p\kappa_p)^n * \mu_\alpha(\varphi) - \frac{1}{p}\sum_{n=k+1}^{\infty}(p\kappa_p)^n * \mu_\alpha(\varphi)\right\} \\ &\geqq \limsup_{\alpha}\left(\kappa + \frac{1}{p}\varepsilon\right) * \mu_\alpha(\varphi) - \eta . \end{aligned}$$

Hence

$$\left(\kappa + \frac{1}{p}\varepsilon\right) * \mu(\varphi) \geqq \limsup_{\alpha} (\kappa * \mu_\alpha)(\varphi) ,$$

which proves (3), and the proof is completed.

The following lemma is due to M. Itô [4].

LEMMA 2. *Let $\kappa \neq 0$ have a resolvent and $\sigma \neq 0$ be a positive measure in $D^+(\kappa)$ such that $\kappa * \sigma = \kappa$ in G, and a be a point of the support* $\mathrm{supp}(\sigma)$ *of σ. Then for every open neighborhood $\omega(a)$ of a, the balayaged measure $\varepsilon_{\omega(a)}$ of ε verifies $\kappa * \varepsilon_{\omega(a)} = \kappa$ in G.*

Proof. The positive measure $\mu = \kappa - \kappa * \varepsilon_{\omega(a)}$ vanishes in a neighborhood ω of 0. In fact, if $\mathrm{supp}(\mu)$ contains 0, it contains absurdly a, since

$$\mu = \kappa * \sigma - \kappa * \sigma * \varepsilon_{\omega(a)} = \mu * \sigma .$$

Consequently $\kappa * \varepsilon = \kappa * \varepsilon_{\omega(a)}$ in ω which contains $\mathrm{supp}(\varepsilon)$, and hence by Proposition 2, $\kappa \geqq \kappa * \varepsilon_{\omega(a)}$. This proves $\kappa = \kappa * \varepsilon_{\omega(a)}$.

4. Positive idempotents.

THEOREM. *Let $\sigma \neq 0$ be a positive measure such that $\sigma * \sigma$ is well defined and $\sigma * \sigma = \sigma$ in G. Then σ is a unit Haar measure of a compact subgroup of G.*

Proof. First we shall show that supp (σ) is compact. Setting for $p>0$

$$\sigma_p = \frac{1}{p+1}\sigma ,$$

we see that σ has a resolvent $\{\sigma_p\}$. Hence we can balayage ε to open neighborhoods of points in supp (σ) with respect to σ. If supp (σ) is not compact, there exists a net $\{\mu_\alpha\}$ of balayaged measures of ε, which converges vaguely to 0. By Lemma 2, $\sigma * \mu_\alpha = \sigma$ in G and by Lemma 1, $\{\sigma * \mu_\alpha\}$ converges vaguely to 0. Hence $\sigma = 0$ which contradicts our assumption.

Now σ being supported by a compact set, the equality $\sigma * \sigma = \sigma$ gives $(\hat{\sigma})^2 = \widehat{\sigma * \sigma} = \hat{\sigma}$ and hence $\hat{\sigma}$ has 1 and 0 as its values. We denote by Γ the dual group of G and set

$$\Gamma' = \{\gamma \in \Gamma \, ; \, \hat{\sigma}(\gamma) = 1\} .$$

Then $\hat{\sigma}$ being a positive definite continuous function, Γ' is a closed subgroup of Γ. It will be seen that σ is the unit Haar measure of the compact subgroup $G' = \{x \in G \, ; \, (x, \gamma) = 1 \text{ for every } \gamma \in \Gamma'\}$, where (x, γ) denotes the value of the character γ at x. In fact, Γ' is the annihilator of G' and $\hat{\sigma}$ is constant on every coset of Γ', and hence σ is supported by G' (cf. [5]). We note then that $\hat{\sigma}$ is the characteristic function of Γ' to conclude our assumption.

5. Some consequences.

Let $\kappa \neq 0$ be a positive measure with resolvent. From (1) it follows that $\{p\kappa_p\}$ is vaguely bounded. We note that as $p \to \infty$, $\{p\kappa_p\}$ converges vaguely to a unit Haar measure of a compact subgroup of G. In fact, let σ be a vaguely adherent positive measure of $\{p\kappa_p\}$. Then for every $p>0$

$$\kappa_p = \sigma * \kappa_p , \tag{5}$$

since by (1) and by Lemma 1

$$\begin{aligned} \kappa_p &= \lim_q (\kappa_p - \kappa_q) \\ &= \lim_q q\kappa_q * \kappa_p - \lim_q p\kappa_q * \kappa_p \\ &= \sigma * \kappa_p . \end{aligned}$$

Hence for any adherent measure σ' of $\{p\kappa_p\}$

$$\sigma = \sigma * \sigma' = \sigma' * \sigma = \sigma' .$$

Thus $\{p\kappa_p\}$ converges vaguely to a positive idempotent $\sigma\neq 0$. Hence by our theorem it converges to a unit Haar measure of a compact subgroup of G.

First we consider the case $\sigma=\varepsilon$.

LEMMA 3. *Suppose that κ has a resolvent $\{\kappa_p\}$ such that $\{p\kappa_p\}$ converges vaguely to ε as $p\to\infty$. Then κ satisfies the unicity principle, that is, $\kappa*\mu=\kappa*\nu$ holds if and only if $\mu=\nu$.*

Proof. If μ and ν are positive measures of $D^+(\kappa)$ and $\kappa*\mu=\kappa*\nu$, then by (1), $p\kappa_p*\mu=p\kappa_p*\nu$. Hence $\mu=\varepsilon*\mu=\lim_p p\kappa_p*\mu=\lim_p p\kappa_p*\nu=\nu$.

A positive measure κ is called a *Hunt kernel*, when it has the following integral representation

$$\kappa=\int_0^\infty \alpha_t dt\,,$$

where $\{\alpha_t\,;\,t\geqq 0\}$ is a vaguely continuous semigroup with $\alpha_0=\varepsilon$. The resolvent $\{\kappa_p\}$ of κ is given by

$$\kappa_p=\int_0^\infty e^{-pt}\alpha_t dt\,,$$

and $\{p\kappa_p\}$ converges vaguely to ε (cf. [3]). Conversely if κ has a resolvent $\{\kappa_p\}$ such that $\{p\kappa_p\}$ converges vaguely to ε, then it satisfies the unicity principle. Hence we can construct the representing semigroup as in [3]. Thus we have

PROPOSITION 3. *Let κ be a positive measure with resolvent $\{\kappa_p\}$. Then it is a Hunt kernel if and only if $\{p\kappa_p\}$ converges vaguely to ε as $p\to\infty$.*

COROLLARY. *Suppose that G has no other compact subgroup than $\{0\}$. Then a positive measure $\kappa\neq 0$ is a Hunt kernel if and only if κ has a resolvent.*

Now we consider the case $\sigma\neq\varepsilon$, σ being a unit Haar measure of a compact subgroup H of G. We denote by G/H the factor space and by π the canonical mapping $G\to G/H$.

PROPOSITION 4. *Let κ be a positive measure with resolvent. Then there exist a compact subgroup H of G and a Hunt kernel $\kappa^\flat$ on G/H such that for every continuous function φ with compact support on G*

$$\int_{G/H}\varphi^\flat d\kappa^\flat=\int_G \varphi\, d\kappa\,, \tag{6}$$

where $\varphi^\flat$ is the continuous function on G/H defined by

$$\varphi^\flat(\pi(x))=\int_H \varphi(x+y)d\sigma(y)\,.$$

These H and $\kappa^\flat$ are uniquely determined by κ.

Proof. We suppose that $\{p\kappa_p\}$ converges vaguely to σ, a unit Haar measure on a compact subgroup H. For every point $y\in H$,

$$\kappa*\varepsilon_y=\kappa*\sigma*\varepsilon_y=\kappa*\sigma=\kappa\,, \qquad \kappa_p*\varepsilon_y=\kappa_p\,.$$

Consequently there exist positive measures $\kappa^\flat$, $\kappa_p^\flat$ such that

$$\int_{G/H}\varphi^\flat d\kappa^\flat=\int_G\varphi\, d\kappa\,, \qquad \int_{G/H}\varphi^\flat d\kappa_p^\flat=\int_G\varphi\, d\kappa$$

(cf. [1]). It is easily seen that $\{\kappa_p^\flat\}$ is a resolvent of $\kappa^\flat$ by using the fact: for a continuous function f with compact support on G/H, $(f\circ\pi)^\flat=f$. It is also seen that $\{p\kappa_p^\flat\}$ converges vaguely to ε on G/H, so that $\kappa^\flat$ is a Hunt kernel on G/H. In order to verify the uniqueness we assume the representation (6) by a compact subgroup H and a Hunt kernel $\kappa^\flat$ on G/H. Then

$$\begin{aligned}\lim_p p\int\varphi\, d\kappa_p&=\lim_p p\int\varphi^\flat d\kappa_p^\flat\\ &=\varphi^\flat(\pi(0))=\int_H\varphi(y)d\sigma(y)\,.\end{aligned}$$

Hence $\{p\kappa_p\}$ converges vaguely to the unit measure on H. Thus H and $\kappa^\flat$ are uniquely determined by κ. This completes the proof.

For a positive measure μ on G, let $\pi\circ\mu$ be the positive measure on G/H defined by

$$\int_{G/H}fd(\pi\circ\mu)=\int f(\pi(x))d\mu(x)\,.$$

We remark that under the same assumption of Proposition 4, it holds that $\kappa*\mu=\kappa*\nu$ ($\mu,\nu\in D^+(\kappa)$) if and only if $\pi\circ\mu=\pi\circ\nu$. In fact, if $\kappa*\mu=\kappa*\nu$, then by the convergence theorem in [3], $\sigma*\mu=\sigma*\nu$ and hence

$$\begin{aligned}\int fd(\pi\circ\mu)&=\int(f\circ\pi)d\mu\\ &=\iint(f\circ\pi)(x+y)d\mu(x)d\sigma(y)\\ &=\sigma*\mu(f\circ\pi)=\sigma*\pi(f\circ\pi)=\int fd(\pi\circ\nu)\end{aligned}$$

for every continuous function with compact support on G/H. Conversely suppose that $\pi\circ\mu=\pi\circ\nu$. Then $(\check{\mu}*\varphi)^\flat=(\check{\nu}*\varphi)^\flat$ and $\kappa*\mu(\varphi)=\kappa*\nu(\varphi)$.

We shall also remark that with respect to κ of Proposition 4 every positive κ-excessive measure s ($p\kappa_p*s\leqq s$ for every $p>0$) is decomposed as follows:

$$s=\kappa*\mu+\nu\,,$$

where $\mu\in D^+(\kappa)$, $\nu\in D^+(\kappa_p)$ for every $p>0$ and $p\kappa_p*\nu=\nu$, and $\pi\circ\mu$ and ν are uniquely determined by s.

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