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NEARLY SASAKIAN STRUCTURES

Dedicated to Professor Y. Komatu on his sixtieth birthday

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1. Introduction. In [1, 2, 3] the authors studied almost contact manifolds with Killing structure tensors (called nearly cosymplectic) and showed that if this structure is normal, it is cosymplectic; in particular the (almost) contact distribution is integrable. In this note, the notion of a nearly Sasakian structure is introduced. It is shown that a normal nearly Sasakian structure is Sasakian and hence in particular is contact. In addition, it is shown that a hypersurface of a nearly Kähler manifold is nearly Sasakian if and only if it is quasi-umbilical with respect to the (almost) contact form. In particular, S^5 properly imbedded in S^6 inherits a nearly Sasakian structure which is not Sasakian.

2. Almost contact manifolds. A $(2n+1)$-dimensional C^∞ manifold M^{2n+1} is said to have an almost contact structure with an associated Riemannian metric g if there exist on M^{2n+1} a tensor field φ of type (1, 1), a unit vector field ξ and dual 1-form η with respect to g which satisfy

$$\varphi^2=-I+\eta\otimes\xi\,,\qquad \varphi\xi=0\,,$$
$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y)\,, \tag{2.1}$$

I being the identity tensor. We define a fundamental 2-form Φ by

$$\Phi(X, Y)=g(\varphi X, Y)\,. \tag{2.2}$$

If M^{2n+1} has an almost contact structure, $M^{2n+1}\times R$ can be given an almost complex structure defined by

$$J(X, fd/dt)=(\varphi X-f\xi, \eta(X)d/dt)\,, \tag{2.3}$$

where f is a C^∞ function defined on $M^{2n+1}\times R$. If this almost complex structure is integrable, then the almost contact structure is said to be *normal*. Let $[J, J]$ denote the Nijenhuis torsion of J. Sasaki and Hatakeyama [6] computed $[J, J]((X, 0), (Y, 0))$ and $[J, J]((X, 0), (0, d/dt))$ which gave rise to four tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ given by

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$$(2.4)\qquad \begin{aligned} N^{(1)}(X, Y) &= [\varphi, \varphi](X, Y) + d\eta(X, Y)\xi\,, \\ N^{(2)}(X, Y) &= (\mathcal{L}_{\varphi X}\eta)(Y) - (\mathcal{L}_{\varphi Y}\eta)(X)\,, \\ N^{(3)}(X) &= (\mathcal{L}_{\xi}\varphi)X\,, \\ N^{(4)}(X) &= (\mathcal{L}_{\xi}\eta)(X)\,, \end{aligned}$$

where $\mathcal{L}$ denotes Lie differentiation. The result is that J is integrable if and only if $N^{(1)}=0$; in particular $N^{(1)}=0$ implies that $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ also vanish.

A $(2n+1)$-dimensional manifold is said to have a contact structure if it carries a 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere and it is known that a contact manifold carries an associated almost contact metric structure (φ, ξ, η, g) with $\Phi=\frac{1}{2}d\eta$, called a *contact metric structure*. On a contact manifold the tensors $N^{(2)}$ and $N^{(4)}$ vanish and if ξ is a Killing vector field with respect to g, $N^{(3)}=0$ [6]. If an almost contact metric structure is both normal and contact metric, it is said to be *Sasakian*, equivalently

$$(2.5)\qquad (\nabla_X\varphi)Y = -g(X, Y)\xi + \eta(Y)X\,,$$

where ∇ denotes covariant differentiation with respect to the Riemannian connection of g. From this one can deduce that for a Sasakian structure

$$(2.6)\qquad \nabla_X\xi = \varphi X\,.$$

Finally consider briefly an almost Hermitian structure (J, G):

$$(2.7)\qquad J^2=-I\,, \qquad G(JX, JY)=G(X, Y)\,.$$

Let $\bar{\nabla}$ denote the Riemannian connection of G. Then J is Killing if and only if

$$(2.8)\qquad (\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0\,.$$

An almost Hermitian structure with J Killing is said to be *nearly Kähler* [5] or *almost Tachibana* [7] and if such a J is integrable, the structure is Kählerian.

3. Nearly Sasakian manifolds.

An almost contact metric structure (φ,ξ,η,g) is said to be *nearly Sasakian* if

$$(3.1)\qquad (\nabla_X\varphi)Y + (\nabla_Y\varphi)X = -2g(X, Y)\xi + \eta(X)Y + \eta(Y)X\,.$$

PROPOSITION 3.1. *On a nearly Sasakian manifold the vector field ξ is Killing.*

Proof. First of all putting $X=\xi$, $Y=\xi$ in (3.1) we find

$$(\nabla_\xi\varphi)\xi=0 \quad \text{or} \quad \varphi\nabla_\xi\xi=0\,,$$

from which applying φ and using (2.1) and $\eta(\nabla_\xi\xi)=0$, we find

$$(3.2)\qquad \nabla_\xi\xi=0 \quad \text{and} \quad \nabla_\xi\eta=0\,.$$

Now applying ∇_ξ to the last equation of (2.1) we find

$$g((\nabla_\xi\varphi)X, \varphi Y)+g(\varphi X, (\nabla_\xi\varphi)Y)=0\,,$$

or, using (3.1),

$$g(-(\nabla_X\varphi)\xi-2g(\xi, X)\xi+X+\eta(X)\xi, \varphi Y)$$
$$+g(\varphi X, -(\nabla_Y\varphi)\xi-2g(\xi, Y)\xi+Y+\eta(Y)\xi)=0\,,$$

or

$$g(\varphi\nabla_X\xi+X, \varphi Y)+g(\varphi X, \varphi\nabla_Y\xi+Y)=0\,,$$

from which

$$g(\nabla_X\xi, Y)+g(\nabla_Y\xi, X)=0\,,$$

which shows that ξ is Killing.

THEOREM 3.2. *For a nearly Sasakian structure normality is equivalent to contact metric. In particular a normal nearly Sasakıan strucrure ıs Sasakian.*

Proof. We compute

$$\begin{aligned}\eta(N^{(1)}(X, Y))&=\eta((\nabla_{\varphi X}\varphi)Y-(\nabla_{\varphi Y}\varphi)X)+d\eta(X, Y)\\&=\eta(-(\nabla_Y\varphi)\varphi X-2g(\varphi X, Y)\xi+(\nabla_X\varphi)\varphi Y\\&\qquad+2g(\varphi Y, X)\xi)+d\eta(X, Y)\\&=(\nabla_Y\eta)(\varphi^2X)-(\nabla_X\eta)(\varphi^2Y)-4\Phi(X, Y)+d\eta(X, Y)\\&=-(\nabla_Y\eta)(X)+(\nabla_X\eta)(Y)-4\Phi(X, Y)+d\eta(X, Y)\\&=2d\eta(X, Y)-4\Phi(X, Y)\,.\end{aligned}$$

Thus if $N^{(1)}(X, Y)=0$ we have $\Phi=\frac{1}{2}d\eta$.

Conversely if a nearly Sasakian structure (φ, ξ, η, g) is also a contact metric structure, $\Phi=\frac{1}{2}d\eta$, so $d\Phi=0$. Therefore

$$\begin{aligned}0&=(\nabla_X\Phi)(Y, Z)-(\nabla_Y\Phi)(X, Z)+(\nabla_Z\Phi)(X, Y)\\&=g((\nabla_X\varphi)Y, Z)-g((\nabla_Y\varphi)X, Z)+g((\nabla_Z\varphi)X, Y)\\&=g((\nabla_X\varphi)Y, Z)-g(-(\nabla_X\varphi)Y-2g(X, Y)\xi+\eta(X)Y+\eta(Y)X, Z)\\&\quad+g(-(\nabla_X\varphi)Z-2g(X, Z)\xi+\eta(X)Z+\eta(Z)X, Y)\\&=3g((\nabla_X\varphi)Y, Z)+3g(X, Y)\eta(Z)-3g(X, Z)\eta(Y)\end{aligned}$$

and hence

$$(\nabla_X\varphi)Y=-g(X, Y)\xi+\eta(Y)X$$

and the structure is Sasakian and consequently normal.

We state expressions for $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ on a nearly Sasakian manifold; these are obtained by direct computation.

$$N^{(2)}(X, Y)=-4[g(\varphi\nabla_X\xi, Y)+g(X, Y)-\eta(X)\eta(Y)],$$

$$N^{(3)}(X)=3[\varphi\nabla_X\xi-\eta(X)\xi+X],$$
$$N^{(4)}(X)=0.$$

In particular, $N^{(2)}=0$ if and only if $\nabla_X\xi=\varphi X$ and hence in view of Theorem 3.2, on a nearly Sasakian manifold the vanishing of $N^{(1)}$ and $N^{(2)}$ are equivalent. Similarly the vanishing of $N^{(1)}$ and $N^{(3)}$ are equivalent on a nearly Sasakian manifold.

4. Hypersurfaces of nearly Kähler manifolds.

Let M^{2n+2} be an almost Hermitian manifold with structure tensors (J, G) and Riemannian connection $\bar{\nabla}$. Let $\iota: M^{2n+1}\rightarrow M^{2n+2}$ be a C^∞ orientable hypersurface and C a unit normal. The induced metric g is given by $g(X, Y)=G(\iota_*X, \iota_*Y)$ and its Riemannian connection ∇ is governed by the Gauss-Weingarten equations

$$\bar{\nabla}_{\iota_*X}\iota_*Y=\iota_*\nabla_XY+h(X, Y)C, \qquad \bar{\nabla}_{\iota_*X}C=-\iota_*HX, \tag{4.1}$$

where h denotes the second fundamental form and H the corresponding Weingarten map.

A hypersurface is said to be *quasi-umbilical* [4] if $h(X, Y)=\alpha g(X, Y)+\beta u(X)u(Y)$ where α and β are functions on the hypersurface and u a non-vanishing 1-form.

Y. Tashiro [8] showed that the hypersurface M^{2n+1} inherits an almost contact metric structure (φ, ξ, η, g) given by

$$J\iota_*X=\iota_*\varphi X+\eta(X)C, \qquad JC=-\iota_*\xi \tag{4.2}$$

and g the induced metric.

THEOREM 4.1. *Let M^{2n+1} be a hypersurface of a nearly Kähler manifold M^{2n+2}. Then the induced structure on M^{2n+1} is nearly Sasakian if and only if*

$$h(X, Y)=g(X, Y)+(h(\xi, \xi)-1)\eta(X)\eta(Y). \tag{4.3}$$

Proof. By the nearly Kähler condition

$$(\bar{\nabla}_{\iota_*X}J)\iota_*Y+(\bar{\nabla}_{\iota_*Y}J)\iota_*X=0,$$

that is,

$$\bar{\nabla}_{\iota_*X}J\iota_*Y-J\bar{\nabla}_{\iota_*X}\iota_*Y+\bar{\nabla}_{\iota_*Y}J\iota_*X-J\bar{\nabla}_{\iota_*Y}\iota_*X=0.$$

Substituting $J\iota_*X=\iota_*\varphi X+\eta(X)C$ and using the Gauss-Weingarten equations we can reduce the above to

$$(\nabla_X \varphi)Y+(\nabla_Y \varphi)X+2h(X, Y)\xi-\eta(X)HY-\eta(Y)HX=0 .$$

Clearly if $h(X, Y)=g(X, Y)+(h(\xi, \xi)-1)\eta(X)\eta(Y)$ M^{2n+1} is nearly Sasakian. Conversely if the structure is nearly Sasakian we have

$$-2g(X, Y)\xi+\eta(X)Y+\eta(Y)X+2h(X, Y)\xi-\eta(X)HY-\eta(Y)HX=0 ,$$

from which

$$2h(X, Y)=2g(X, Y)-2\eta(X)\eta(Y)+\eta(X)h(Y, \xi)+\eta(Y)h(X, \xi) .$$

Setting $Y=\xi$ we find that

$$h(X, \xi)=h(\xi, \xi)\eta(X)$$

and therefore

$$2h(X, Y)=2g(X, Y)-2\eta(X)\eta(Y)+2h(\xi, \xi)\eta(X)\eta(Y) ,$$

or

$$h(X, Y)=g(X, Y)+(h(\xi, \xi)-1)\eta(X)\eta(Y)$$

as desired.

As an example we show that the 5-dimensional sphere S^5 has a nearly Sasakian structure which is not Sasakian; in particular this is not the usual almost contact metric structure on an odd-dimensional sphere.

First consider the unit sphere S^6 in R^7 with its vector product $\times$ induced from the Cayley algebra. Letting N denote the unit outer normal and ϕ the imbedding, $\phi_* JX=N\times\phi_* X$ defines an almost complex structure J on S^6 which, as is well known, is nearly Kähler with respect to the induced metric. Now consider S^5 umbilically imbedded in S^6 at a "latitude" of 45° and with unit normal C such that the second fundamental form $h(X, Y)=g(X, Y)$. Then by Theorem 4.1 we see that the induced structure on S^5 from the nearly Kähler structure on S^6 is nearly Sasakian. However on a Sasakian manifold all sectional curvatures of plane sections containing ξ are equal to 1. Thus since the induced metric on S^5 has constant curvature 2, the induced nearly Sasakian structure is not Sasakian.

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