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ON THE SHARP FORM OF THE THREE-CIRCLES THEOREM

Dedicated to Professor Yûsaku Komatu

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1. It might seem surprising that anything remains to be said about the Three-Circles Theorem especially since various treatments of the sharp form of this result have been given by Teichmüller [3], Heins [1] and Robinson [2]. We regard the problem in the form that for $f(z)$ regular (single-valued) for $1\leqq|z|\leqq R$ and satisfying the bounds $|f(z)|\leqq 1$ for $|z|=1$, $|f(z)|\leqq M$ for $|z|=R$ the sharp bound of $|f(r)|$, $1<r<R$ is to be determined. Heins' solution is indeed quite implicit. Teichmüller gave an explicit expression for the sharp bound and a corresponding expression for the extremal function, further deducing from this several rather evident properties of this function. The approach of Robinson is in certain respects the most penetrating. We make slight changes in his notation to conform to that used here. Taking R fixed he observed that for $1<M\leqq R$ the extremal function, further assumed normalized so that $f(r)>0$ and denoted by $\varphi(z, M)$, will be univalent with a simple geometric nature. He further observed that for $R^n<M\leqq R^{n+1}$, n an integer $\geqq 1$, the extremal function will be given by $z^n\varphi(z, MR^{-n})$. He also expressed these extremal functions in terms of theta functions and deduced certain results from this representation. The final statement of his paper suggests that he did not articulate the sharp bound in the form given by Teichmüller although it appears a certain reordering of his considerations would have provided this.

None of the above procedures provides a complete description of the geometric form of the extremal functions and it is with this that we will deal. Further we will show that Teichmüller's explicit bound is an immediate consequence of the simplest properties of the extremal functions.

2. We recall first an important lemma which was the basis of Robinson's method.

LEMMA 1. *If $f(z)$ is regular for $1\leqq|z|\leqq R$ except perhaps for a simple pole at $-b$, $1<b<R$, and if $|f(z)|\leqq 1$ for $|z|=1$ and $|z|=R$ then $|f(z)|\leqq 1$ for $1<z<R$. If the equality holds at any point of this segment, then $f(z)$ is constant.*

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We now proceed to describe geometrically a family of functions which will prove to be the extremal functions for the Three-Circles Theorem. In this we will take M fixed >1 and determine corresponding functions for all values of $R>1$.

Definition 1. Let $\mathfrak{R}(M^{1/n})$ for n a positive integer denote the Riemann surface covering $1<|w|<M$ which is the image of $1<|z|<M^{1/n}$ by the function $f(z, M^{1/n}) \equiv z^n$. We will denote points of this surface by the function value in a simple shorthand notation. Let $\mathcal{S}(n, t)$, $1 \leqq t < M$, denote the Riemann surface obtained by cross-joining $\mathcal{R}(M^{1/n})$ to the disc $|w|<M$ along the segment of points $f(z, M^{1/n})$, z real and negative, which satisfy $t<|f(z, M^{1/n})|<M$. This is a doubly-connected schlichtartig surface, the image of $1<|z|<R(n, t)$ by a function $F(z, n, t)$ (which extends to be regular on $|z|=1$ with $|F(z, n, t)|=1$ there) normalized to be real and positive for z real and positive. Let $\mathcal{S}(0, 1)$ denote the disc $|w|<M$. Let $\mathcal{T}(n, \theta)$, n a non-negative integer, $0<\theta<\pi$, denote the Riemann surface obtained by slitting $\mathcal{S}(n, 1)$ along the arc covering the points w of modulus 1 with $\pi-\theta \leqq \arg w \leqq \pi+\theta$, n odd, $-\theta \leqq \arg w \leqq \theta$, n even. This is a doubly-connected schlichtartig surface the image of $1<|z|<R'(n, \theta)$ by a function $G(z, n, \theta)$ (which extends to be regular on $|z|=1$ with $|G(z, n, \theta)|=1$ there) normalized to be real and positive for z real and positive.

LEMMA 2. *As t decreases on the interval $[1, M)$, $R(n, t)$ decreases strictly over the interval $[R(n, 1), M^{1/n})$ where $R(n, 1)>M^{1/(n+1)}$. As θ increases on the interval $(0, \pi)$, $R'(n, \theta)$ decreases strictly over the interval $(M^{1/(n+1)}, R(n, 1))$ where we set by convention $R(0, 1)=+\infty$. Thus every value greater than 1 other than $M^{1/m}$, m a positive integer, is assumed exactly once by $R(n, t)$, $1 \leqq t<M$, as n runs through the positive integers, or $R'(n, \theta)$, $0<\theta<\pi$, as n runs through the non-negative integers.*

It follows by applying the equality statement of Lemma 1 to the quotients of any two functions among the $F(z, n, t)$, $G(z, n, \theta)$ that no two values among the $R(n, t)$, $R'(n, \theta)$ can be equal. Suppose now we have proved Lemma 2 for all values $n<N$. Let t_1, t_2 be values in $(1, M)$ with $t_1>t_2$. Magnifying $\mathcal{S}(N, t_2)$ in the ratio t_1/t_2 with centre the origin we obtain a surface $\mathcal{S}'(N, t_2)$ which can be regarded as lying with $\mathcal{S}(N, t_1)$ in a common Riemann covering surface. It is then clear that $\mathcal{S}(N, t_1)$, $\mathcal{S}'(N, t_2)$ can be related by a quasiconformal mapping whose dilation tends to 1 as t_2 tends to t_1 (or vice versa). Thus $R(N, t)$ is continuous for t on $(1, M)$. It is readily seen that $\lim_{t\to M} R(N, t)=M^{1/N}$, $\lim_{t\to 1} R(N, t) =R(N, 1)$. Since $R(N, t)$ cannot take any value greater than $M^{1/N}$ or any value less than $M^{1/N}$ more than once, it must decrease strictly as t decreases on $[1, M)$. Further each surface $\mathcal{T}(N, \theta)$, $0<\theta<\pi$, can be regarded as imbedded in $\mathcal{S}(N, 1)$ and for $0<\theta_1<\theta_2<\pi$, $\mathcal{T}(N, \theta_2)$ can be regarded as imbedded in $\mathcal{T}(N, \theta_1)$. It is readily seen that $R'(N, \theta)$ varies continuously with θ on $(0, \pi)$ and that $\lim_{\theta\to 0} R'(N, \theta)=R(N, 1)$, $\lim_{\theta\to\pi} R'(N, \theta)=M^{1/(N+1)}$. Thus $R'(N, \theta)$ decreases strictly on the interval $(M^{1/(N+1)}, R(N, 1))$ as θ increases on $(0, \pi)$. In particular $R(N, 1) > M^{1/(N+1)}$. The result of Lemma 2 follows by induction.

Definition 2. For every value of R greater than 1 and distinct from $M^{1/m}$, m a positive integer, we denote the function among $F(z, n, t)$, $G(z, n, \theta)$ for which $R(n, t)=R$ or $R'(n, \theta)=R$ by $f(z, R)$. This combines with the notation introduced in Definition 1 to define $f(z, R)$ for all $R>1$. If we wish to make explicit the role of M we denote the function in question by $f(z, R, M)$.

It is immediate that $f(z, R, M)$ extends to be regular on $1\leqq|z|\leqq R$ with $|f(z, R, M)|=1$ on $|z|=1$, $|f(z, R, M)|=M$ on $|z|=R$.

3. THEOREM 1. *Given positive numbers $R>1$, $M>1$ let $f(z)$ be regular for $1\leqq|z|\leqq R$ and satisfy $|f(z)|\leqq 1$ for $|z|=1$, $|f(z)|\leqq M$ for $|z|=R$. Then for $1<|z|<R$*

$$|f(z)|\leqq f(|z|, R, M)$$

with equality for $z=re^{i\theta}$, $1<r<R$, θ real, only if

$$f(z)=e^{i\varphi}f(e^{-i\theta}z, R, M),$$

φ real.

We apply Lemma 1 to the function $f(e^{i\theta}z)/f(z, R, M)$ at the point $z=r$, the conditions for that result being clearly satisfied. Then

$$|f(re^{i\theta})|\leqq f(r, R, M)$$

and equality occurs only for $f(ze^{i\theta})=e^{i\varphi}f(z, R, M)$ that is

$$f(z)=e^{i\varphi}f(e^{-i\theta}z, R, M).$$

COROLLARY 1. *For $R^{m-1}<M<R^m$, m a positive integer $f(z, R, M)$ has in $1\leqq|z|\leqq R$ a unique simple zero at the point $-R^m/M$. Thus*

$$\log|f(z, R, M)|=\omega(z)\log M-g(z, -R^m/M)$$

where $\omega(z)$ denotes the harmonic measure at z of $|z|=R$ with respect to $1<|z|<R$ and $g(z, -R^m/M)$ denotes the Green's function of $1<|z|<R$ with pole at $-R^m/M$.

Explicitly, for $f(z)$ satisfying the conditions of Theorem 1,

$$\log|f(z)|\leqq(\log M)(\log|z|)/\log R-g(|z|, -R^m/M)$$

with equality (occurring for z positive) only if

$$\log|f(z)|=\omega(z)\log M-g(z, -R^m/M).$$

The final statement is Teichmüller's result.

Evidently $\omega(z)=\log|z|/\log R$. It is clear from the construction of $f(z, R, M)$ that it has a unique simple zero at a point p with $-R<p<-1$. We denote

$|z|=R$ by C_1, $|z|=1$ by C_2 and a small simple closed curve on which $|f(z, R, M)|$ is constant containing p by γ. Let D be the domain bounded by C_1, C_2, γ and let these curves be sensed positively on the boundary of D. Let $u(z)=\log|f(z, R, M)|$. We denote the differential conjugate to the differential η by $*\eta$. Then

$$0=\iint_D(d\omega*du-du*d\omega)=\iint_D d(\omega*du-u*d\omega)$$

and by Stokes' Theorem

$$\left(\int_{C_1}+\int_{C_2}+\int_\gamma\right)(\omega*du-u*d\omega)=0\,.$$

Now

$$\int_{C_1}\omega*du=2\pi m\,,\qquad \int_{C_2}\omega*du=0\,,\qquad \int_\gamma\omega*du=-2\pi\omega(p)\,,$$

$$\int_{C_1}u*d\omega=(\log M)(2\pi/\log R)\,,\qquad \int_{C_2}u*d\omega=0\,,\qquad \int_\gamma u*d\omega=0\,.$$

Thus

$$2\pi\omega(p)=2\pi m-2\pi\log M/\log R$$

or

$$\log|p|=m\log R-\log M$$

and $|p|=R^m/M$. The other statements follow at once.

4. It is worth observing that, while the construction of Lemma 2 requires some form of uniformization theorem, we can obtain a quite elementary proof of Teichmüller's result as follows. We need only construct $f(z, R, M)$ for $R>M$ and utilize Robinson's remark that for $R^m<M<R^{m+1}$, $f(z, R, M)=z^m f(z, R, MR^{-m})$. This is enough to obtain the properties of $f(z, R, M)$ utilized in the proof of Corollary 1.

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