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INTEGRABILITY CONDITIONS FOR POLYNOMIAL STRUCTURES

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0. Let f be a tensor field of type (1, 1) defined on a differentiable manifold and satisfying there a polynomial equation

$$f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n I = 0$$

with constant coefficients. Under the assumption that the polynomial $\xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi + a_n$ has only simple roots we give two necessary and sufficient conditions for the integrability of the tensor field f. The integrability conditions presented in the paper generalize those known for various structures on manifolds (e.g. almost complex, almost contact, almost product structure etc.). All differentiable structures involved are supposed to be of class C^∞.

1. We consider a connected differentiable manifold M. A tensor field f of type (1, 1) on M is called *polynomial structure on M* if it satisfies the equation

$$R(f) = f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n I = 0$$

where $a_1, \cdots, a_n$ are real numbers and I denotes the identity tensor of type (1, 1). By f^k we understand here the composition $\underbrace{f \circ \cdots \circ f}_{k \times}$. The polynomial $R(\xi) = \xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi + a_n$ we shall call *characteristic polynomial* of the structure. We suppose moreover that $R(\xi)$ is the minimal polynomial of the endomorphism $f_x : T_x(M) \to T_x(M)$ for any $x \in M$.

Decompose the polynomial $R(\xi)$ into the prime factors

$$R(\xi) = R_1'(\xi) \cdots R_r'(\xi) R_1''(\xi) \cdots R_s''(\xi)$$

where

$$R_i'(\xi) = (\xi - b_i)^{k_i}; \qquad k_i \geqq 1, \quad i = 1, \cdots, r,$$

$$R_j''(\xi) = (\xi^2 + 2c_j\xi + d_j)^{l_j}; \qquad l_j \geqq 1, \quad c_j^2 - d_j < 0, \quad j = 1, \cdots, s.$$

The polynomials $\xi - b_i$; $i = 1, \cdots, r$, as well as the polynomials $\xi^2 + 2c_j\xi + d_j$; $j = 1, \cdots, s$ are pairwise distinct.

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Let $T_x(M)$ denote the tangent space of M at x. We define $r+s$ subspaces $(D'_i)_x$ and $(D''_j)_x$ of $T_x(M)$ by

$$(D'_i)_x=\mathrm{Ker}\, R'_i(f_x)\,, \qquad (D''_j)_x=\mathrm{Ker}\, R''_j(f_x)\,.$$

Both $(D'_i)_x$ and $(D''_j)_x$ are obviously invariant under f_x, and the tangent space $T_x(M)$ can be decomposed

$$T_x(M)=(D'_1)_x\oplus \cdots \oplus(D'_r)_x\oplus(D''_1)_x\oplus \cdots \oplus(D''_s)_x\,,$$

$R'_i(\xi)$ and $R''_j(\xi)$ are the minimal polynomials of the restrictions of f_x to $(D'_i)_x$ and $(D''_j)_x$ respectively.

PROPOSITION 1. *There exist uniquely determined polynomials Q'_i, Q''_j such that for any $x\in M$ there is*

(i) $(P'_i)^2_x=(P'_i)_x\,,\quad (P''_j)^2_x=(P''_j)_x\,,$
$(P'_{i_1})_x(P'_{i_2})_x=(P'_{i_2})_x(P'_{i_1})=0$ for $i_1\neq i_2\,,$
$(P''_{j_1})_x(P''_{i_2})_x=(P''_{j_2})_x(P''_{j_1})_x=0$ for $j_1\neq j_2\,,$
$(P'_i)_x(P''_j)_x=(P''_j)_x(P'_i)_x=0\,,$
$\sum_{i=1}^{r}(P'_i)_x+\sum_{j=1}^{s}(P''_j)_x=I_x\,,$
$\mathrm{Im}\,(P'_i)_x=(D'_i)_x\,,\quad \mathrm{Im}\,(P''_j)_x=(D''_j)_x$

where $(P'_i)_x=Q'_i(f_x)$, $(P''_j)_x=Q''_j(f_x)$.

(ii) $\deg Q'_i<\deg R$, $\deg Q''_j<\deg R$, *where* deg *denotes degree of polynomial.*

Proof. Let us introduce the polynomials $V'_i=\dfrac{R}{R'_i}$, $V''_j=\dfrac{R}{R''_j}$, $i=1,\cdots,r$; $j=1,\cdots,s$. Their greatest common divisor is obviously equal to 1 and thus we can find polynomials W'_i, W''_j such that

$$\sum_{i=1}^{r}W'_iV'_i+\sum_{j=1}^{s}W''_jV''_j=1\,.$$

Writing $W'_i=S'_iR'_i+T'_i$, $W''_j=S''_jR''_j+T''_j$ with $\deg T'_i<\deg R'_i$ and $\deg T''_j<\deg R''_j$ we set

$$Q'_i=T'_iV'_i\,,\qquad Q''_j=T''_jV''_j\,.$$

The details of the proof are left to the reader.

Proposition 1 implies immeditely that P'_i, P''_j are tensor fields of class C^∞, and hence we get easily

COROLLARY. $\dim D'_i$ $(\dim D''_j)$ *is constant on M and thus $(D'_1,\cdots, D'_r, D''_1,\cdots, D''_s)$ is an almost product structure.*

We shall call it *almost product structure associated with the polynomial structure f*. Obviously P'_i, P''_j are the projectors corresponding to this almost product structure.

2. Let N be a differentiable manifold and g, h be two tensor fields of type (1, 1) on N such that $gh=hg$. We introduce a tensor field on N by

$$\{g, h\}(X, Y)=[gX, hY]+gh[X, Y]-g[X, hY]-h[gX, Y]$$

where X, Y are vector fields on N. One can easily check that this definition is good.

Now let an almost complex structure J be given on N. We recall that a vector field X on N is called infinitesimal automorphism of J if there is $L_X J=0$ where L_X denotes the Lie derivative along X. In other words X is an infinitesimal automorphism of J if and only if for any vector field Y on N there is

$$0=(L_X J)(Y)=L_X(JY)-JL_X Y=[X, JY]-J[X, Y].$$

PROPOSITION 2. *Let X be an infinitesimal automorphism of J, and let g be a tensor field of type* (1, 1) *such that $gJ=Jg$. Then gX is an infinitesimal automorphism of J if and only if there is $\{g, J\}(X, Y)=0$ for any vector field Y.*

Proof follows immeditely from the equality

$$\begin{aligned}(L_{gX}J)(Y)&=[gX, JY]-J[gX, Y]\\ &=\{g, J\}(X, Y)-gJ[X, Y]+g[X, JY]\\ &=\{g, J\}(X, Y).\end{aligned}$$

From now to the end of the parahraph we shall suppose that the almost complex structure J on N is integrable, i. e. that N is a complex manifold. In such a case there is an isomorphism from the Lie algebra of all infinitesimal automorphisms of J onto the Lie algebra of all holomorphic vector fields on N given by $X \to Z=\frac{1}{2}(X-iJX)$. Via this isomorphism we get from Proposition 2.

COROLLARY 2. *Let N be a manifold with a complex structure J on it, and let g be a tensor field of type* (1, 1) *on N such that $gJ=Jg$ and $\{g, J\}=0$. Now if Z is a holomorphic vector field on N, then the field gZ is also holomorphic.*

Consider now an almost product structure $(D_1, \cdots, D_t)$ on N with all the distributions D_i invariant under J, i. e. $J(D_i)=D_i$, $i=1, \cdots, t$. Such a structure we may call *complex almost product structure.* If we denote by P_i the corresponding projectors, we can easily see that there is $P_i J=JP_i$. Let $\dim D_i=2n_i$, $n=\sum_{i=1}^{t} n_i$. We shall say that a complex almost product structure $(D_1, \cdots, D_t)$ is integrable if to any point of N there exists its open neighbourhood U with a complex chart $(z_1, \cdots, z_n)$ on it such that for any $y \in U$, $1 \leqq i \leqq t$ there is

$$\begin{aligned}(D_i)_y=\{X\in T_y(N)\,;\ &(dz_j)_y(X)=0,\\ &1=\leqq j\leqq n_1+\cdots+n_{i-1},\ n_1+\cdots+n_i+1\leqq j\leqq n\}.\end{aligned}$$

To avoid possible confusion, remark here that X denotes real tangent vector at y.

PROPOSITION 3. *Let $(D_1, \cdots, D_t)$ be a complex almost product structure on a complex manifold N. Then $(D_1, \cdots, D_t)$ is integrable if and only if*

(i) *$[P_i, P_j]=0$, $1 \leqq i, j \leqq t$ where $[\ ,\ ]$ denotes Nijenhuis torsion,*

(ii) *$\{P_i, J\}=0$, $1 \leqq i \leqq t$.*

Proof. It is easy to see that the both conditions are necessary. Let us prove that they are also sufficient. Take any $x \in N$. By virtue of (ii) and Corollary 2 we can find an open neighbourhood U of x and lineraly independent holomorphic 1-forms $\omega_1, \cdots, \omega_n$ no it such that for any $y \in U$, $1 \leqq i \leqq t$ there is

$$(D_i)_y = \{X \in T_y(N)\,;\ (\omega_j)_y(X)=0\,;\\ 1 \leqq j \leqq n_1 + \cdots + n_{i-1},\ n_1 + \cdots + n_i + 1 \leqq j \leqq n\}\,.$$

For $n_{i-1}+1 \leqq j \leqq n_i$ (we take $n_0=0$) we can write

$$d\omega_j = \sum_{u=n_{i-1}+1}^{n_{i-1}+n_i} \phi_j^u \wedge \omega_u + \sum_{u<v} a_j^{uv} \omega_u \wedge \omega_v$$

where ϕ_j^u are holomorphic 1-forms and a_j^{uv} holomorphic functions on U. Recall that the condition (i) is a necessary and sufficient condition for the integrability of the almost product structure $(D_1, \cdots, D_t)$ considered as a real one (see [3]). Hence we obtain $a_j^{uv}=0$ for the all u, v in question. Now applying the complex version of the Frobenius theorem we find that there exist a smaller neighbourhood $U' \subseteq U$ of x and holomorphic functions $g_1, \cdots, g_n$ defined on U' such that for $n_{i-1}+1 \leqq j \leqq n_{i-1}+n_i$, $i=1, \cdots, t$ there is

$$\omega_j = \sum_{u=n_{i-1}+1}^{n_{i-1}+n_i} h_j^u \, dg_u$$

where h_j^u are holomorphic functions on U'. It is clear that $(g_1, \cdots, g_n)$ is a complex chart on U' having the properties required in the definition of integrability of a complex almost product structure $(D_1, \cdots, D_t)$.

3. Beginning with this section we shall deal with those polynomial structures only, the characteristic polynomial of which has only simple roots. Keeping the notations from paragraph 1 we have then

$$R_i'(\xi) = \xi - b_i \qquad ; \qquad i=1, \cdots, r\,,$$

$$R_j''(\xi) = \xi^2 + 2c_j\xi + d_j\,; \qquad j=1, \cdots, s\,.$$

We denote $n_i' = \dim D_i'$, $n_j'' = \dfrac{1}{2} \dim D_j''$, $\tilde{n} = \sum_{i=1}^{r} n_i'$, $\tilde{\tilde{n}} = \sum_{j=1}^{s} n_j''$, $n = \tilde{n} + 2\tilde{\tilde{n}} = \dim M$. The restriction f_j'' of f to D_j'' satisfies $f_j''^2 + 2c_j f_j'' + d_j I_j = 0$, where I_j denotes the identity automorphism of D_j'', and we can introduce an almost complex structure J_j'' on D_j'' by setting

$$J_j''=\frac{f_j''+c_jI_j}{\sqrt{d_j-c_j^2}}\,.$$

On the other hand from this formula we obtain $f_j''=\sqrt{d_j-c_j^2}\,J_j''-c_jI_j$. Therefore it is quite natural to take the following

DEFINITION 1. *A polynomial structure f on M is called integrable if to any point of M there exists its open neighborhood U with a chart $(x_1,\cdots,x_n)$ on it such that the matrix expression of f with respect to this chart is*

$$\begin{pmatrix} c_1I_{n_1'} & & & & & \\ & \ddots & & & 0 & \\ & & c_rI_{n_r'} & & & \\ & & & K_1 & & \\ & 0 & & & \ddots & \\ & & & & & K_s \end{pmatrix} \quad \textit{where} \quad K_j=\left(\begin{array}{c|c} -c_jI_{n_j''} & \sqrt{d_j-c_j^2}\,I_{n_j''} \\ \hline -\sqrt{d_j-c_j^2}\,I_{n_j''} & -c_jI_{n_j''} \end{array}\right)$$

I_k denotes here the unite (k,k)-matrix.

We are going now to give necessary and sufficient conditions for the integrability of a polynomial structure. Let us define a tensor field Φ on M by

$$\Phi=\sum_{j=1}^{s}\frac{f+c_jI}{\sqrt{d_j-c_j^2}}P_j''\,.$$

One finds easily that Φ satisfies the equation $\Phi^3+\Phi=0$. We may call Φ *almost contact structure associated with the polynomial structure f*. More details about such structure can be found for example in [2]. Φ is a polynomial structure on M with characteristic polynomial $R(\xi)=\xi(\xi^2+1)$, with $r=1$, $s=1$, $k_1=1$, $l_1=1$ (that is of type we have restricted ourselves to), $P_1'=\Phi^2+I$, $P_1''=-\Phi^2$. According to Definition 1, Φ is integrable if to any point of M there exists its open neighbourhood U with a chart $(x_1,\cdots,x_n)$ on it such that the matrix expression of Φ with respect to this chart is

$$\left(\begin{array}{c|cc} O_{n_1'} & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & I_{n_1''} \\ & -I_{n_1''} & 0 \end{array}\right)\,.$$

where $O_{n_1'}$ denotes the zero (n_1',n_1')-matrix, $n_1'=\dim\operatorname{Ker}\Phi$, $n_1''=\dim\operatorname{Ker}(\Phi^2+I)$. Φ is integrable if and only if $[\Phi,\Phi]=0$, where $[\ ,\ ]$ denotes again the Nijenhuis torsion (see [2]).

THEOREM 1. *A polynomial structure f the characteristic phlynomial of which has only simple roots is integrable if and only if the following conditions are satisfied*

(i) $[P'_{i_1}, P'_{i_2}]=0,\ 1\leqq i_1, i_2\leqq r$; $[P''_{j_1}, P''_{j_2}]=0,\ 1\leqq j_1, j_2\leqq s$.
$[P'_i, P''_j]=0,\ 1\leqq i\leqq r$; $1\leqq j\leqq s$.

(ii) $[\Phi, \Phi]=0$.

(iii) $\{P''_j, \Phi\}=0,\ 1\leqq j\leqq s$.

Proof. As usual it can be easily seen that the above conditions are necessary. Thus we must prove they are also sufficient.

Let $x\in M$ be arbitrary point. By (i) the almost product structure $(D'_1, \cdots, D'_r, D''_1\oplus\cdots\oplus D''_s)$ is integrable, i. e. we can find an open neighbourhood U of x with a chart $(x_1, \cdots, x_{\tilde{n}}, y_1, \cdots, y_{2\tilde{\tilde{n}}})$ on it such that for any $y\in U$, $1\leqq i\leqq r$ there is

$$(D'_i)_y=\{X\in T_y(M)\,;\ (dx_k)_y(X)=0,\ (dy_l)_y(X)=0,$$
$$1\leqq k\leqq n_1+\cdots+n_{i-1},\ n_1+\cdots+n_i+1\leqq k\leqq\tilde{n},\ 1\leqq l\leqq 2\tilde{\tilde{n}}\}$$

and moreover

$$(D''_1\oplus\cdots\oplus D''_s)_y=\{X\in T_y(M)\,;\ (dx_k)_y(X)=0,\ 1\leqq k\leqq\tilde{n}\}\ .$$

The matrix expressions of P'_i, P''_j, Φ with respect to this chart are

$$P'_i=\left(\begin{array}{ccc:c} 0 & 0 & 0 & \\ 0 & I'_{n_i} & 0 & 0 \\ 0 & 0 & 0 & \\ \hdashline & 0 & & 0 \end{array}\right) \qquad P''_j=\left(\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \tilde{P}''_j \end{array}\right)$$

$$\Phi=\left(\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \tilde{\Phi} \end{array}\right)$$

where P''_j, Φ are $(2\tilde{n}, 2\tilde{n})$-matrices, the entries of which are functions of the variables $x_1, \cdots, x_{\tilde{n}}, y_1, \cdots, y_{2\tilde{n}}$. In the next we shall show that in fact they are functions of $y_1, \cdots, y_{2\tilde{n}}$ only.

Let $1\leqq k\leqq\tilde{n}$, $n_1+\cdots+n_{i-1}+1\leqq k\leqq n_1+\cdots+n_i$ be arbitrary. Then we have

$$0=[P'_i, P''_j]\Big(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}\Big)=\Big[\frac{\partial}{\partial x_k}, \sum_{m=1}^{\tilde{n}}(p''_j)_l^m\frac{\partial}{\partial y_m}\Big]$$
$$-P'_i\Big[\frac{\partial}{\partial x_k}, \sum_{m=1}^{\tilde{n}}(p''_j)_l^m\frac{\partial}{\partial y_m}\Big]-P''_j\Big[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}\Big]$$
$$=\sum_{m=1}^{\tilde{n}}\frac{\partial(p''_j)_l^m}{\partial x_k}\frac{\partial}{\partial y_m}$$

which implies $\frac{\partial(p_j'')_l^m}{\partial x_k}=0$, where $(p_j'')_l^m$ are entries of the matrix $\tilde{P}_j''$. Further using (ii) we get along the same lines

$$0=[\Phi, \Phi]\Big(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}\Big)=-\Phi\Big[\frac{\partial}{\partial x_k}, \sum_{m=1}^{\tilde{n}} \Phi_l^m \frac{\partial}{\partial y_m}\Big]$$
$$=-\sum_{m=1}^{\tilde{n}} \frac{\partial \Phi_l^m}{\partial x_k} \Phi\Big(\frac{\partial}{\partial y_m}\Big)$$

which imples again $\frac{\partial \Phi_l^m}{\partial x_k}=0$, where Φ_l^m are entries of the matrix $\tilde{\Phi}$.

Finally by (i), (ii), (iii), and by virtue of Proposition 3 it is not difficult to see that we can perform a coordinate change

$$\begin{aligned} x_{\tilde{n}+1} &= \varphi_{\tilde{n}+1}(y_1, \cdots, y_{2\tilde{n}}) \\ &\vdots \\ x_n &= \varphi_n \ (y_1, \cdots, y_{2\tilde{n}}) \end{aligned}$$

in such a way that the chart $(x_1, \cdots, x_n)$ has the properties required in Definition 1.

4. We start this paragraph with the following

LEMMA 1. *Let Φ be a polynomial structure with characteristic polynomial $R(\xi)=\xi(\xi^2+1)$. Then Φ is integrable if and only if there exists a symmetric liner connection ∇ such that $\nabla\Phi=0$.*

Proof. Suppose first that there exists a aymmetric connection ∇ with $\nabla\Phi=0$. We have

$$\frac{1}{2}[\Phi, \Phi](X, Y)=(\nabla_{\Phi X}\Phi)(Y)-(\nabla_{\Phi Y}\Phi)(X)-\Phi(\nabla_X\Phi)(Y)+\Phi(\nabla_Y\Phi)(X)=0$$

and thus by virtue of [2] (or in a little more complicated way also by virtue of our Theorem 1) the structure is integrable.

Conversely let Φ be integrable. Take any symmetric connection $\hat{\nabla}$ and define a new connection ∇ by

$$\nabla_X Y=\hat{\nabla}_X Y+Q(X, Y)$$

where

$$\begin{aligned} 4Q(X, Y)=&6\Phi^2(\hat{\nabla}_X\Phi^2)(Y)-2\Phi(\hat{\nabla}_X\Phi)(Y)+4(\hat{\nabla}_X\Phi^2)(Y) \\ &+4\Phi^2(\hat{\nabla}_Y\Phi^2((X)+4(\hat{\nabla}_Y\Phi^2)(X)+\Phi^2(\hat{\nabla}_{\Phi Y}\Phi)(X) \\ &+\Phi(\hat{\nabla}_{\Phi Y}\Phi^2)(X)+3\Phi^2(\hat{\nabla}_{\Phi^2 Y}\Phi^2)(X)+\Phi(\hat{\nabla}_{\Phi^2 Y}\Phi)(X) \\ &+4(\hat{\nabla}_{\Phi^2 Y}\Phi^2)(X)\,. \end{aligned}$$

(This rather complicated formula can be found using the results of [1], §§ 3, 4).

Calculation shows that $\nabla\Phi=0$ and that the torsion tensor $T(X, Y)$ of ∇ is equal to

$$8T(X, Y)=-3[\Phi^2, \Phi^2](\Phi^2X, \Phi^2Y)-[\Phi^2, \Phi^2](X, Y)$$
$$-2\Phi[\Phi, \Phi^2](X, Y)-\Phi^2[\Phi, \Phi](X, Y).$$

But because of the integrability of Φ, $T(X, Y)$ obviously vanishes, showing thus that ∇ is symmetric.

The main result of this paragraph is

THEOREM 2. *A polynomial structure f the characteristic polynomial of which has only simple roots is integrable if and only if there exists a symmetric linear connection ∇ such that $\nabla f=0$.*

Proof. Let ∇ be a symmetric connection such that $\nabla f=0$. The projectors P'_i, P''_j are polynomials in f, and thus we have $\nabla P'_i=0$, $\nabla P''_j=0$. Similarly from the definition of the almost contact structure Φ associated with f we get $\nabla\Phi=0$. Now it is an easy calculation to find that the conditions of Theorem 1 are satisfied. Theorem 1 implies then the integrability of f.

If f is integrable, then the almost product structure associated with f is also integrable. Therefore we can find a symmetric connection $\hat{\nabla}$ such that $\hat{\nabla}P'_i=0$, $\hat{\nabla}P''_j=0$. Construct a connection ∇ in the same way as in the proof of Lemma 1, taking for Φ the almost contact structure associated with f. There is $\nabla\Phi=0$, the integrability of f implies that of Φ, and thus we get from the proof of Lemma 1 that ∇ is symmetric.

Now we are going to prove that $\nabla P=0$ where P denotes P'_i or P''_j. Because of $\hat{\nabla}P=0$ it is sufficient to prove $Q(X, PY)-PQ(X, Y)=0$. We get

$$4Q(X, PY)-4PQ(X, Y)=3\{P, \Phi^2\}(\Phi^2X, \Phi^2Y)+4\{P, \Phi^2\}(\Phi^2X, Y)$$
$$-18\{P, \Phi^2\}(X, \Phi^2Y)-18\{P, \Phi^2\}(X, \Phi Y)$$
$$-18\Phi^2\{P, \Phi\}(X, \Phi Y)-18\Phi\{P, \Phi\}(X, \Phi^2Y)=0$$

as a consequence of the simultaneous integrability of the associated almost product and almost contact structures.

We have thus found a symmetric connection ∇ such that $\nabla\Phi=0$, $\nabla P'_i=0$, $\nabla P''_j=0$. From the obvious formula

$$f=\sum_{i=1}^{r}b_iP'_i+\sum_{j=1}^{s}\sqrt{d_j-c_j^2}\,\Phi P''_j$$

we then get $\nabla f=0$. This finishes the proof.

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