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ON CONFORMAL CHANGES OF RIEMANNIAN METRICS

Dedicated to Professor Yûsaku Komatu on his sixtieth birthday

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§ 1. Introduction.

Let M be an n-dimensional connected differentiable manifold and g a Riemannian metric tensor field on M. We denote by (M, g) a Riemannian manifold with the metric tensor field g. Let there be given two metric tensor fields g and g^* on M. If g^* is conformal to g, that is, if there exists a function ρ on M such that $g^*=e^{2\rho}g$, then we call such a change of metric tensor field $g\rightarrow g^*$ a conformal change of metric. In particular, if ρ=constant then the conformal change of metric is said to be homothetic and if $\rho=0$ then the conformal change of metric is said to be isometric.

Let (M, g) and (M', g') be two Riemannian manifolds and $f: M\rightarrow M'$ a diffeomorphism. Then $g^*=f^*g'$ is a Riemannian metric tensor field on M. When g^* is conformal to g, that is, when there exists a function ρ on M such that $g^*=e^{2\rho}g$, we call $f:(M, g)\rightarrow(M', g')$ a conformal transformation. In particular, if ρ=constant then f is called a homothetic transformation or a homothety and if $\rho=0$ then f is called an isometric transformation or an isometry.

If a vector field v on M satisfies

$$L_v g=2\phi g, \tag{1.1}$$

where L_v denotes the Lie derivation with respect to v and ϕ a function on M, then v is called an infinitesimal conformal transformation. The v is said to be homothetic or isometric according as ϕ is a constant or zero.

Given a Riemannian manifold (M, g), we denote by g_{ji}, $\left\{ {h \atop j\ i} \right\}$, ∇_i, $K_{kji}{}^h$, K_{ji} and K, respectively, the components of the metric tensor field g, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\left\{ {h \atop j\ i} \right\}$, the components of the curvature tensor field, the components of the Ricci tensor field and the scalar curvature of (M, g), where and in the sequel, indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. Hereafter we assume that functions under consideration are always differentiable.

When we consider a conformal change of metric

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$$g^*=e^{2\rho}g,$$

if Ω is a quantity formed with g then we denote by Ω^* the quantity formed with g^* by the same rule as that Ω is formed with g.

Recently, Goldberg and Yano [2] studied non-homothetic conformal changes of metrics and obtained the following

THEOREM A. *Let (M, g) be a compact orientable Riemannian manifold of dimension $n>3$ with constant scalar curvature K and admitting a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that $K^*=K$. Then if*

$$\int_M u^{-n+1}G_{ji}u^j u^i dV \geqq 0, \tag{1.2}$$

where

$$G_{ji}=K_{ji}-\frac{K}{n}g_{ji} \tag{1.3}$$

and $u=e^{-\rho}$, $u_i=\nabla_i u$, $u^h=u_i g^{ih}$ and dV denotes the volume element of (M, g), then (M, g) is isometric to a sphere.

Yano and Obata [13] proved following theorems.

THEOREM B. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (\Delta u)KdV=0, \qquad G^*_{ji}G^{*ji}=u^4 G_{ji}G^{ji},$$

where $\Delta u=g^{ji}\nabla_j\nabla_i u$, then (M, g) is conformal to a sphere.

THEOREM C. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ and with K=constant admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$G^*_{ji}G^{*ji}=u^4 G_{ji}G^{ji},$$

then (M, g) is isometric to a sphere.

THEOREM D. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$\int_M (\Delta u)KdV=0, \qquad Z^*_{kjih}Z^{*kjih}=u^4 Z_{kjih}Z^{kjih},$$

where

$$Z_{kji}{}^h=K_{kji}{}^h-\frac{K}{n(n-1)}(\delta_k^h g_{ji}-\delta_j^h g_{ki}), \tag{1.4}$$

then (M, g) is conformal to a sphere.

THEOREM E. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ and with K=constant admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$Z^*_{kjih}Z^{*kjih}=u^4 Z_{kjih}Z^{kjih},$$

then (M, g) *is isometric to a sphere.*

THEOREM F. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$\int_M (\Delta u)K dV=0,$$

$$W^*_{kjih}W^{*kjih}=u^4 W_{kjih}W^{kjih}, \qquad a+(n-2)b\neq 0,$$

where

$$W_{kji}{}^h=aZ_{kji}{}^h+b(\delta_k^h G_{ji}-\delta_j^h G_{ki}+G_k{}^h g_{ji}-G_j{}^h g_{ki}), \tag{1.5}$$

a and b being constants, then (M, g) *is conformal to a sphere.*

THEOREM G. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *and with* $K=$*constant admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$W^*_{kjih}W^{*kjih}=u^4 W_{kjih}W^{kjih}, \qquad a+(n-2)b\neq 0,$$

then (M, g) *is isometric to a sphere.*

THEOREM H. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n\geqq 2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$K^*=K, \qquad L_{du}K=0, \qquad \int_M u^{-n+1}G_{ji}u^j u^i dV\geqq 0,$$

where L_{du} *denotes the Lie derivation with respect to a vector field* $u^h=g^{ih}\nabla_i u$, *then* (M, g) *is isometric to a sphere.*

THEOREM I. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$K^*=K, \qquad L_{du}K=0, \qquad G^*_{ji}G^{*ji}=G_{ji}G^{ji},$$

then (M, g) *is isometric to a sphere.*

(See also Barbance [1].)

THEOREM J. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$K^*=K, \qquad L_{du}K=0, \qquad Z^*_{kjih}Z^{*kjih}=Z_{kjih}Z^{kjih},$$

then (M, g) *is isometric to a sphere.*

(See also Hsiung and Liu [3].)

THEOREM K. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$K^*=K, \qquad L_{du}K=0,$$

$$W^*_{kjih}W^{*kjih}=W_{kjih}W^{kjih}, \qquad a+(n-2)b\neq 0,$$

then (M, g) *is isometric to a sphere.*

(See also Hsiung and Liu [3].)

Yano and Sawaki [14] proved following theorems.

THEOREM L. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$L_{du}K=0, \qquad L_{du}K^*=0,$$

$$u^p G_{ji}G^{ji}=\{(u-1)\varphi+1\}G^*_{ji}G^{*ji},$$

where p is a real number such that $p\leqq 4$ and φ a non-negative function on M, then (M, g) is isometric to a sphere.

THEOREM M. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$L_{du}K=0, \qquad L_{du}K^*=0,$$

$$u^p Z_{kjih}Z^{kjih}=\{(u-1)\varphi+1\}Z^*_{kjih}Z^{*kjih},$$

where p and φ are the same as in Theorem L, then (M, g) is isometric to a sphere.

THEOREM N. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$L_{du}K=0, \qquad L_{du}K^*=0,$$

$$u^p W_{kjih}W^{kjih}=\{(u-1)\varphi+1\}W^*_{kjih}W^{*kjih}, \qquad a+(n-2)b\neq 0,$$

where p and φ are the same as in Theorem L, then (M, g) is isometric to a sphere.

The purpose of the present paper is to prove generalizations of Theorems A$\sim$N.

In the sequel, we need the following two theorems.

THEOREM O (Tashiro [8]). *If a compact Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M such that*

$$\nabla_j\nabla_i u-\frac{1}{n}\Delta u g_{ji}=0,$$

then (M, g) is conformal to a sphere in an $(n+1)$-dimensional Euclidean space.

(See also Ishihara [4], Ishihara and Tashiro [5].)

THEOREM P (Yano and Obata [13]). *If a complete Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M such that*

$$\nabla_j\nabla_i u-\frac{1}{n}\Delta u g_{ji}=0, \qquad L_{du}K=0,$$

then (M, g) is isometric to a sphere in an $(n+1)$-dimensional Euclidean space.

§ 2. Preliminaries.

We consider a conformal change of metric

$$g^*_{ji}=e^{2\rho}g_{ji}. \tag{2.1}$$

First of all, we have

$$\left\{ {h \atop j\ i} \right\}^* = \left\{ {h \atop j\ i} \right\} + \delta_j^h \rho_i + \delta_i^h \rho_j - g_{ji}\rho^h, \tag{2.2}$$

where

$$\rho_i = \nabla_i \rho, \qquad \rho^h = \rho_i g^{ih},$$

from which

$$K^*_{kji}{}^h = K_{kji}{}^h - \delta_k^h \rho_{ji} + \delta_j^h \rho_{ki} - \rho_k{}^h g_{ji} + \rho_j{}^h g_{ki}, \tag{2.3}$$

where

$$\rho_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \frac{1}{2}\rho_t \rho^t g_{ji}, \qquad \rho_j{}^h = \rho_{ji} g^{ih},$$

and consequently

$$K^*_{ji} = K_{ji} - (n-2)\rho_{ji} - \rho_t{}^t g_{ji} \tag{2.4}$$

and

$$e^{2\rho}K^* = K - 2(n-1)\rho_t{}^t, \tag{2.5}$$

where

$$\rho_t{}^t = \Delta\rho + \frac{n-2}{2}\rho_t\rho^t, \qquad \Delta\rho = g^{ji}\nabla_j\rho_i.$$

From (2.3), (2.4) and (2.5) and the definitions of G_{ji}, $Z_{kji}{}^h$ and $W_{kji}{}^h$, we have

$$G^*_{ji} = G_{ji} - (n-2)(\nabla_j\rho_i - \rho_j\rho_i) + \frac{n-2}{n}(\Delta\rho - \rho_t\rho^t)g_{ji}, \tag{2.6}$$

$$\begin{aligned} Z^*_{kji}{}^h = Z_{kji}{}^h &- \delta_k^h(\nabla_j\rho_i - \rho_j\rho_i) + \delta_j^h(\nabla_k\rho_i - \rho_k\rho_i) \\ &- (\nabla_k\rho^h - \rho_k\rho^h)g_{ji} + (\nabla_j\rho^h - \rho_j\rho^h)g_{ki} \\ &+ \frac{2}{n}(\Delta\rho - \rho_t\rho^t)(\delta_k^h g_{ji} - \delta_j^h g_{ki}) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} W^*_{kji}{}^h = W_{kji}{}^h + \{a+(n-2)b\}\Big\{ &-\delta_k^h(\nabla_j\rho_i - \rho_j\rho_i) + \delta_j^h(\nabla_k\rho_i - \rho_k\rho_i) \\ &- (\nabla_k\rho^h - \rho_k\rho^h)g_{ji} + (\nabla_j\rho^h - \rho_j\rho^h)g_{ki} \\ &+ \frac{2}{n}(\Delta\rho - \rho_t\rho^t)(\delta_k^h g_{ji} - \delta_j^h g_{ki})\Big\}. \end{aligned} \tag{2.8}$$

If we put

(2.9) $$u=e^{-\rho}, \qquad u_i=\nabla_i u,$$

then we have

(2.10) $$\nabla_j u_i=-u(\nabla_j\rho_i-\rho_j\rho_i)$$

and

(2.11) $$\Delta u=-u(\Delta\rho-\rho_t\rho^t),$$

and consequently

(2.12) $$K^*=u^2K+2(n-1)u\Delta u-n(n-1)u_tu^t,$$

(2.13) $$G^*_{ji}=G_{ji}+(n-2)P_{ji},$$

(2.14) $$Z^*_{kji}{}^h=Z_{kji}{}^h+Q_{kji}{}^h$$

and

(2.15) $$W^*_{kji}{}^h=W_{kji}{}^h+\{a+(n-2)b\}Q_{kji}{}^h,$$

where

(2.16) $$P_{ji}=u^{-1}\left(\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}\right),$$

(2.17) $$Q_{kji}{}^h=\delta_k^h P_{ji}-\delta_j^h P_{ki}+P_k{}^h g_{ji}-P_j{}^h g_{ki}$$

and

$$P_j{}^h=P_{ji}g^{ih}.$$

From (2.16) and (2.17), we obtain

(2.18) $$P_{ji}P^{ji}=u^{-2}\left\{(\nabla^j u^i)(\nabla_j u_i)-\frac{1}{n}(\Delta u)^2\right\}$$

and

(2.19) $$Q_{kjih}Q^{kjih}=4(n-2)P_{ji}P^{ji}$$

respectively.

We also have, from (2.13), (2.14) and (2.15),

(2.20) $$G^*_{ji}G^{*ji}=u^4\{G_{ji}G^{ji}+2(n-2)G_{ji}P^{ji}+(n-2)^2P_{ji}P^{ji}\},$$

(2.21) $$Z^*_{kjih}Z^{*kjih}=u^4\{Z_{kjih}Z^{kjih}+8G_{ji}P^{ji}+4(n-2)P_{ji}P^{ji}\}$$

and

(2.22) $$W^*_{kjih}W^{*kjih}=u^4[W_{kjih}W^{kjih}+8\{a+(n-2)b\}^2G_{ji}P^{ji} +4(n-2)\{a+(n-2)b\}^2P_{ji}P^{ji}]$$

respectively. For the expression $G_{ji}P^{ji}$ in (2.20), (2.21) and (2.22), we have, from (2.16),

(2.23)
$$G_{ji}P^{ji}=u^{-1}G_{ji}\nabla^{j}u^{i},$$
where $\nabla^{j}=g^{ji}\nabla_{i}$.

§ 3. Lemmas.

LEMMA 1 (Lichnerowicz [6], Satō [7], Yano [9, 11]). *For a vector field v^h on a compact orientable Riemannian manifold (M, g), we have*

(3.1)
$$\int_M\left(g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_t v^t\right)v_h dV$$
$$+\frac{1}{2}\int_M\left(\nabla^j v^i+\nabla^i v^j-\frac{2}{n}\nabla_t v^t g^{ji}\right)$$
$$\times\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_s v^s g_{ji}\right)dV=0.$$

Proof. By a straightforward computation, we have
$$\nabla_i\left\{\left(\nabla^i v^h+\nabla^h v^i-\frac{2}{n}\nabla_t v^t g^{ih}\right)v_h\right\}$$
$$=\left(g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_i v^i\right)v_h$$
$$+\frac{1}{2}\left(\nabla^j v^i+\nabla^i v^j-\frac{2}{n}\nabla_t v^t g^{ji}\right)$$
$$\times\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_s v^s g_{ji}\right),$$
and consequently, integrating over M, we have (3.1).

REMARK. If a vector field v^h defines an infinitesimal conformal transformation, then we have
$$L_v g_{ji}=2\rho g_{ji},$$
that is,
$$\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_t v^t g_{ji}=0.$$
From this, we can deduce

(3.2)
$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_t v^t=0.$$

Formula (3.1) shows that this is not only necessary but also sufficient in order that the vector field v^h defines an infinitesimal conformal transformation in a compact orientable Riemannian manifold.

LEMMA 2 (Yano [10]). *For a function u on a compact orientable Riemannian manifold (M, g), we have*

(3.3)
$$\int_M \left(g^{ji}\nabla_j\nabla_i u^h + K_i{}^h u^i + \frac{n-2}{n}\nabla^h \Delta u\right) u_h dV$$
$$+2\int_M \left(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\right)\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\right) dV = 0$$

and

(3.4)
$$\int_M \left\{(g^{ji}\nabla_j\nabla_i u^h + K_i{}^h u^i) u_h - \frac{n-2}{n}(\Delta u)^2\right\} dV$$
$$+2\int_M \left(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\right)\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\right) dV = 0,$$

where $u_i = \nabla_i u$, $u^h = u_i g^{ih}$ *and* $\Delta u = g^{ji}\nabla_j\nabla_i u$.

Proof. Putting $v^h = u^h$ in (3.1) and using $\nabla^j u^i = \nabla^i u^j$, we obtain (3.3). (3.4) follows from (3.3) because of

$$\int_M (\nabla^h \Delta u) u_h dV = -\int_M (\Delta u)^2 dV.$$

LEMMA 3 (Yano [10]). *For a function u on a Riemannian manifold (M, g), we have*

(3.5)
$$\nabla^h \Delta u = g^{ji}\nabla_j\nabla_i u^h - K_i{}^h u^i,$$

that is,

(3.6)
$$g^{ji}\nabla_j\nabla_i u^h = \nabla^h \Delta u + K_i{}^h u^i.$$

Proof. We have

$$\begin{aligned}\nabla_h(\Delta u) &= \nabla_h(g^{ji}\nabla_j u_i) = g^{ji}\nabla_h\nabla_j u_i \\ &= g^{ji}(\nabla_j\nabla_h u_i - K_{hji}{}^t u_t) \\ &= g^{ji}\nabla_j\nabla_i u_h - K_h{}^t u_t,\end{aligned}$$

from which (3.5) follows.

LEMMA 4. *For a function u on a compact orientable Riemannian manifold (M, g), we have*

(3.7)
$$\int_M \left(K_{ji}u^j u^i + \frac{n-1}{n} u^h \nabla_h \Delta u\right) dV$$
$$+\int_M \left(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\right)\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\right) dV = 0$$

and

(3.8)
$$\int_M \left\{K_{ji}u^j u^i - \frac{n-1}{n}(\Delta u)^2\right\} dV$$
$$+\int_M \left(\nabla^j u^i - \frac{1}{n}\Delta u g^{ji}\right)\left(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\right) dV = 0.$$

Proof. Substituting (3.6) into (3.3), we have (3.7), and substituting (3.6) into (3.4), we have (3.8).

LEMMA 5. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a conformal change of metric $g^* = e^{2\rho} g$, then, for any real number p, we have*

$$\begin{aligned}(3.9)\quad &\int_M u^{p-1} G_{ji} u^j u^i dV \\ &+(p+n-2)\int_M u^{p-2}(\nabla_j u_i) u^j u^i dV + \frac{1}{2n}\int_M (u^{p-2} L_{du} K^* - u^p L_{du} K) dV \\ &- \frac{p+n-2}{2}\int_M u^{p-3}(u_t u^t)^2 dV - \frac{p+n-2}{2n(n-1)}\int_M u^{p-3} u_t u^t K^* dV \\ &+ \frac{p+n-2}{2n(n-1)}\int_M u^{p-1} u_t u^t K dV + \int_M u^{p+1} P_{ji} P^{ji} dV = 0,\end{aligned}$$

$$\begin{aligned}(3.10)\quad &\int_M u^{p-1} K_{ji} u^j u^i dV \\ &- \frac{p+n-2}{n}\int_M u^{p-1}(\Delta u)^2 dV - \frac{(p-1)(p+n-2)}{n}\int_M u^{p-2} u_t u^t \Delta u\, dV \\ &+ \frac{p-1}{n(n-1)}\int_M u^{p-1} u_t u^t K dV - \frac{p-1}{2n(n-1)}\int_M (u^{p-2} L_{du} K^* - u^p L_{du} K) dV \\ &+ \int_M u^{p+1} P_{ji} P^{ji} dV = 0\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad &\int_M u^{p-1} K_{ji} u^j u^i dV \\ &+ \frac{p+n-2}{n}\int_M u^{p-1} u^i \nabla_i(\Delta u) dV + \frac{p-1}{n(n-1)}\int_M u^{p-1} u_t u^t K dV \\ &- \frac{p-1}{2n(n-1)}\int_M (u^{p-2} L_{du} K^* - u^p L_{du} K) dV + \int_M u^{p+1} P_{ji} P^{ji} dV = 0.\end{aligned}$$

In particular, if $p = -n+2$ then

$$\begin{aligned}(3.12)\quad &\int_M u^{-n+1} G_{ji} u^j u^i dV \\ &+ \frac{1}{2n}\int_M (u^{-n} L_{du} K^* - u^{-n+2} L_{du} K) dV + \int_M u^{-n+3} P_{ji} P^{ji} dV = 0,\end{aligned}$$

and if $p=1$ then

$$(3.13)\quad \int_M K_{ji} u^j u^i dV - \frac{n-1}{n}\int_M (\Delta u)^2 dV + \int_M u^2 P_{ji} P^{ji} dV = 0$$

and

$$(3.14)\quad \int_M K_{ji} u^j u^i dV + \frac{n-1}{n}\int_M u^i \nabla_i(\Delta u) dV + \int_M u^2 P_{ji} P^{ji} dV = 0.$$

Proof. We first have

$$\begin{aligned}\nabla_j(u^{p-1}u_i\nabla^j u^i)&=(p-1)u^{p-2}(\nabla^j u^i)u_j u_i+u^{p-1}(\nabla_j u_i)(\nabla^j u^i)+u^{p-1}u_i\nabla_j\nabla^j u^i\\&=(p-1)u^{p-2}(\nabla^j u^i)u_j u_i+u^{p-1}(\nabla_j u_i)(\nabla^j u^i)\\&\quad+u^{p-1}K_{ji}u^j u^i+u^{p-1}u_i\nabla^i(\Delta u)\,,\end{aligned}$$

where we have used (3.6), that is,

$$\nabla_j\nabla^j u^i=K_t{}^i u^t+\nabla^i\Delta u\,,$$

and consequently, integrating over M, we have

$$\begin{aligned}\int_M u^{p-1}(\nabla_j u_i)(\nabla^j u^i)dV+(p-1)\int_M u^{p-2}(\nabla^j u^i)u_j u_i dV&\\+\int_M u^{p-1}K_{ji}u^j u^i dV&\\+\int_M u^{p-1}u_i\nabla^i(\Delta u)dV=0\,.&\end{aligned}\tag{3.15}$$

Similarly, computing $\nabla_i(u^{p-1}u^i\Delta u)$ and integrating over M, we have

$$\begin{aligned}(p-1)\int_M u^{p-2}u_t u^t\Delta u\,dV+\int_M u^{p-1}(\Delta u)^2 dV&\\+\int_M u^{p-1}u^i\nabla_i(\Delta u)dV=0\,.&\end{aligned}\tag{3.16}$$

By using (2.18), (3.15) and (3.16), we get

$$\begin{aligned}\int_M u^{p+1}P_{ji}P^{ji}dV&=\int_M u^{p-1}(\nabla_j u_i)(\nabla^j u^i)dV-\frac{1}{n}\int_M u^{p-1}(\Delta u)^2 dV\\&=-(p-1)\int_M u^{p-2}(\nabla^j u^i)u_j u_i dV-\int_M u^{p-1}K_{ji}u^j u^i dV\\&\quad+\frac{p-1}{n}\int_M u^{p-2}u_t u^t\Delta u\,dV-\frac{n-1}{n}\int_M u^{p-1}u^i\nabla_i(\Delta u)dV\,.\end{aligned}\tag{3.17}$$

On the other hand, from (2.12), we have

$$\Delta u=\frac{1}{2(n-1)}(u^{-1}K^*-uK)+\frac{n}{2}u^{-1}u_t u^t\,,\tag{3.18}$$

from which

$$\begin{aligned}\nabla_i(\Delta u)&=-\frac{1}{2(n-1)}(u^{-2}u_iK^*+u_iK)+\frac{1}{2(n-1)}(u^{-1}\nabla_iK^*-u\nabla_iK)\\&\quad-\frac{n}{2}u^{-2}u_iu_tu^t+nu^{-1}(\nabla_iu_t)u^t\,.\end{aligned}\tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17) and using

$$K_{ji}=G_{ji}+\frac{K}{n}g_{ji}\,,$$

we have (3.9).

Substituting

$$u^{p-3}u_t u^t K^* = 2(n-1)u^{p-2}u_t u^t \Delta u - n(n-1)u^{p-3}(u_t u^t)^2 + u^{p-1}u_t u^t K$$

which can be obtained from (3.18) into

$$\begin{aligned}\int_M u^{p-1}u^i \nabla_i(\Delta u)dV &= n\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV \\ &-\frac{1}{2(n-1)}\int_M u^{p-3}u_t u^t K^* dV - \frac{1}{2(n-1)}\int_M u^{p-1}u_t u^t K dV \\ &+\frac{1}{2(n-1)}\int_M (u^{p-2}L_{du}K^* - u^p L_{du}K)dV - \frac{n}{2}\int_M u^{p-3}(u_t u^t)^2 dV\end{aligned}$$

which follows from (3.19), we have

$$\begin{aligned}&\int_M u^{p-1}u^i \nabla_i(\Delta u)dV \\ &= n\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV - \int_M u^{p-2}u_t u^t \Delta u\, dV \\ &-\frac{1}{n-1}\int_M u^{p-1}u_t u^t K dV + \frac{1}{2(n-1)}\int_M (u^{p-2}L_{du}K^* - u^p L_{du}K)dV,\end{aligned}$$

and consequently, by using

$$\int_M u^{p-1}u^i \nabla_i(\Delta u)dV = -(p-1)\int_M u^{p-2}u_t u^t \Delta u\, dV - \int_M u^{p-1}(\Delta u)^2 dV \tag{3.20}$$

which is equivalent to (3.16), we obtain

$$\begin{aligned}\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV &= -\frac{p-2}{n}\int_M u^{p-2}u_t u^t \Delta u\, dV \\ &-\frac{1}{n}\int_M u^{p-1}(\Delta u)^2 dV + \frac{1}{n(n-1)}\int_M u^{p-1}u_t u^t K dV \\ &-\frac{1}{2n(n-1)}\int_M (u^{p-2}L_{du}K^* - u^p L_{du}K)dV.\end{aligned} \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.17), we get (3.10). From (3.16) and (3.20), we have (3.11) immediately.

LEMMA 6. *If a compact orientable Riemannian manifold (M, g) admits a conformal change of metric $g^* = e^{2\rho}g$, then, for any real number p,*

$$\begin{aligned}&\int_M (u^{p-3}G^*_{ji}G^{*ji} - u^{p+1}G_{ji}G^{ji})dV \\ &+2(n-2)p\int_M u^{p-1}G_{ji}u^j u^i dV + \frac{(n-2)^2}{n}\int_M u^p L_{du}K dV \\ &-(n-2)^2\int_M u^{p+1}P_{ji}P^{ji}dV = 0.\end{aligned} \tag{3.22}$$

In particular, if $p=-n+2$ then

$$\int_M (u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})dV \tag{3.23}$$
$$-2(n-2)^2\int_M u^{-n+1}G_{ji}u^ju^idV+\frac{(n-2)^2}{n}\int_M u^{-n+2}L_{du}KdV$$
$$-(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV=0\,,$$

and if $p=0$ then

$$\int_M (u^{-3}G^*_{ji}G^{*ji}-uG_{ji}G^{ji})dV \tag{3.24}$$
$$+\frac{(n-2)^2}{n}\int_M L_{du}KdV-(n-2)^2\int_M uP_{ji}P^{ji}dV=0\,.$$

Proof. Using (2.20) and (2.23), we have

$$\int_M (u^{p-3}G^*_{ji}G^{*ji}-u^{p+1}G_{ji}G^{ji})dV \tag{3.25}$$
$$=2(n-2)\int_M u^pG_{ji}\nabla^ju^idV+(n-2)^2\int_M u^{p+1}P_{ji}P^{ji}dV\,.$$

On the other hand, calculating $\nabla^j(u^pG_{ji}u^i)$ and using

$$\nabla^jG_{ji}=\frac{n-2}{2n}\nabla_iK\,,$$

we have

$$\nabla^j(u^pG_{ji}u^i)=pu^{p-1}G_{ji}u^ju^i+\frac{n-2}{2n}u^pu^i\nabla_iK+u^pG_{ji}\nabla^ju^i\,,$$

and consequently, integrating over M, we have

$$\int_M u^pG_{ji}\nabla^ju^idV=-p\int_M u^{p-1}G_{ji}u^ju^idV-\frac{n-2}{2n}\int_M u^pu^i\nabla_iKdV\,. \tag{3.26}$$

Substituting this into (3.25), we have (3.22) to be proved.

LEMMA 7. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then*

$$\int_M (u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})dV \tag{3.27}$$
$$+\frac{(n-2)^2}{n}\int_M u^{-n}L_{du}K^*dV+(n-2)^2\int_M u^{-n+3}P_{ji}P^{ji}dV=0\,.$$

Proof. Adding (3.12) $\times 2(n-2)^2$ and (3.23), we have (3.27).

LEMMA 8. *If a compact orientable Riemannian manifold (M,g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then, for any real number p,*

$$(3.28)\qquad \int_M (u^{p-3}Z^*{}_{kjih}Z^{*kjih} - u^{p+1}Z_{kjih}Z^{kjih})dV + 8p\int_M u^{p-1}G_{ji}u^j u^i dV + \frac{4(n-2)}{n}\int_M u^p L_{du}K dV - 4(n-2)\int_M u^{p+1}P_{ji}P^{ji}dV = 0 .$$

In particular, if $p=-n+2$ *then*

$$(3.29)\qquad \int_M (u^{-n-1}Z^*{}_{kjih}Z^{*kjih} - u^{-n+3}Z_{kjih}Z^{kjih})dV - 8(n-2)\int_M u^{-n+1}G_{ji}u^j u^i dV + \frac{4(n-2)}{n}\int_M u^{-n+2}L_{du}K dV - 4(n-2)\int_M u^{-n+3}P_{ji}P^{ji}dV = 0 ,$$

and if $p=0$ *then*

$$(3.30)\qquad \int_M (u^{-3}Z^*{}_{kjih}Z^{*kjih} - uZ_{kjih}Z^{kjih})dV + \frac{4(n-2)}{n}\int_M L_{du}K dV - 4(n-2)\int_M uP_{ji}P^{ji}dV = 0 .$$

Proof. Using (2.21) and (2.23), we have

$$(3.31)\qquad \int_M (u^{p-3}Z^*{}_{kjih}Z^{*kjih} - u^{p+1}Z_{kjih}Z^{kjih})dV - 8\int_M u^p G_{ji}\nabla^j u^i dV - 4(n-2)\int_M u^{p+1}P_{ji}P^{ji}dV = 0 .$$

Substituting (3.26) into (3.31), we have (3.28).

LEMMA 9. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n\geqq 2$ *admits a conformal change of metric* $g^*=e^{2\rho}g$, *then*

$$(3.32)\qquad \int_M (u^{-n-1}Z^*{}_{kjih}Z^{*kjih} - u^{-n+3}Z_{kjih}Z^{kjih})dV + \frac{4(n-2)}{n}\int_M u^{-n}L_{du}K^* dV + 4(n-2)\int_M u^{-n+3}P_{ji}P^{ji}dV = 0 .$$

Proof. (3.32) follows from (3.12) and (3.29).

LEMMA 10. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n\geqq 2$ *admits a conformal change of metric* $g^*=e^{2\rho}g$, *then, for any real number* p,

$$(3.33)\qquad \int_M (u^{p-3}W^*{}_{kjih}W^{*kjih} - u^{p+1}W_{kjih}W^{kjih})dV + 8\{a+(n-2)b\}^2 p\int_M u^{p-1}G_{ji}u^j u^i dV$$

$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^p L_{du}K dV$$

$$-4(n-2)\{a+(n-2)b\}^2\int_M u^{p+1}P_{ji}P^{ji}dV=0\,.$$

In particular, if $p=-n+2$ then

$$\int_M (u^{-n-1}W^*{}_{kjih}W^{*kjih}-u^{-n+3}W_{kjih}W^{kjih})dV \tag{3.34}$$

$$-8(n-2)\{a+(n-2)b\}^2\int_M u^{-n+1}G_{ji}u^j u^i dV$$

$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^{-n+2}L_{du}K dV$$

$$-4(n-2)\{a+(n-2)b\}^2\int_M u^{-n+1}P_{ji}P^{ji}dV=0\,,$$

and if $p=0$ then

$$\int_M (u^{-3}W^*{}_{kjih}W^{*kjih}-uW_{kjih}W^{kjih})dV \tag{3.35}$$

$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M L_{du}K dV$$

$$-4(n-2)\{a+(n-2)b\}^2\int_M uP_{ji}P^{ji}dV=0\,.$$

Proof. Using (2.22) and (2.23), we have

$$\int_M (u^{p-3}W^*{}_{kjih}W^{*kjih}-u^{p+1}W_{kjih}W^{kjih})dV \tag{3.36}$$

$$-8\{a+(n-2)b\}^2\int_M u^p G_{ji}\nabla^j u^i dV$$

$$-4(n-2)\{a+(n-2)b\}^2\int_M u^{p+1}P_{ji}P^{ji}dV=0\,.$$

Substituting (3.26) into (3.36), we have (3.33).

LEMMA 11. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$, then*

$$\int_M (u^{-n-1}W^*{}_{kjih}W^{*kjih}-u^{-n+3}W_{kjih}W^{kjih})dV \tag{3.37}$$

$$+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^{-n}L_{du}K^* dV$$

$$+4(n-2)\{a+(n-2)b\}^2\int_M u^{-n+3}P_{ji}P^{ji}dV=0\,.$$

Proof. (3.37) follows from (3.12) and (3.34).

LEMMA 12. *Suppose that a Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a conformal change of metric $g^*=e^{2\rho}g$ and f and f^* are non-negative functions on M such that*

$$u^p f=\{u^q+(u^r-1)\varphi\}f^*, \tag{3.38}$$

where p is a real number such that $p\leqq 4$, q and r non-negative numbers and φ a non-negative function on M. Then

$$(u^{-n-1}f^*-u^{-n+3}f)-(u^{-3}f^*-uf)\geqq 0. \tag{3.39}$$

Proof. We have

$$\begin{aligned}
&(u^{-n-1}f^*-u^{-n+3}f)-(u^{-3}f^*-uf)\\
&\quad=u^{-n-1}(1-u^{n-2})(f^*-u^4f)\\
&\quad=u^{-n-1}(1-u^{n-2})(f^*-u^qf^*-u^pf+u^qf^*+u^pf-u^4f)\\
&\quad=u^{-n-1}(1-u^{n-2})(1-u^q)f^*-u^{-n-1+p}(1-u^{n-2})(1-u^{4-p})f\\
&\qquad+u^{-n-1}(1-u^{n-2})(1-u^r)\varphi f^*.
\end{aligned}$$

We can easily prove that

$$(1-u^{n-2})(1-u^q)\geqq 0, \qquad (1-u^{n-2})(1-u^{4-p})\geqq 0, \qquad (1-u^{n-2})(1-u^r)\geqq 0,$$

and consequently that (3.39) holds.

§ 4. Propositions.

PROPOSITION 1. *If a compact Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M, then*

$$(\nabla_j u_i)(\nabla^j u^i)\geqq \frac{1}{n}(\Delta u)^2, \tag{4.1}$$

equality holding if and only if (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then the equality holds if and only if (M, g) is isometric to a sphere.

Proof. (4.1) is equivalent to

$$\Big(\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}\Big)\Big(\nabla^j u^i-\frac{1}{n}\Delta u g^{ji}\Big)\geqq 0,$$

and consequently equality in (4.1) holds if and only if

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0,$$

that is, by Theorem O, if and only if (M, g) is conformal to a sphere. The latter part of this proposition follows from Theorem P.

PROPOSITION 2. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M such that*

$$K_i{}^h u^i + \frac{n-1}{n}\nabla^h \Delta u = 0, \tag{4.2}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or K=constant, then (M, g) is isometric to a sphere.

Proof. From (3.5), we have

$$g^{ji}\nabla_j\nabla_i u^h - K_i{}^h u^i - \nabla^h \Delta u = 0.$$

Adding (4.2)$\times 2$ and this relation, we have

$$g^{ji}\nabla_j\nabla_i u^h + K_i{}^h u^i + \frac{n-2}{n}\nabla^h \Delta u = 0.$$

Thus, by the Remark to Lemma 1, we see that the vector field u^h on M defines an infinitesimal conformal transformation and consequently that

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji} = 0.$$

Thus, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 3. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$K_i{}^h u^i + \frac{n-1}{n}\nabla^h \Delta u = 0,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or K=constant, then (M, g) is isometric to a sphere.

Proof. This is an immediate consequence of Proposition 2. But, an another proof is as follows. From (3.14) and (4.2), we have $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji} = 0,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 4. *If a compact orientable Riemannian manifold (M, g) of dimension $n\geqq 2$ admits a non-constant function u on M such that*

$$\int_M K_{ji}u^j u^i dV \geqq \frac{n-1}{n}\int_M (\Delta u)^2 dV, \tag{4.3}$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or K=constant, then (M, g) is isometric to a sphere.

Proof. From (3.8) and (4.3), we have

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 ,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 5. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

$$\int_M K_{ji} u^j u^i dV \geqq \frac{n-1}{n} \int_M (\Delta u)^2 dV ,$$

then (M, g) is conformal to a sphere. If moreover $L_{du} K = 0$ or K=constant, then (M, g) is isometric to a sphere.

Proof. This is an immediate consequence of Proposition above. But, we can give an another proof. From (3.13) and the above relation, we find $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 ,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

(For Propositions 2∼5, see Yano and Hiramatu [12].)

PROPOSITION 6. *If a compact orientable Riemannian manifold (M, g) of dimension $n \geqq 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

$$\int_M u^{-n+1} G_{ji} u^j u^i dV + \frac{1}{2n} \int_M (u^{-n} L_{du} K^* - u^{-n+2} L_{du} K) dV \geqq 0 , \tag{4.4}$$

then (M, g) is conformal to a sphere. If moreover $L_{du} K = 0$ or K=constant, then (M, g) is isometric to a sphere.

Proof. By using (3.12) and (4.4), we have $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0 ,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. We have the latter part of the proposition by Theorem P.

The latter part of the proposition above is a generalization of Theorems A and H.

PROPOSITION 7. *If a compact orientable Riemannian manifold (M, g) of dimension $n > 2$ admits a non-homothetic conformal change of metric $g^* = e^{2\rho} g$ such that*

(4.5) $$\int_M (u^{-3}G^*_{ji}G^{*ji}-uG_{ji}G^{ji})dV+\frac{(n-2)^2}{n}\int_M L_{du}KdV\leqq 0\,,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. By using (3.24) and (4.5), we have $P_{ji}=0$, that is,

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0\,,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. Using Theoroem P, we can prove the latter part of the proposition.

The first part of Proposition 7 is a generalization of Theorem B because of

$$\int_M (\Delta u)KdV=-\int_M L_{du}KdV\,,$$

and the latter part a generalization of Theorem C.

PROPOSITION 8. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

(4.6) $$\int_M (u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})dV+\frac{(n-2)^2}{n}\int_M u^{-n}L_{du}K^*dV\geqq 0\,,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.27) and Theorems O and P.

PROPOSITION 9. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

(4.7) $$\int_M (u^{-3}Z^*_{kjih}Z^{*kjih}-uZ_{kjih}Z^{kjih})dV+\frac{4(n-2)}{n}\int_M L_{du}KdV\leqq 0,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=$constant, then (M, g) is isometric to a sphere.

Proof. This follows from (3.30) and Theorems O and P.

The first part of this proposition is a generalization of Theorem D and the latter part is a generalization of Theorem E.

PROPOSITION 10. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

(4.8) $$\int_M (u^{-n-1}Z^*_{kjih}Z^{*kjih}-u^{-n+3}Z_{kjih}Z^{kjih})dV+\frac{4(n-2)}{n}\int_M u^{-n}L_{du}K^*dV\geqq 0\,,$$

then (M, g) *is conformal to a sphere. If moreover* $L_{du}K=0$ *or* $K=$*constant, then* (M, g) *is isometric to a sphere.*

Proof. This follows from (3.32) and Theorems O and P.

PROPOSITION 11. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$\int_M (u^{-3}W^*_{kjih}W^{*kjih}-uW_{kjih}W^{kjih})dV + \frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M L_{du}K\,dV \leqq 0\,, \qquad a+(n-2)b\neq 0\,, \tag{4.9}$$

then (M, g) *is conformal to a sphere. If moreover* $L_{du}K=0$ *or* $K=$*constant, then* (M, g) *is isometric to a sphere.*

Proof. This follows from (3.35) and Theorems O and P.

The first part of Proposition 11 generalizes Theorem F and the latter part generalizes Theorem G.

PROPOSITION 12. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$\int_M (u^{-n-1}W^*_{kjih}W^{*kjih}-u^{-n+3}W_{kjih}W^{kjih})dV + \frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M u^{-n}L_{du}K^*dV \geqq 0\,, \qquad a+(n-2)b\neq 0\,, \tag{4.10}$$

then (M, g) *is conformal to a sphere. If moreover* $L_{du}K=0$ *or* $K=$*constant, then* (M, g) *is isometric to a sphere.*

Proof. This follows from (3.37) and Theorems O and P.

PROPOSITION 13. *If a compact orientable Riemannian manifold* (M, g) *of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$u^pG_{ji}G^{ji}=\{u^q+(u^r-1)\varphi\}G^*_{ji}G^{*ji} \tag{4.11}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0\,, \tag{4.12}$$

where p *is a real number such that* $p\leqq 4$, q *and* r *non-negative numbers and* φ *a* *non-negative function on M, then (M, g) is conformal to a sphere. If moreover* $L_{du}K=0$ *or* K*=constant, then (M, g) is isometric to a sphere.*

Proof. Subtracting (3.24) from (3.27), we obtain

$$\int_M \{(u^{-n-1}G^*_{ji}G^{*ji}-u^{-n+3}G_{ji}G^{ji})-(u^{-3}G^*_{ji}G^{*ji}-uG_{ji}G^{ji})\}dV \tag{4.13}$$
$$+\frac{(n-2)^2}{n}\int_M (u^{-n}L_{du}K^*-L_{du}K)dV$$
$$+(n-2)^2\int_M (u^{-n+3}+u)P_{ji}P^{ji}dV=0 .$$

By Lemma 12, from (4.11), (4.12) and (4.13), we have $P_{ji}=0$, that is,

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. By using Theorem P, we can prove the latter part of this proposition.

The latter part of Proposition 13 is a generalization of Theorem L.

COROLLARY 1. *If a compact orientable Riemannian manifold (M, g) of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$G_{ji}G^{ji}=G^*_{ji}G^{*ji} \tag{4.14}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0 ,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ *or* K*=constant, then (M, g) is isometric to a sphere.*

Proof. Putting $p=q=r=0$ in (4.11), we have (4.14), and consequently this corollary follows immediately from Proposition 13.

The latter part of this corollary is a generalization of Theorem I.

PROPOSITION 14. *If a compact orientable Riemannian manifold (M, g) of dimension* $n>2$ *admits a non-homothetic conformal change of metric* $g^*=e^{2\rho}g$ *such that*

$$u^p Z_{kjih}Z^{kjih}=\{u^q+(u^r-1)\varphi\}Z^*_{kjih}Z^{*kjih} \tag{4.15}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0 ,$$

where p, q, r *and* φ *are the same as in Proposition 13, then (M, g) is conformal to a sphere. If moreover* $L_{du}K=0$ *or* K*=constant, then (M, g) is isometric to a* *sphere.*

Proof. Subtracting (3.30) from (3.32), we have

$$\int_M \{(u^{-n-1}Z^*{}_{kjih}Z^{*kjih}-u^{-n+3}Z_{kjih}Z^{kjih}) -(u^{-3}Z^*{}_{kjih}Z^{*kjih}-uZ_{kjih}Z^{kjih})\}dV + \frac{4(n-2)}{n}\int_M (u^{-n}L_{du}K^*-L_{du}K)dV + 4(n-2)\int_M (u^{-n+3}+u)P_{ji}P^{ji}dV=0. \tag{4.16}$$

Using Lemma 12, (4.12), (4.15) and (4.16), we have $P_{ji}=0$, that is,

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}=0,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. By using Theorem P, we can prove the latter part of the proposition.

The latter part of Proposition 14 is a generalization of Theorem M.

COROLLARY 2. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$Z^*{}_{kjih}Z^{*kjih}=Z_{kjih}Z^{kjih} \tag{4.17}$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV \geqq 0,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=constant$, then (M, g) is isometric to a sphere.

Proof. Putting $p=q=r=0$ in (4.15), we get (4.17), and consequently Corollary 2 follows immediately from Proposition 14.

The latter part of Corollary 2 generalizes Theorem J.

PROPOSITION 15. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$u^p W_{kjih}W^{kjih}=\{u^q+(u^r-1)\varphi\}W^*{}_{kjih}W^{*kjih}, \tag{4.18}$$

$$a+(n-2)b\neq 0$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV \geqq 0,$$

where p, q, r and φ are the same as in Proposition 13, then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=constant$, then (M, g) is isometric to a sphere.

Proof. Subtracting (3.35) from (3.37), we have

$$\begin{aligned}(4.19)\qquad &\int_M \{(u^{-n-1}W^*_{kjih}W^{*kjih}-u^{-n+3}W_{kjih}W^{kjih})\\ &\qquad -(u^{-3}W^*_{kjih}W^{*kjih}-uW_{kjih}W^{kjih})\}dV\\ &+\frac{4(n-2)}{n}\{a+(n-2)b\}^2\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\\ &+4(n-2)\{a+(n-2)b\}^2\int_M (u^{-n+3}+u)P_{ji}P^{ji}dV=0\,.\end{aligned}$$

By using Lemma 12, from (4.12), (4.18) and (4.19), we have $P_{ji}=0$, that is,

$$\nabla_j u_i-\frac{1}{n}\Delta u g_{ji}=0\,,$$

and consequently, by Theorem O, (M, g) is conformal to a sphere. By using Theorem P, we can prove the latter part of Proposition 15.

The latter part of Proposition 15 is a generalization of Theorem N.

COROLLARY 3. *If a compact orientable Riemannian manifold (M, g) of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{2\rho}g$ such that*

$$(4.20)\qquad W^*_{kjih}W^{*kjih}=W_{kjih}W^{kjih}\,,\qquad a+(n-2)b\neq 0$$

and

$$\int_M (u^{-n}L_{du}K^*-L_{du}K)dV\geqq 0\,,$$

then (M, g) is conformal to a sphere. If moreover $L_{du}K=0$ or $K=constant$, then (M, g) is isometric to a sphere.

Proof. Putting $p=q=r=0$ in (4.18), we get (4.20), and consequently Corollary 3 follows immediately from Proposition 15.

The latter part of Corollary 3 is a generalization of Theorem K.

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