M. OZAWA
KŌDAI MATH. SEM. REP.
27 (1976), 1—10

ON CERTAIN CRITERIA FOR THE LEFT-PRIMENESS OF ENTIRE FUNCTIONS, II

BY MITSURU OZAWA

1. Introduction. In our previous paper [5] we had proved two general theorems guaranteeing the left-primeness of entire functions. The first one may be stated in the following manner:

THEOREM A. *Let $F(z)$ be an entire function of finite order whose derivative $F'(z)$ has infinitely many zeros. Assume that the equations $F(z)=c$ and $F'(z)=0$ have only finitely many common roots for any constant c. Then $F(z)$ is left-prime in entire sense.*

Although this has a wide range of applicability, there are lots of defects, for example, this does not work to the function $z\sin z+z$. The function has infinitely many double zeros and hence $F(z)=0$ and $F'(z)=0$ have infinitely many common roots. We shall now fill up this kind of defect. Our theorems are the following.

THEOREM 1. *Let $F(z)$ be an entire function of finite order. Assume that for a certain constant A $F(z)=A$ has at least one but at most finitely many simple roots and has infinitely many multiple roots all of whose multiplicities are the same. Assume further that the equations $F(z)=c$, $F'(z)=0$ have only a finite number of common roots for any $c\neq A$. Then $F(z)$ is left-prime in entire sense.*

THEOREM 2. *Let $F(z)$ be such an entire function that for a constant A $F(z)=A$ has at least one but at most finitely many simple roots and has infinitely many multiple roots all of whose multiplicities are the same. Assume that*

$$N(r, A, F)-\bar{N}(r, A, F)\geqq Km(r, F)$$

and

$$N(r, 0, F')-(N(r, A, F)-\bar{N}(r, A, F))\geqq km(r, F)$$

for some $K, k>0$. Assume further that $F(z)=c$, $F'(z)=0$ have only a finite number of common roots for any $c\neq A$. Then $F(z)$ is left-prime in entire sense.

Firstly we should remark that any entire function has neither three perfectly branched values nor a finite Picard exceptional value and a perfectly branched value. Here we call a a perfectly branched value of $F(z)$ when $F(z)=a$ has

Received Jan. 15, 1974.

infinitely many multiple roots except for a finite number of simple roots. We shall make use of the above fact repeatedly.

2. Proof of Theorem 1. Suppose that $F(z)=f(g(z))$ with transcendental f and g. Then by Pólya's result $\rho(f)=0$, where $\rho(f)$ indicates the order of f. Assume that $f(w)-A$ has three simple zeros w_1, w_2, w_3. Then $g(z)=w_j$ should have only finitely many simple roots by the finiteness of simple zeros of $F(z)-A$. This is impossible. Further at least one zero of $f(w)-A$ should be simple, since $F(z)-A$ has at least one simple zero. Hence there occur three possibilities:

1) $f(w)-A=B(w-w_1)M(w)^{\mu}$,

2) $f(w)-A=B(w-w_1)(w-w_2)^{\lambda}M(w)^{\mu}$, $\lambda\geqq 2$,

3) $f(w)-A=B(w-w_1)(w-w_2)M(w)^{\mu}$,

where $M(w)$ has infinitely many simple zeros and $\mu\neq\lambda$. Here μ is the order of multiple zeros of $F(z)-A$. In what follows we shall prove the existence of a w_0 for which $f'(w_0)=0$ but $f(w_0)\neq A$. Suppose this is not the case. Then any zero of $f'(w)$ satisfies $f(w)=A$ and every multiple root of $f(w)=A$ satisfies $f'(w)=0$.

The case 1). In this case

$$\frac{f'^{\mu}}{(f-A)^{\mu-1}}=\frac{1}{B^{\mu-1}(w-w_1)^{\mu-1}}.$$

By integration of this equation we have

$$\mu(f-A)^{1/\mu}=B^{(1-\mu)/\mu}\mu[(w-w_1)^{1/\mu}-d].$$

We put $w=w_1$. Then $f(w_1)=A$ implies $d=0$. Thus

$$f-A=B^{(1-\mu)}(w-w_1).$$

This is impossible.

The case 2). In this case $(g(z)-w_2)^{\lambda}$ has infinitely many zeros, whose orders should be equal to μ. Hence $\lambda s=\mu$, $s\geqq 2$. Then consider

$$\frac{f'^{\mu}}{(f-A)^{\mu-1}}=\frac{1}{B^{\mu-1}(w-w_1)^{\mu-1}(w-w_2)^{\mu-\lambda}}.$$

Integrating this equation we have

$$\mu(f-A)^{1/\mu}=B^{(1-\mu)/\mu}\int\frac{dw}{(w-w_1)^{1-1/\mu}(w-w_2)^{1-\lambda/\mu}}$$
$$=B^{(1-\mu)/\mu}(L(w)-d).$$

We put $w=w_1$ or $w=w_2$. Then $d=L(w_1)=L(w_2)$. Thus $L(w_2)-L(w_1)=0$. However

$$L(w_2)-L(w_1)=\int_{w_1}^{w_2}\frac{dw}{(w-w_1)^{1-1/\mu}(w-w_2)^{1-\lambda/\mu}}$$

$$=\frac{1}{(-1)^{1-\lambda/\mu}(w_2-w_1)^{1-1/\mu-\lambda/\mu}}\int_0^1\frac{dW}{W^{1-1/\mu}(1-W)^{1-\lambda/\mu}}$$
$$\neq 0\,.$$

This is a contradiction.

The case 3). Consider

$$\frac{f'^{\mu}}{(f-A)^{\mu-1}}=\frac{1}{B^{\mu-1}\{(w-w_1)(w-w_2)\}^{\mu-1}}\,.$$

Similarly as in the case 2) we have

$$0=L(w_2)-L(w_1)$$
$$=\int_{w_1}^{w_2}\frac{dw}{\{(w-w_1)(w-w_2)\}^{1-1/\mu}}$$
$$=\frac{1}{(-1)^{1-1/\mu}(w_2-w_1)^{1-2/\mu}}\int_0^1\frac{dW}{\{W(1-W)\}^{1-1/\mu}}$$
$$\neq 0\,.$$

This is untenable.

Hence in all the cases there exists a w_0 such that $f'(w_0)=0$ but $f(w_0)\neq A$. Then consider $g(z)=w_0$. At all the roots of $g(z)=w_0$ $F(z)=f(w_0)\neq A$ and $F'(z)=0$, which have only a finite number of common roots. Hence $g(z)=w_0$ has only finitely many roots, that is, w_0 is a Picard exceptional value of g. However g has already at least one perfectly branched value w_1. This is impossible. Therefore $F(z)$ is pseudo-prime in entire sense.

Suppose that $F(z)=f(g(z))$ with a non-linear polynomial f and entire g. Then similarly as in the above we have three possibilities:

$$f-A=B(w-w_1)(w-w_2)^{\mu}\cdots(w-w_s)^{\mu}\,,\qquad s\geqq 2\,,$$
$$f-A=B(w-w_1)(w-w_2)^{\lambda}(w-w_3)^{\mu}\cdots(w-w_s)^{\mu}\,,\qquad \lambda\geqq 2\,,\quad s\geqq 2$$

and

$$f-A=B(w-w_1)(w-w_2)(w-w_3)^{\mu}\cdots(w-w_s)^{\mu}\,,\qquad s\geqq 2\,.$$

We shall prove the existence of w_0 such that $f'(w_0)=0$ but $f(w_0)\neq A$. In each case we put x the number of roots of $f'=0$ other than the ones satisfying $f(w)=A$. We compute the degree of f' in two manners. Then we have

$$\mu(s-1)=(\mu-1)(s-1)+x$$

in the first case,

$$\mu(s-2)+\lambda=(\mu-1)(s-2)+\lambda-1+x$$

in the second case and

$$\mu(s-2)+1=(\mu-1)(s-2)+x$$

in the third case. In every case we have

$$x=s-1\geqq 1.$$

This gives the desired result. Then considering $g(z)=w_0$ and hence the equations $F(z)=f(w_0)\neq A$, $F'(z)=0$ and remarking the existence of a perfectly branched value w_1 of g we have a contradiction. q. e. d.

We can release our assumptions on the roots of $F(z)=A$ in the following manner: $F(z)-A=B(z-z_1)^{\lambda_1}\cdots(z-z_s)^{\lambda_s}L(z)^{\mu}$ for a certain A, where λ_j and μ are coprime for each j $(1\leqq j\leqq s)$, $s\geqq 1$ and $L(z)$ has only infinitely many simple zeros.

3. Proof of Theorem 2. Suppose that $F=f(g)$ with transcendental f and g. Then we have again three possibilities:

1) $f(w)-A=B(w-w_1)M(w)^{\mu}$,
2) $f(w)-A=B(w-w_1)(w-w_2)^{\lambda}M(w)^{\mu}$, $\lambda\geqq 2$

and

3) $f(w)-A=B(w-w_1)(w-w_2)M(w)^{\mu}$.

Here $M(w)$ is transcendental entire. In all the cases we shall firstly prove the existence of infinitely many zeros of $M(w)$. In the case 1)

$$N(r, A, F)\leqq N(r, w_1, g)+\mu\sum_{j=2}^{p}N(r, w_j, g)$$
$$\leqq(\mu p-\mu+1)m(r, g).$$

However

$$N(r, A, F)-\bar{N}(r, A, F)\geqq Km(r, F)$$
$$\geqq Ksm(r, g)(1-\varepsilon)$$

for $r\notin E_g$, which is of finite measure. This gives a contradiction, since s is arbitrary. Hence $M(w)$ has infinitely many simple zeros in this case. Quite similarly we can prove the existence of infinitely many (simple) zeros of $M(w)$ in the remaining two cases.

Next we shall prove the existence of at least one w_0, for which $f'(w_0)=0$ but $f(w_0)\neq A$. Suppose this is not the case. Then in the case 1)

$$N(r, 0, f'(g))=N(r, A, F(g))-\bar{N}(r, A, f(g)).$$

Therefore for $r\notin E_g$

$$N(r, 0, F')=N(r, 0, f'(g))+N(r, 0, g')$$
$$\leqq N(r, A, F)-\bar{N}(r, A, F)+N(r, 0, g')$$
$$\leqq N(r, 0, F')-km(r, F)+m(r, g)(1+\varepsilon).$$

Thus

$$km(r, F) \leqq m(r, g)(1+\varepsilon) .$$

However

$$km(r, F) \geqq ksm(r, g)(1-\varepsilon)$$

for $r \notin E_g$. Hence

$$ks \leqq 1 ,$$

which is impossible, since s is arbitrary. This gives the desired existence of w_0, for which $f'(w_0)=0$ but $f(w_0) \neq A$ in the case 1). Similarly we can prove the desired existence of w_0 in the remaining two cases. Once the existence of w_0 such that $f'(w_0)=0$, $f(w_0) \neq A$ is acertained, the remaining part of the proof is quite similar as in Theorem 1. Then we have the pseudo-primeness of F in entire sense. If $F(z)=f(g(z))$ with a non-linear polynomial f and entire g, we can prove the existence of $w_0 : f'(w_0)=0$, $f(w_0) \neq A$ as in Theorem 1 and then we have the left-primeness of F. q. e. d.

As in the case of Theorem 1 we can release the condition on the roots of $F=A$.

4. Applications.

COROLLARY 1. *$P(z)(\sin z+1)$ is prime, where $P(z)$ is a polynomial of odd degree.*

Proof. Let A be zero. Then

$$F(z)=P(z)\left(\frac{e^{iz/2}+ie^{-iz/2}}{\sqrt{2i}}\right)^2 .$$

Since $P(z)$ is of odd degree, there is at least one zero of P of odd multiplicity. Consider

$$F(z)=P(z)(\sin z+1)=c \neq 0 ,$$

$$F'(z)=P'(z)(\sin z+1)+P(z) \cos z=0 .$$

Then we have

$$2P(z)^3=CP(z)^2+CP'(z)^2 .$$

This has only finitely many roots. Hence $F(z)$ is left-prime in entire sense.

Consider the distribution of zeros of $F(z)=f(g(z))$ with a non-linear polynomial g. Then g should be quadratic. Let $g(z)$ be $\alpha(z-a)^2+b$. In the present case a should be either $2n\pi+5\pi/2$ or $2n\pi+3\pi/2$, where n is an integer. Hence with $x=z-a$

$$\begin{aligned} P(a+x)\{\sin(a+x)+1\} &= F(a+x)=f(\alpha x^2+b)=f(\alpha(-x)^2+b) \\ &= F(a-x)=P(a-x)\{\sin(a-x)+1\} . \end{aligned}$$

However $\sin(a+x)+1=\sin(a-x)+1$. Hence $P(a+x)=P(a-x)$. By comparing the leading coefficients of both sides we have a contradiction. Thus we have the right-primeness of F in entire sense. Hence F is prime by Gross' theorem [2]. q. e. d.

By the above proof we can say that $P(z)(\sin z+1)$ is prime if $P(z)$ does not have the form $Q(z)^2$ and does not satisfy $P(a-z)=P(a+z)$ for any $a=2n\pi+2\pi+\pi/2$ or $2n\pi+\pi+\pi/2$. These conditions are necessary as the following examples show:

$$z^2(\sin z+1)=\left(z\frac{e^{iz/2}+ie^{-iz/2}}{\sqrt{i2}}\right)^2.$$

$$z(z-\pi)(\sin z+1)=\left\{\left(w-\frac{\pi^2}{4}\right)(\cos\sqrt{w}+1)\right\}\circ\left(\frac{\pi}{2}-z\right)^2.$$

COROLLARY 2. *$z(e^{\alpha z}+e^{\beta z})^2$ is prime if either $\alpha\beta\neq0$, $\alpha\beta^{-1}$ is real or $\alpha=0$, $\beta\neq0$ or $\alpha=\beta=0$.*

Proof. We firstly consider the case $\alpha\neq\beta$, $\alpha\beta\neq0$. Consider

$$F(z)=z(e^{\alpha z}+e^{\beta z})^2=c\neq0,$$

$$F'(z)=\{(2z\alpha+1)e^{\alpha z}+(2z\beta+1)e^{\beta z}\}(e^{\alpha z}+e^{\beta z})=0.$$

Then

$$(\alpha-\beta)z=\log\frac{z+1/2\beta}{z+1/2\alpha}+\log\frac{\beta}{\alpha}+\pi i+2p\pi i$$

and

$$\alpha z=\log\frac{z+1/2\beta}{z}-\frac{1}{2}\log z+\log\frac{\beta\sqrt{c}}{\beta-\alpha}+2q\pi i.$$

Taking their real parts we have

$$\left(1-\frac{\beta}{\alpha}\right)\Re(\alpha z)=\log\frac{|z+1/2\beta|}{|z+1/2\alpha|}+\log\left|\frac{\beta}{\alpha}\right|$$

and

$$\Re(\alpha z)=\log\frac{|z+1/2\beta|}{|z|}+\log\frac{|\beta\sqrt{c}|}{|\beta-\alpha|}-\frac{1}{2}\log|z|.$$

If $z\to\infty$, then $|\Re(\alpha z)|$ is bounded but by the second equation $\Re(\alpha z)\to-\infty$. This is impossible. Hence the equations $F=c\neq0$, $F'=0$ have only finitely many common roots. Hence F is left-prime in entire sense.

We next consider the right-primeness. To this end we consider the distribution of zeros of F. The set of zeros of F is $\{0\}$ and $\{(2p-1)\pi i/(\alpha-\beta)\}$ $p=0, \pm1, \cdots$. The latter ones are of order 2. Hence $F(z)=f(g(z))$ with a polynomial g implies that g is of degree two or one. Since the set of zeros of F is symmetric with respect to the origin only, $g(z)$ must be az^2+b if $\deg g=2$. Then $F(-z)=F(z)$. This implies that

$$e^{-\alpha z}+e^{-\beta z}=+i(e^{\alpha z}+e^{\beta z})\,.$$

This is impossible by the impossibility of Borel's identity [1], [3]. Hence F is right-prime in entire sense. Hence F is prime.

Next we consider the case $\alpha=\beta\neq 0$. We may consider the function $ze^{\alpha z}$. This has 0 as a Picard exceptional valve. Hence $ze^{\alpha z}$ is pseudo-prime [4]. Then the remaining part is almost trivial. The case $\alpha=\beta=0$ is trivial.

If $\alpha=0$, we consider

$$F=z(e^{\beta z}+1)^2=c\neq 0\,.$$

$$F'=\{2z(\beta e^{\beta z}+1)+e^{\beta z}+1\}(e^{\beta z}+1)=0\,.$$

By cancelling out $e^{\beta z}$ we have

$$c(1+2\beta z)^2=4\beta^2 z^3\,,$$

which has only three roots. Hence F is left-prime in entire sense. The right-primeness is almost similarly proved as in the general case. Hence F is prime.

q. e. d.

COROLLARY 3.

$$z\Big(\int_0^z e^{-t^2}dt+z\Big)^2$$

is prime.

Proof. It is necessary to prove that

$$\int_0^z e^{-t^2}dt+z$$

has only finitely many multiple zeros and infinitely many simple zeros. However this fact was proved already in [5]. Let us consider

$$F=z\Big(\int_0^z e^{-t^2}dt+z\Big)^2=c\neq 0\,,$$

$$F'=\Big(\int_0^z e^{-t^2}dt+z\Big)^2+2z(e^{-z^2}+1)\Big(\int_0^z e^{-t^2}dt+z\Big)=0\,.$$

Then

$$\int_0^z e^{-t^2}dt+z=\frac{\sqrt{c}}{\sqrt{z}}\,.$$

Hence

$$e^{-z^2}=-\frac{2z\sqrt{z}+\sqrt{c}}{az\sqrt{z}}\,,$$

which implies

$$-z^2=\log\frac{2z\sqrt{z}+\sqrt{c}}{2z\sqrt{z}}+\pi i+2p\pi i\,,$$

$$\Re(z^2)=-\log\frac{|z\sqrt{z}+\sqrt{c}/2|}{|z\sqrt{z}|}\,.$$

Let z be $re^{i\theta}$. Then if $z\to\infty$, $r^2\cos 2\theta\to 0$. Therefore $\theta\to\pi/4, 3\pi/4, 5\pi/4$ or $7\pi/4$ if $r\to\infty$. We firstly consider the case $\theta\to\pi/4$. Then $r^2\cos 2\theta\to 0$ as $r\to\infty$ implies $r^2(\theta-\pi/4)\to 0$. In this case

$$\left|\int_{re^{i\pi/4}}^{re^{i\theta}} e^{-t^2}dt\right|\leqq\left|\int_{\pi/4}^{\theta} re^{-r^2\cos 2\phi}d\phi\right|\leqq 2r|\theta-\pi/4|\to 0 \qquad (r\to\infty).$$

Further

$$\left|\int_0^{re^{i\pi/4}} e^{-t^2}dt\right|=\left|\int_0^r \cos s^2 ds-i\int_0^r \sin s^2 ds\right|\leqq M.$$

Hence

$$\left|z-\frac{\sqrt{c}}{\sqrt{z}}\right|\leqq\left|\int_0^{re^{i\pi/4}} e^{-t^2}dt\right|+\left|\int_{re^{i\pi/4}}^{re^{i\theta}} e^{-t^2}dt\right|$$

implies that

$$\left|z-\frac{\sqrt{c}}{\sqrt{z}}\right|$$

is bounded as $z\to\infty$ along the solutions of $e^{-z^2}=-(2z\sqrt{z}+\sqrt{c})/2z\sqrt{z}$ being near the ray $\theta=\pi/4$. This is impossible. The same holds in the other three cases. Hence F is left-prime in entire sense.

The proof fo the right-primeness of F is quite similar as in the one of

$$\int_0^z e^{-t^2}dt+z$$

in [5]. We shall make use of the result in [5]. Firstly we have only one possibility for the right-factor, that is, g is a quadratic polynomial, if F is not right-prime. Then by the symmetry with respect to the origin $g(z)$ should be αz^2+b. Hence $F(z)=F(-z)$. However it is immediate to prove $F(-z)=-F(z)$. This is impossible. Hence F is right-prime. q. e. d.

COROLLARY 4. *$z(e_2(z)-p(z))^2$ is prime, where $e_2(z)=\exp e^z$ and $p(z)$ is a non-zero polynomial.*

Proof. Firstly it is necessary to prove that $e_2(z)-p(z)$ has only finitely many multiple zeros and infinitely many simple zeros. This was already proved in [5] for any non-constant $p(z)$. If $p(z)$ is a non-zero constant, this is almost trivial.

Since $e_2(z)-p(z)$ has only finitely many multiple zeros together with infinitely many simple zeros,

$$\begin{aligned} N(r, 0, F)-\bar{N}(r, 0, F) &\geqq N(r, 0, e_2(z)-p(z))(1-\varepsilon) \\ &\geqq m(r, e_2(z))(1-\varepsilon) \\ &\geqq \frac{1}{2}m(r, F)(1-\varepsilon) \end{aligned}$$

for $r\notin E_\phi$, which is of finite measure, and

$$\begin{aligned} &N(r, 0, F')-N(r, 0, F)+\overline{N}(r, 0, F) \\ &\qquad =N(r, p+2zp', (1+2ze^z)e_2(z)) \\ &\qquad \geqq m(r, \phi)(1-\varepsilon)\geqq m(r, e_2(z))(1-\varepsilon) \\ &\qquad \geqq \frac{1}{2}m(r, F)(1-\varepsilon) \end{aligned}$$

for $r\notin E_\phi$, which is of finite measure. Here

$$\phi=\frac{e_2(z)}{e_2(z)-p(z)}, \qquad \phi=\frac{e_2(z)(1+2ze^z)}{e_2(z)(1+2ze^z)-p(z)-2zp'(z)}.$$

Now let us consider

$$\begin{cases} F=z(e_2(z)-p(z))^2=c\neq 0 \\ F'=\{(2ze^z+1)e_2(z)-p(z)-2zp'(z)\}\{e_2(z)-p(z)\}=0. \end{cases}$$

We now assume that $p(z)$ is not a constant. Then

$$e_2(z)=p(z)+\frac{\sqrt{c}}{\sqrt{z}}$$

and

$$e^z=\frac{2z\sqrt{z}\,p'(z)-\sqrt{c}}{2z(\sqrt{c}+\sqrt{z}\,p(z))}.$$

Hence

$$\Re z=\log\left|\frac{2z\sqrt{z}\,p'(z)-\sqrt{c}}{2z(\sqrt{z}\,p(z)+\sqrt{c})}\right|\to-\infty$$

as $z\to\infty$. Putting $z=x+\imath y$ we have $x\to-\infty$. Further

$$e^x\cos y=\log\left|p(z)+\frac{\sqrt{c}}{\sqrt{z}}\right|\to+\infty$$

as $z\to\infty$. But $x\to-\infty$ implies the boundedness of $e^x\cos y$. This is impossible. We next assume that $p(z)$ is a non-zero constant a. Then

$$e_2(z)=a+\frac{\sqrt{c}}{\sqrt{z}}$$

and

$$e^z=-\frac{\sqrt{c}}{az\sqrt{z}+z\sqrt{c}}$$

imply

$$\log\left(a+\frac{\sqrt{c}}{\sqrt{z}}\right)+2p\pi i=-\frac{\sqrt{c}}{az\sqrt{z}+\sqrt{c}\,z}$$

and

$$-\frac{\sqrt{c}}{az\sqrt{z}+\sqrt{c}\,z}=\log a+\frac{\sqrt{c}}{a\sqrt{z}}-\frac{1}{2}\left(\frac{\sqrt{c}}{a\sqrt{z}}\right)^2+\cdots+2p\pi\imath.$$

Hence as $z\to\infty$ $\log a+2p\pi i=0$, that is, $a=1$. This implies $p=0$. Hence

$$1-\frac{\sqrt{c}}{2\sqrt{z}}+\cdots=-\frac{1}{z+\sqrt{c}\sqrt{z}},$$

which is impossible. Hence in both cases $F=c\neq 0$, $F'=0$ have at most a finite number of common roots. Therefore F is left-prime in entire sense.

For the right-primeness of F we remark the following fact:

$$e^x \cos y=\log|p(z)|$$

is satisfied by the non-zero roots of $F(z)=0$. Hence for $x\leqq x_0$ there are at most finitely many solutions of $F(z)=0$ if $p(z)$ is not a constant. Here x_0 is arbitrary. Then we can conclude the right-primeness of F as in [5], Corollary 3. If $p(z)$ is a non-zero constant a, then

$$e^z=\log a+2p\pi i,$$

$$e^x \cos y=\log|a|.$$

Therefore there is no solution in $x\leqq -x_0$ for a sufficiently large x_0 and there are infinitely many roots in $x\geqq x_0$. Therefore we can conclude the right-primeness of F in entire sense in a quite similar manner. Thus $F(z)$ is prime. q. e. d.

BIBLIOGRAPHY

[1] BOREL, E., Sur les zéros des fonctions entières, Acta Math., **20** (1897), 357–396.

[2] GROSS, F., Factorization of entire functions which are periodic mod g, Indian Journ. pure Appl. Math., **2** (1971), 561–571.

[3] NEVANLINNA, R., Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen, Acta Math., **48** (1926), 367–391.

[4] OZAWA, M., On the solution of the functional equation $f\circ g(z)=F(z)$, Kōdai Math. Sem. Rep., **20** (1968), 159–162.

[5] OZAWA, M., On certain cirteria for the left-primeness of entire functions. Kōdai Math. Sem. Rep., **26** (1975), 304–317.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY,
OH-OKAYAMA, MEGURO-KU, TOKYO,
JAPAN