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ON CONFORMAL RIGIDITY OF A RIEMANN SURFACE

Dedicated to Professor Yûsaku Komatu on his 60th birthday

BY TAKAO KATO

1. A Riemann surface is said to be conformally rigid if the preservation of a certain condition attached to the Riemann surface implies that an analytic self-mapping is an automorphism. The notion of the homotopical or the homological (conformal) rigidity was treated by Huber [2], Landau and Osserman [4], Marden, Richards and Rodin [5] and Jenkins and Suita [3]. Jenkins and Suita have made clear the relationship between these two types of rigidity.

In this paper we shall give a criterion for a Riemann surface to be homotopically rigid. Furthermore, we shall introduce the notion of the weakly homological rigidity in a similar manner and shall treat the relationship among these three types of rigidity.

2. A Riemann surface W is said to be homotopically (resp. homologically or weakly homologically) rigid if every analytic self-mapping of W, which preserves the homotopical (resp. homological or weakly homological) non-triviality, reduces to an automorphism of W. Let $\mathfrak{S}$ denote the class of Riemann surfaces every non-constant analytic self-mapping of which reduces to a univalent mapping. Then we have

THEOREM 1. *If W is of $O_{HD}\cap\mathfrak{S}$, then it is homotopically rigid. $O_{HD}\cap\mathfrak{S}$ cannot be replaced with $\mathfrak{S}$.*

Heins [1] proved

THEOREM A. *Every Riemann surface of class O_G and having the non-abelian fundamental group is of class $\mathfrak{S}$.*

Therefore, we have

THEOREM 2. *If W is of class O_G and has the non-abelian fundamental group, then it is homotopically rigid.*

Next we shall show a criterion to be weakly homologically rigid. That is

THEOREM 3. *If W is of positive finite genus but not a torus, then it is weakly*

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homologically rigid.

Finally, we shall treat the relationship among three types of conformal rigidity. Jenkins and Suita [3] showed

THEOREM B. *If a Riemann surface is homotopically rigid, it is homologically rigid. There is a Riemann surface which is homologically rigid but not homotopically rigid.*

We shall show

THEOREM 4. *There is a Riemann surface which is homotopically rigid but not weakly homologically rigid. Furthermore, there is a Riemann surface which is weakly homologically rigid but not homologically rigid.*

3. To prove Theorem 1 we shall prepare a result related to an analytic mapping between two algebroid surfaces. Ozawa [7] introduced another sort of rigidity with respect to an analytic mapping between two algebroid surfaces. Let R and S be two ultrahyperelliptic surfaces defined by the equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions having no zero other than an infinite number of simple zeros. Let $\mathfrak{P}_R$ and $\mathfrak{P}_S$ be the projection mappings $(z, y)\to z$ and $(w, u)\to w$, respectively. Let f be an analytic mapping from R into S. Then we say that f is rigid in the sense of projection mapping ($\mathfrak{P}$-rigid) if f satisfies $\mathfrak{P}_S\circ f(p)=\mathfrak{P}_S\circ f(q)$ for every pair of p and q, so that $\mathfrak{P}_R p=\mathfrak{P}_R q$.

Let

$$G_1(x)=G\left(\frac{1+x}{1-x}\right).$$

Suppose that each zero of G_1 lies in the unit disk and that

$$\varlimsup_{r\to 1}\frac{N(r, 0, G_1)}{\log\dfrac{1}{1-r}}=\infty .$$

Let W be a subregion of R, so that $\mathfrak{P}_R(W)$ is the whole z-plane and $\mathfrak{P}_R(R-W)$ lies in the left half plane $\operatorname{Re} z<0$. Then we have

THEOREM 5. *If there exists an analytic mapping from W into S, then it is $\mathfrak{P}$-rigid.*

To prove this theorem we need the following

LEMMA (cf. Ozawa [7]). *There is no solution of an equation of the following form*

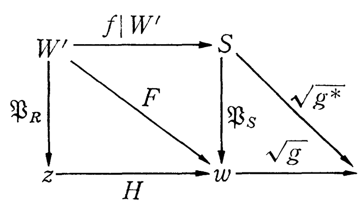
$$g\circ(h_1(x)+h_2(x)\sqrt{G_1(x)})=(L_1(x)+L_2(x)\sqrt{G_1(x)})^2$$

for any two entire functions L_1 and L_2 in the unit disk, where $h(x)=h_1(x)+h_2(x)\sqrt{G_1(x)}$ is an algebroid entire function in the unit disk satisfying

$$\overline{\lim_{r\to1}}\frac{T(r, h)}{\log\dfrac{1}{1-r}}=\infty\,.$$

Proof. We note that Nevanlinna-Selberg's second fundamental theorem and the ramification relation remain true in the unit disk if $\overline{\lim}\, T(r, h)/\log(1/(1-r))=\infty$ holds (Nevanlinna [6], Selberg [8], Valiron [9]). Then we can prove this lemma by the same method of Ozawa's.

4. Proof of Theorem 5. Let f be an analytic mapping from W into S. Let W' be a subregion of W which lies over $\mathrm{Re}\, z>0$. If we restrict f to W', again by Ozawa's method we obtain the same schema as in his paper. Let



$$h(x)\equiv H\left(\frac{1+x}{1-x}\right).$$

Then we have a functional equation

$$g\circ(h_1(x)+h_2(x)\sqrt{G_1(x)})=(L_1(x)+L_2(x)\sqrt{G_1(x)})^2$$

in the unit disk, where $h(x)=h_1(x)+h_2(x)\sqrt{G_1(x)}$.

Suppose $h_2\not\equiv0$. From Valiron [9] we have

$$N(r, 0, G_1)\leqq 7T(r, h)\,.$$

This contradicts Lemma. Therefore, $h(x)$ is single valued.

Since, $f|W'$ is $\mathfrak{P}$-rigid on W', we can conclude that f is $\mathfrak{P}$-rigid on W by the analytic continuation.

This yields immediately

COROLLARY. *Let W be as in Theorem* 5. *Every analytic self-mapping of W is univalent. i.e. $W\in\mathfrak{S}$.*

Proof. $H(z)$ in the proof of Theorem 5 can be extended to an entire function in the whole z-plane. Hence, every analytic self-mapping of W can be extended to a self-mapping of R in the unique way. It is known that R is of O_G. Then by Theorem A we have $W\in\mathfrak{S}$.

5. Proof of Theorem 1. Let $W\in O_{HD}\cap\mathfrak{S}$. Suppose that f is a non-constant analytic self-mapping of W which preserves the homotopical non-triviality. Since

$W \in \mathfrak{S}$, f is univalent. Hence $f(W)$, the image of W under f, is of class O_{HD}. Therefore, $W-f(W)$ is a totally disconnected subset of W. Suppose that there is a point $p \in W-f(W)$. We take a small topological disk $\varDelta$ in W centred at p, so that $\partial\varDelta$, the boundary of $\varDelta$, lies in $f(W)$ completely. Since the inverse image of $\varDelta \cap f(W)$ does not simply connected, the outer boundary γ of it is not null homotop in W. But $f(\gamma)=\partial\varDelta$ is null homotop. This contradicts the homotopical condition of f. Therefore, f is an automorphism of W.[1)]

6. (continued) To show the latter half we shall construct an example.

Let E_1 be the whole z-plane less the slits $\{x+mi\,;\,2n-1 \leqq x \leqq 2n\}$ ($n=1, 2, 3, \cdots$ and $m=0, \pm 1, \pm 2, \cdots$). Let E_2 be the E_1 less the disks $\{|z+1-mi|<1/3\}$ ($m=-1, -2, -3, \cdots$) and the points $\{z=mi\}$ ($m=0, 1, 2, \cdots$). We construct the desired Riemann surface W by joining E_1 to E_2 along their corresponding slits in the standard manner.

Then there is a diffining entire function G which satisfies the hypothesis of Theorem 5. Hence, by Corollary to Theorem 5, W is of class $\mathfrak{S}$.

On the other hand, let f be an analytic self-mapping of W, such that $z \to z+i$. Then f preserves the weakly homological non-triviality but is not an automorphism.

7. Proof of Theorem 3. Let f be an analytic self-mapping of W. Suppose that f is not an automorphism. Then f^n tends to a point on W or an ideal boundary component of W uniformly on every compact subset of W (Heins [1]). Since every boundary component of W is planar, f can not preserve weakly homological non-triviality of non-dividing cycle. Hence, W is weakly homologically rigid.

8. Proof of Theorem 4. To prove this theorem we shall show two examples.

One of those is a Riemann surface which is homotopically rigid but not weakly homologically rigid.

Let W be the Riemann surface defined by the equation $y^2=\cos \pi x$ less the points over $x=0, 1, 2, \cdots$. Then it is of class O_G. Hence, it is homotopically rigid by Theorem 2. If we consider a mapping $x \to x+1$, it is evident that W is not weakly homologically rigid.

For an example which is weakly homologically rigid but not homologically rigid, it is sufficient to show an open Rieman surface of finite positive genus which is not homologically rigid. Such a surface was given by Jenkins and Suita [3, p. 47]. The reader will consult with their paper.

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