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SINGULAR SETS OF SOME INFINITELY GENERATED KLEINIAN GROUPS

Dedicated to Professor Yûsaku Komatu on his 60th birthday

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§1. Preliminaries and Notations.

1. Let $\{K_j\}_{j=1}^p$ and $\{H_i, H_i'\}_{i=p+1}^\infty$ be an infinite number of circles external to one another in the extended complex plane $\tilde{C}=\{z; |z|\leqq\infty\}$, where $\{H_i, H_i'\}_{i=p+1}^q$ tend to only a finite point Q for $q\to\infty$. Let B be a domain bounded by these circles. Without loss of generality we may assume that these circles are contained in some closed disc $D_0=\{z; |z|\leqq\rho_0\}$.

Let $\{T_j\}_{j=1}^p$ be the elliptic transformations with period 2 corresponding to $\{K_j\}_{j=1}^p$, each of which transforms the outside of K_j onto the inside of itself. Let $\{T_i\}_{i=p+1}^\infty$ be the system of hyperbolic or loxodromic transformations, each T_i of which transforms the outside of H_i' onto the inside of H_i. Then the system $\mathfrak{G}=\{T_i, T_i^{-1}\}_{i=1}^\infty$ $(T_i=T_i^{-1}, 1\leqq i\leqq p)$ generates an infinitely generated discontinuous group denoted by G and we call $\mathfrak{G}$ the generator system of G, where T_i^{-1} denotes the inverse of T_i.

The purpose of this paper is to investigate the singular set E of G. Take a positive integer q $(>p)$ and consider a subset $\mathfrak{G}_N=\{T_j\}_{j=1}^p\cup\{T_i, T_i^{-1}\}_{i=p+1}^q$ $(N=2q-p)$ of $\mathfrak{G}$. Then $\mathfrak{G}_N$ generates a finitely generated subgroup G_N of G. If we denote by B_N a domain bounded by $\{K_j\}_{j=1}^p\cup\{H_i, H_i'\}_{i=p+1}^q$ $(N=2q-p)$, it is well known that B_N coincides with a fundamental domain of G_N. We gave some results with respect to the singular set E_N of G_N by using the relations between E_N and the computing functions on G_N ([1]). We shall get G from G_N for $N\to\infty$. It is natural to try whether we can extend the results for G_N to ones for G. Unfortunately, we can not extend those in the same way, for the behavior of the accumulation of the circles to Q gives the complicated difficulty. Therefore we must impose some restrictions with respect to the accumulation of circles and henceforth we shall consider only such groups.

2. Denote by $r(H)$ the radius of a circle $H\in\{H_i, H_i'\}_{i=p+1}^\infty$ and assume that there exists some positive constant K independent of H such that it holds

$$\text{(A)}\qquad \frac{r(H)}{l(H)}\leqq K,$$

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where $l(H)=\inf|z-\zeta|$ and the infimum is taken for all points $z\in H$ and for all points ζ on any circle from $\{H_i, H_i'\}_{i=p+1}^{\infty}-\{H\}$.

Defining the product ST in G by $ST(z)=S(T(z))$, we can write any element U of G in the form

$$U=T_{i_n}\cdots T_{i_2}T_{i_1} \qquad (T_{i_j}\in\mathfrak{G}\ (1\leqq j\leqq n)\,;\ T_{i_{j+1}}^{-1}\neq T_{i_j}).$$

We call the positive integer n the grade of U and for simplicity we use the notation $S_{(n)}$ to clarify the grade of U.

Since we can let the generator T_i $(\in\mathfrak{G})$ correspond to the boundary circle H_i, we shall denote by C_{T_i} and $C_{T_i^{-1}}$ the circles H_i and H_i', and further by D_{T_i} and $D_{T_i^{-1}}$ the closed discs bounded by C_{T_i} and $C_{T_i^{-1}}$, respectively. Then it is obvious that $C_{T_i}=T_i(C_{T_i^{-1}})$.

Consider the image $S_{(n)}(B_N)$ by any element $S_{(n)}=T_{i_n}\cdots T_{i_2}T_{i_1}$ $(\in G_N)$. It is easily seen that $S_{(n)}(B_N)$ is bounded by an outer boundary circle $S_{(n)}(C_{T_{i_1}^{-1}})$ and $(N-1)$ inner boundary circles $S_{(n)}(C_{T_j})$ $(T_j\neq T_{i_1}^{-1}, T_j\in\mathfrak{G}_N)$. We shall call such inner boundary circles the circles of grade n. It is obvious that the number of all circles of grade n for G_N is equal to $N(N-1)^n$. Circles $\{K_j\}_{j=1}^{p}$ and $\{H_i, H_i'\}_{i=p+1}^{q}$, which bounds B_N, are of grade 0. The circles of grade n with respect to G can be defined in the same way.

Now let us impose a restriction with respect to the accumulation of circles for G. Consider the circle $C_{T_i}: |z-\alpha(T_i)|=r_{T_i}$ of radius r_{T_i} with center $\alpha(T_i)$ for any T_i $(\in\mathfrak{G})$. Take some boundary circle C_{T_j} $(T_j\neq T_i)$ of B and denote the distance from $\alpha(T_i)$ to C_{T_j} by $\rho_j(T_i)$, that is,

$$\rho_j(T_i)=\inf_{z\in C_{T_j}}|z-\alpha(T_i)|. \tag{1.1}$$

It is obvious that $\rho_j(T_i)>r(T_i)$ and from the property (A)

$$\frac{r(T_i)}{\rho_j(T_i)}\leqq\frac{K}{K+1}. \tag{1.2}$$

We assume that there exists a positive constant $K_1(\alpha)$ depending only on some positive number α $(0<\alpha<2)$ satisfying

$$W(T_j, \alpha)=\sum_{T_i\in\mathfrak{G}}{}'\left(\frac{r(T_i)}{\rho_j(T_i)^2}\right)^{\alpha}\leqq K_1(\alpha)\,, \tag{B}$$

where $\sum'_{T_i\in\mathfrak{G}}$ denotes the sum with respect to all T_i $(\neq T_j)$. Then we can determine the unique number α_0 $(\geqq 0)$ such that

$$\alpha_0=\inf\{\alpha\,;\,K_1(\alpha)<+\infty\}\,. \tag{1.3}$$

We note that α_0 is always equal to 0 for G_N.

From now we shall call such discontinuous group with these properties (A) and (B) the Kleinian group with properties (A) and (B) and denote it by $G(K, \alpha_0)$, which has the generator system $\mathfrak{G}(K, \alpha_0)$. For brevity we also denote it by $G^*(\alpha_0)$ with $\mathfrak{G}^*(\alpha_0)$. We can easily give the examples of such groups $G^*(\alpha_0)$, for example, by fuchsian groups with an infinite number of generators.

3. Consider two arbitrary transformations T and S of G. We assume that $S\neq T^{-1}$. Denote by I_S, I_T and I_{ST} the isometric circles of S, T and ST, respectively. Let R_S, R_T and R_{ST} be radii of I_S, I_T and I_{ST}, respectively. As to these values, the relation

$$R_{ST}=\frac{R_S\cdot R_T}{|T(\infty)-S^{-1}(\infty)|} \tag{1.4}$$

holds.

Let $S_{(n)}=T_{i_n}\cdots T_{i_2}T_{i_1}$ $(T_{i_j}\in\mathfrak{G}_N)$ be any element of G_N and assume that $T_{i_1}^{-1}\neq T$ for a fixed element T $(\in\mathfrak{G}_N)$ and take any point $z\in D_T$. If we put

$$S_{(n)}(z)=\frac{az+b}{cz+d} \qquad (ad-bc=1)\,,$$

we obtain easily

$$\left|\frac{dS_{(n)}(z)}{dz}\right|^{\frac{\mu}{2}}=\left(\frac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu} \qquad (0<\mu<4)\,,$$

where $S_{(n)}^{-1}$ denotes the inverse $(S_{(n)})^{-1}=T_{i_1}^{-1}\cdots T_{i_n}^{-1}$ of $S_{(n)}$. Here we note that $z\in D_T$ and $S_{(n)}^{-1}(\infty)\in D_{T_{i_1}^{-1}}\neq D_T$.

Forming the sum of $(N-1)^n$ terms with respect to all $S_{(n)}$ $(\in G_N)$ such that $T_{i_1}^{-1}\neq T$ and $T_{i_j}\neq T_{i_{j+1}}^{-1}$ $(1\leqq j\leqq n-1)$, we had the function

$$\chi_{n,N}^{(\mu;T)}(z)=\sum_{S_{(n)}\in G_N}\left(\frac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu}, \tag{1.5}$$

and called $\chi_{n,N}^{(\mu;T)}(z)$ the μ-dimensional computing function of order n on T and there exist N computing functions $\chi_{n,N}^{(\mu;T)}(z)$ corresponding to the choice of T from $\mathfrak{G}_N$ ([1]). The domain of definition of $\chi_{n,N}^{(\mu;T)}(z)$ is D_T.

Since each term in the sum $\chi_{n,N}^{(\mu;T)}(z)$ is positive, $\chi_{n,N}^{(\mu;T)}(z)$ has necessarily the unique limit containing the infinity for any $z\in D_T$, if N tends to the infinity. Thus we can define the function

$$\lim_{N\to\infty}\chi_{n,N}^{(\mu;T)}(z)=\lim_{N\to\infty}\sum_{S_{(n)}\in G_N}\left(\frac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu}=\sum_{S_{(n)}\in G}\left(\frac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu}, \tag{1.6}$$

and we shall call it the μ-dimensional limiting computing function of order n on T and denote it by $\chi_{n,\infty}^{(\mu;T)}(z)$. Using the relation (1.4), we obtain

$$\begin{cases}\chi_{n,N}^{(\mu;T)}(S_{(m)}(\infty))=\displaystyle\sum_{S_{(n)}\in G_N}\left(\frac{R_{S_{(n+m)}}}{R_{S_{(m)}}}\right)^{\mu},\\[2ex] \chi_{n,\infty}^{(\mu;T)}(S_{(m)}(\infty))=\displaystyle\sum_{S_{(n)}\in G_\infty}\left(\frac{R_{S_{(n+m)}}}{R_{S_{(m)}}}\right)^{\mu},\end{cases} \tag{1.7}$$

where $S_{(n)}S_{(m)}=S_{(n+m)}$.

Now let us give the following definition.

DEFINITION. Let $\{\chi_{n,\infty}^{(\mu;T)}(z)\}$ $(n=1, 2, \cdots)$ be the sequence of the μ-dimensional limiting computing functions on $T\in\mathfrak{G}$. If it holds

(1.8) $$\lim_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)=0 \qquad (\text{or } \infty)$$

for some element T of $\mathfrak{G}$ and some point $z\in D_T$, we call G the μ-convergent (or divergent) type. If it holds

(1.9) $$0<\underline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)\leqq\overline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)<\infty$$

for some T $(\in\mathfrak{G})$ and some point $z\in D_T$, we call G the μ-finite type. We have already given these concepts for $\chi_{n,N}^{(\mu;T)}(z)$ in the former paper ([1]).

4. In the former paper [1] we obtained the following important proposition with respect to E_N.

PROPOSITION 1. *The following three propositions are equivalent to one another:* (1) *The sequence* $\{\chi_{n,N}^{(\mu;T^*)}(z)\}$ $(n=1, 2, \cdots)$ *on some fixed* T^* $(\in\mathfrak{G}_N)$ *diverges (or converges to zero) at some singular point* $z_0\in E\frown D_{T^*}$, *that is,*

(1.10) $$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T)}(z_0)=\infty \quad (\text{or } 0) \qquad \text{for some } z_0\in E\frown D_{T^*}.$$

(2) *It holds for any* T $(\in\mathfrak{G}_N)$

(1.11) $$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T)}(z)=\infty \qquad (\text{or } 0)$$

uniformly on D_T.

(3) $$M_{\mu/2}(E_N)=\infty \quad (\text{or } 0).$$

Now we consider a linear transformation

$$S(z)=\frac{\alpha z+\beta}{\gamma z+\delta}, \qquad \alpha\delta-\beta\gamma=1$$

and a circle C: $|z-z_0|=r$. Then we have the following proposition ([5]).

PROPOSITION 2. *If* $S^{-1}(\infty)=-\delta/\gamma$ *lies outside* C, *the radius* r' *of the image* C' *of* C *by* S *is equal to*

(1.12) $$\frac{1}{|\gamma|^2}\,\frac{r}{(\rho^2-r^2)},$$

where ρ *denotes the distance* $|S^{-1}(\infty)-z_0|$.

Proof. Put $\theta=\arg\{(z-z_0)/(S^{-1}(\infty)-z_0)\}$ $(z\in C)$. Then we have

$$r'=\frac{1}{2\pi}\int_C\left|\frac{dS(z)}{dz}\right||dz|=\frac{1}{2\pi}\int_C\frac{|dz|}{|\gamma z+\delta|^2}$$
$$=\frac{1}{2\pi|\gamma|^2}\int_0^{2\pi}\frac{rd\theta}{\rho^2-2\rho r\cos\theta+r^2}=\frac{1}{|\gamma|^2}\,\frac{r}{(\rho^2-r^2)}.$$

q. e. d.

Now let us seek for the relation between radii of circles of grade n and the radius of the isometric circle.

Let $S_{(n)}=T_{i_n}\cdots T_{i_2}T_{i_1}$ ($T_{i_j}\in\mathfrak{G}^*(\alpha_0)$) be any element of $G^*(\alpha_0)$. An infinite number of circles of grade n corresponding to $S_{(n)}$ are the circles $S_{(n)}(C_{T_j})$ ($T_j\neq T_{i_1}^{-1}$) bounded by the circle $S_{(n)}(C_{T_{i_1}^{-1}})$. Since the pole of $S_{(n)}$, that is, $S_{(n)}^{-1}(\infty)$ $=T_{i_1}^{-1}\cdots T_{i_n}^{-1}(\infty)$ is contained in $C_{T_{i_1}^{-1}}$, it holds for any T_j ($\neq T_{i_1}^{-1}$)

$$|S_{(n)}^{-1}(\infty)-\alpha(T_j)|>\rho(T_j)\,, \tag{1.13}$$

where $\alpha(T_j)$ and $\rho(T_j)$ are defined in No. 2. Hence we see from Proposition 2 that the radius of the circle $S_{(n)}(C_{T_j})$ of grade n is smaller than

$$\frac{r(T_j)}{\rho(T_j)^2-r(T_j)^2}\cdot R^2_{S(n)}\,. \tag{1.14}$$

Since $r(T_j)/\rho(T_j)\leqq K/(K+1)$ by (1.2), we have

$$\frac{r(T_j)}{\rho(T_j)^2-r(T_j)^2}R^2_{S(n)}\leqq(K+1)\frac{r(T_j)}{\rho(T_j)^2}R^2_{S(n)}\,. \tag{1.15}$$

Take a positive number μ ($0<\mu<4$) such that $\alpha_0<\mu/2$ and denote by $\sum'_{T_j\in\mathfrak{G}^*(\alpha_0)} r_{S(n)}(T_j)$ the sum of radii of circles $S_{(n)}(C_{T_j})$ of grade n for all T_j ($\neq T_{i_1}^{-1}$) by any element $S_{(n)}\in G^*(\alpha_0)$, we obtain easily from the property (B) and (1.15) the following proposition.

PROPOSITION 3. *It holds for any element* $S_{(n)}$ ($\in G^*(\alpha_0)$)

$$\sum'_{T_j\in\mathfrak{G}^*(\alpha_0)}(r_{S(n)}(T_j))^{\frac{\mu}{2}}<K(G^*(\alpha_0),\,\mu)(R_{S(n)})^{\mu}\,, \tag{1.16}$$

where $K(G^*(\alpha_0),\mu)$ *is a constant depending only on* $G^*(\alpha_0)$ *and* μ.

§ 2. Local property of the limiting computing function.

5. The main purpose of this paper is to extend Proposition 1 for a Kleinian group $G^*(\alpha_0)$ with properties (A) and (B). For this purpose we shall give the following Lemma which is also the extension of Lemma 1 in [1].

LEMMA 1. *Let* $G^*(\alpha_0)$ *be a Kleinian group with properties* (A) *and* (B).
(i) *There exist positive constants* $K_1(G^*(\alpha_0),\mu)$ *and* $K_2(G^*(\alpha_0),\mu)$ *depending only on* $G^*(\alpha_0)$ *and* μ ($>2\alpha_0$) *such that it holds*

$$K_1(G^*(\alpha_0),\,\mu)\chi^{(\mu;T)}_{n,\infty}(z_0)\leqq\chi^{(\mu;T)}_{n,\infty}(z)\leqq K_2(G^*(\alpha_0),\,\mu)\chi^{(\mu;T)}_{n,\infty}(z_0) \tag{2.1}$$

for any element $T\in\mathfrak{G}^*(\alpha_0)$ *and any points* z *and* $z_0\in D_T$. (ii) *There exists a positive constant* $K_3(G^*(\alpha_0),T^*,\mu)$ *depending only on* $G^*(\alpha_0)$, T^* *and* μ ($>2\alpha_0$) *such that it holds for any elements* T *and* T^* ($\in\mathfrak{G}^*(\alpha_0)$; $T^*\neq T^{-1}$) *and any points* z ($\in D_T$) *and* z^* ($\in D_{T^*}$)

$$\chi^{(\mu;T)}_{n+1,\infty}(z)\geqq K_3(G^*(\alpha_0),\,T^*,\,\mu)\chi^{(\mu;T^*)}_{n,\infty}(z^*)\,. \tag{2.2}$$

Remark. In the finite case, $K_3(G^*(\alpha_0),T^*,\mu)$ can be replaced by a constant

$K_3(G_N, \mu)$. In this case we could show in [1] that the μ-divergence (or convergence) type of G_N is the uniform property in D_T and D_{T^*} with respect to the sequence $\{\chi_{n,N}^{(\mu;T)}(z)\}$ and $\{\chi_{n,N}^{(\mu;T^*)}(z^*)\}$ $(n=1, 2, \cdots)$ for different elements T and T^* $(\in \mathfrak{G}_N)$, that is, for example, we can take a common order N_0 for any positive number M such that

$$\chi_{n,N}^{(\mu;T)}(z)>M \quad \text{and} \quad \chi_{n,N}^{(\mu;T^*)}(z^*)>M \qquad \text{for} \quad n>N_0\,.$$

But in our case, we can no more take such common order N_0 for the μ-convergence type and this means that N_0 depends not only on $G^*(\alpha_0)$ and μ but also each element of $\mathfrak{G}^*(\alpha_0)$.

Proof of (i). $S_{(n)}=T_{i_n}\cdots T_{i_1}$ $(T_{i_1}^{-1}\neq T)$ be any element of grade n in $G^*(\alpha_0)$ and take any two points z and $z_0 \in D_T$. Denote the closed disc D_T of radius r_T with center α by $D_T: |z-\alpha|\leqq r_T$. Let us denote by l the distance between D_T and $D_{T_{i_1}^{-1}}$. Since $S_{(n)}^{-1}(\infty)$ is contained in $D_{T_{i_1}^{-1}}$, we get easily the following estimation:

$$(2.3)\qquad \begin{aligned}&\left(\frac{R_{S(n)}}{|z-S_{(n)}^{-1}(\infty)|}\Big/\frac{R_{S(n)}}{|z_0-S_{(n)}^{-1}(\infty)|}\right)^{\mu}=\left|\frac{z_0-S_{(n)}^{-1}(\infty)}{z-S_{(n)}^{-1}(\infty)}\right|^{\mu}\\ &=\left(\frac{|z_0-\alpha|+|\alpha-S_{(n)}^{-1}(\infty)|}{|S_{(n)}^{-1}(\infty)-\alpha|-|\alpha-z|}\right)^{\mu}\leqq\left(\frac{2r_T+l+2r_{T_{i_1}^{-1}}}{r_T+l-r_T}\right)^{\mu}.\end{aligned}$$

Hence from the property (A) we have

$$(2.4)\qquad \left(\frac{2r_T+l+2r_{T_{i_1}^{-1}}}{l}\right)^{\mu}\leqq(4K+1)^{\mu}=K_2(G^*(\alpha_0), \mu)\,.$$

Thus we obtain from (2.3) and (2.4)

$$(2.5)\qquad \left(\frac{R_{S(n)}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu}\leqq K_2(G^*(\alpha_0), \mu)\left(\frac{R_{S(n)}}{|z_0-S_{(n)}^{-1}(\infty)|}\right)^{\mu}.$$

Forming the sum with respect to all $S_{(n)}=T_{i_n}\cdots T_{i_1}$ $(T_{i_1}^{-1}\neq T)$, we have an inequality of the right hand side in (i). Since (2.5) is symmetric with respect to any pair of points z and z_0 contained in D_T, we have

$$K_1(G^*(\alpha_0), \mu)\chi_{n,\infty}^{(\mu;T)}(z_0)\leqq\chi_{n,\infty}^{(\mu;T)}(z)\leqq K_2(G^*(\alpha_0), \mu)\chi_{n,\infty}^{(\mu;T)}(z_0)\,.$$

Proof of (ii). Take any element $S_{(n+1)}=S_{(n)}T^*=T_{i_n}\cdots T_{i_1}T^*$ of grade $n+1$ such that $T^*\neq T^{-1}$ and $T_{i_1}\neq T^{*-1}$ $(T, T_{i_1}, T^*\in\mathfrak{G}^*(\alpha_0))$. Since

$$\left|\frac{dS_{(n+1)}(z)}{dz}\right|^{\frac{\mu}{2}}=\left(\left|\frac{dS_{(n)}(z^*)}{dz^*}\right|\times\left|\frac{dT^*(z)}{dz}\right|\right)^{\frac{\mu}{2}}, \qquad z^*=T^*(z)\,,$$

we have from the definition of the computing function (1.5)

$$(2.6)\qquad \begin{aligned}&\sum_{S(n+1)\in G^*(\alpha_0)}\left(\frac{R_{S(n+1)}}{|z-S_{(n+1)}^{-1}(\infty)|}\right)^{\mu}\\ &=\sum_{S(n+1)\in G^*(\alpha_0)}\left(\frac{R_{S(n)}}{|T^*(z)-S_{(n)}^{-1}(\infty)|}\right)^{\mu}\left(\frac{R_{T^*}}{|z-T^{*-1}(\infty)|}\right)^{\mu}.\end{aligned}$$

Since all circles $\{K_j\}_{j=1}^{p}$ and $\{H_i, H_i'\}_{i=p+1}^{\infty}$ are contained in a closed disc $D_0=\{z; |z|\leqq\rho_0\}$, it holds $|z-T^{*-1}(\infty)|\leqq 2\rho_0$. Hence we have

$$(2.7)\qquad \sum_{S_{(n+1)}\in G^*(\alpha_0)}\left(\frac{R_{S_{(n+1)}}}{|z-S_{(n+1)}^{-1}(\infty)|}\right)^{\mu}\geqq\left(\frac{R_{T^*}}{2\rho_0}\right)^{\mu}\sum_{S_{(n)}\in G^*(\alpha_0)}\left(\frac{R_{S_{(n)}}}{|T^*(z)-S_{(n)}^{-1}(\infty)|}\right)^{\mu}.$$

Thus from (2.7) we have for any elements T and T^* $(\in\mathfrak{G}^*(\alpha_0); T\neq T^{*-1})$

$$\chi_{n+1,\infty}^{(\mu;T)}(z)\geqq K_3(G^*(\alpha_0), T^*, \mu)\chi_{n,\infty}^{(\mu;T^*)}(z^*).$$

q. e. d.

6. By using Lemma 1 we have the following theorem.

THEOREM 1. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *If $G^*(\alpha_0)$ is the μ-divergent* (*or convergent*) *type, then it holds*

$$\lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T^*)}(z)=\infty \qquad (or\ 0)$$

for any T^* $(\in\mathfrak{G}^*(\alpha_0))$ and any point $z\in D_{T^*}$.

Proof. From the assumption we have

$$(2.8)\qquad \lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z_0)=\infty \qquad (\text{or } 0)$$

for some element T $(\in\mathfrak{G}^*(\alpha_0))$ and some point $z_0\in D_T$.

We divide the proof of this theorem into two cases of μ-divergent and μ-convergent types. At first let us prove the case of μ-divergent type. There exists from (ii) of Lemma 1 and (2.8) some positive $N_0=N_0(T, M)$ depending only on T and arbitrarily large number M such that it holds for any T^* $(\in\mathfrak{G}^*(\alpha_0))$ and any $n>N_0$

$$(2.9)\qquad \chi_{n+1,\infty}^{(\mu;T^*)}(z^*)>K_3(G^*(\alpha_0), T, \mu)M.$$

Hence it is easily seen from (i) of Lemma 1 that

$$(2.10)\qquad \lim_{n\to\infty}\chi_{n+1,\infty}^{(\mu;T^*)}(z^*)=\infty$$

uniformly on D_T. Since T is a fixed element and T^* is any element of $\mathfrak{G}^*(\alpha_0)$, we note from (2.9) that the uniform convergence in (2.10) is independent of T^*.

Next we shall prove the case of μ-convergent type. In this case there exists from (ii) of Lemma 1 and (2.8) some positive integer $N_0=N_0(T, \varepsilon)$ depending only on T and arbitrarily small ε such that it holds for any T^* $(\in\mathfrak{G}^*(\alpha_0))$ and any $n>N_0$

$$(2.11)\qquad \varepsilon>K_3(G^*(\alpha_0), T^*, \mu)\chi_{n,\infty}^{(\mu;T^*)}(z^*).$$

Hence also it is easily seen from (i) of Lemma 1 that

$$(2.12)\qquad \lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T^*)}(z^*)=0$$

uniformly on D_{T^*}. But we find from (2.11) that the uniform convergence in (2.12) holds for each D_{T^*} only. q. e. d.

7. By using Theorem 1 we have the following theorem.

THEOREM 2. *Let $G^*(\alpha_0)$ be a Kleinian groups with properties* (A) *and* (B). *If $G^*(\alpha_0)$ is the μ-divergent type, then it holds $M_{\mu/2}(E)=\infty$.*

Proof. Take an arbitrarily large number M (>1). Then from Theorem 1 there exists a positive integer $N_0=N_0(M)$ depending only on M such that it holds $\chi_{n,N}^{(\mu;T)}(z)>M$ for any n, $N>N_0$. Hence from the result in [1], it holds for the singular set E_N of the subgroup G_N of G

$$M_{\mu/2}(E_N)=\infty .$$

Since $E\supset E_N$, we have $M_{\mu/2}(E_N)=\infty$. q. e. d.

8. In the finite case, we obtained the result that it holds $M_{\mu/2}(E_N)=0$, if $\lim_{n\to\infty}\chi_{n,N}^{(\mu;T)}(z)=0$. But in our case, it is not clear $\lim_{n\to\infty}\sum_{S(n)\in G^*(\alpha_0)}\{\sum'_{T_j\in\mathfrak{G}^*(\alpha_0)}(r_{S(n)}(T_j))^{\mu/2}\}=0$ implies $M_{\mu/2}(E)=0$. Hence we shall prove the following lemma.

LEMMA 2. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *If $\lim_{n\to\infty}\sum_{S(n)\in G(\alpha_0)}\{\sum'_{T_j\in\mathfrak{G}(\alpha_0)}(r_{S(n)}(T_j))^{\mu/2}\}=0$, then it holds $M_{\mu/2}(E)=0$.*

Remark. This lemma holds also for a Kleinian group which has no properties (A) and (B).

Proof. Denote by $D_{S(n)}(T_j)$ the closed disc bounded by $S_{(n)}(C_{T_j})$, where $S_{(n)}=T_{in}\cdots T_{i1}$ $(T_{ij}\in\mathfrak{G}^*(\alpha_0))$ is any element of grade n of $G^*(\alpha_0)$ and $T_j\neq T_{ij}^{-1}$. Then for the proof it suffices to prove that we can take $\bigcup_{S(n)\in G^*(\alpha_0)}\bigcup_{T_j\in\mathfrak{G}^*(\alpha_0)}D_{S(n)}(T_j)$ as a covering of E. Let $\{\varepsilon_k\}_{k=0}^{\infty}$ $(\varepsilon_k>0)$ be the sequence of numbers such that it holds $\sum_{k=0}^{\infty}(\varepsilon_k)^{\mu/2}<\varepsilon$ for any $\varepsilon>0$ and $\mu/2>\alpha_0$ and consider the images $S_{(n)}(Q)$ of Q by all $S_{(n)}$ $(\in G^*(\alpha_0))$. Describe a circle of radius ε_0 with center Q and denote by $D(Q, \varepsilon_0)$ the closed disc bounded by this circle. Then there exist an infinite number of circles of grade 0 contained in $D(Q, \varepsilon_0)$ completely. Hence there are only a finite number of closed discs which are not contained in $D(Q, \varepsilon_0)$. Some of them have common parts with $D(Q, \varepsilon_0)$, that is, they are not contained completely in $D(Q, \varepsilon_0)$.

Denote these closed discs, for brevity, by

$$D_{T_1},\ D_{T_2},\ \cdots,\ D_{T_{m_0}}, \tag{2.13}$$

where they are bounded by circles of grade 0, that is, the boundary circles of B. The images $T_i(Q)$ $(1\leqq i\leqq m_0)$ $(T_i\in\mathfrak{G}^*(\alpha_0))$ are contained in these discs D_{T_i} $(1\leqq i\leqq m_0)$, respectively. Take ε_i $(1\leqq i\leqq m_0)$ from $\{\varepsilon_k\}_{k=0}^{\infty}$ and describe circles of radii ε_i $(1\leqq i\leqq m_0)$ with centers $T_i(Q)$ $(1\leqq i\leqq m_0)$ and denote the closed discs bounded by these circles by $D(T_i(Q), \varepsilon_i)$ $(1\leqq i\leqq m_0)$.

Consider all closed discs bounded by circles of grade 1 and extract the closed discs which are not contained in $D(T_i(Q), \varepsilon_i)$ $(1\leqq i\leqq m_0)$ from these ones in the same manner as the case of grade 0. Then there are a finite number of such closed discs. We denote them for brevity by

$$D_{S^{(1)}_{(2)}},\ D_{S^{(2)}_{(2)}},\ \cdots,\ D_{S^{(m_1)}_{(2)}}\ . \tag{2.14}$$

The images $S^{(i)}_{(2)}(Q)$ $(1\leqq i\leqq m_1)$ are contained in $D_{S^{(i)}_{(2)}}$ $(1\leqq i\leqq m_1)$. Take ε_{m_0+i} $(1\leqq i\leqq m_1)$ from $\{\varepsilon_k\}_{k=0}^{\infty}$ and describe circles of radii ε_{m_0+i} $(1\leqq i\leqq m_1)$ with centers $S^{(i)}_{(2)}(Q)$ $(1\leqq i\leqq m_1)$ and denote by $D(S^{(i)}_{(2)}(Q),\ \varepsilon_{m_0+i})$ $(1\leqq i\leqq m_1)$ the closed discs bounded by these circles. We get a finite number of closed discs, denoted by $D_{S^{(i)}_{(3)}}$ $(1\leqq i\leqq m_2)$ briefly, which are not contained in $D(S^{(i)}_{(2)},\ \varepsilon_{m_0+i})$ $(1\leqq i\leqq m_1)$ in the above sense. Then

$$\{\bigcup_{i=1}^{m_2} D_{S^{(i)}_{(3)}}\}\cup D(Q,\ \varepsilon_0)\cup\{\bigcup_{i=1}^{m_0} D(T_i(Q),\ \varepsilon_i\}\cup\{\bigcup_{i=1}^{m_1} D(S^{(i)}_{(2)}(Q),\ \varepsilon_{m_0+i})\}$$

is a covering of E.

Continuing this procedure n-times, we can reach to the closed discs of grade n. Then we have as a covering of E

$$\{\bigcup_{i=1}^{m_n} D_{S^{(i)}_{(n+1)}}\}\cup D(Q,\ \varepsilon_0)\cup\{\bigcup_{i=1}^{m_0} D(T_i(Q),\ \varepsilon_i)\}\cup\cdots\cup\{\bigcup_{i=1}^{m_{n-1}} D(S^{(i)}_{(n)}(Q),\ \varepsilon_{m_0+m_1+\cdots+m_{n-2}+i})\}\ .$$

Denoting by $r_{S^{(i)}_{(n)}}$ the radius of $D_{S^{(i)}_{(n+1)}}$ bounded by circle of grade n, we obtain that the sum of radii to the power $\mu/2$ of the above closed discs is equal to

$$\sum_{i=1}^{m_n}(r_{S^{(i)}_{(n)}}{}^{(\mu/2}+\varepsilon_0{}^{\mu/2}+\sum_{k=2}^{n}(\sum_{i=1}^{m_{k-1}}(\varepsilon_{m_0+\cdots+m_{k-2}+i})^{\mu/2})\ , \tag{2.15}$$

which is less than

$$\sum_{S_{(n)}\in G^*(\alpha_0)}(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}'(r_{S_{(n)}}(T_j))^{\mu/2})+\varepsilon\ . \tag{2.16}$$

From the arbitrariness of ε, it holds

$$M_{\mu/2}(E)\leqq\lim_{n\to\infty}\sum_{S_{(n)}\in G^*(\alpha_0)}(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}'(r_{S_{(n)}}(T_j)^{\mu/2})\ . \tag{2.17}$$

Thus we could prove this lemma completely. q. e. d.

9. Now we shall give the following Lemma due to A. F. Beardon ([3]) and H. Takahashi ([4]).

LEMMA 3. *If* $M_{\mu/2}(E\frown D_T)+M_{\mu/2}(E\frown D_{T^{-1}})=0$ *for some element of* $G^*(\alpha_0)$, *then it holds* $M_{\mu/2}(E)=0$. *If* $M_{\mu/2}(E)>0$ (*or* $=\infty$), *then it holds* $M_{\mu/2}(E\frown D_T)>0$ (*or* $=\infty$) *for any* T $(\in\mathfrak{G}^*(\alpha_0))$.

Proof. Denote by $\mathrm{Ext}\,D_T$ and $\mathrm{Int}\,D_T$ the exterior and interior of D_T, respectively. Since the singular set E is invariant under the mapping $T(z)$ for any T $(\in\mathfrak{G}^*(\alpha_0))$, then $E\frown\mathrm{Ext}\,D_T$ is mapped onto $E\frown\mathrm{Int}\,D_{T^{-1}}$ by $T^{-1}(z)$. We assumed in No. 1 that the circles $\{K_j\}_{j=1}^{p}$ and $\{H_i,\ H_i'\}_{i=p+1}^{\infty}$ are contained in some closed disc $D_0=\{z\ ;\ |z|\leqq\rho_0\}$. Hence it is obvious that

$$E \subset \overline{\bigcup_{T \in \mathfrak{G}^*(\alpha_0)} D_T} \subset D_0\,,$$

where $\overline{\bigcup_{T \in \mathfrak{G}^*(\alpha_0)} D_T}$ is the closure of $\bigcup_{T \in \mathfrak{G}^*(\alpha_0)} D_T$.

Take arbitrary two points z_1 and z_2 in $D(T^{-1}, \rho_0)=T^{-1}(D_0) \cap \operatorname{Int} D_{T^{-1}}$ such that the line segment $\overline{z_1 z_2}$ is contained in $D(T^{-1}, \rho_0)$. Since $T(z)$ is holomorphic in $D(T^{-1}, \rho_0)$, we have from the mean value theorem on the holomorphic function the Lipschitz condition

$$|T(z_1)-T(z_2)|<K|z_1-z_2|, \tag{2.18}$$

where $K=K(T, \rho_0)$ is a constant depending only on T and ρ_0. It is obvious that $T(z_i)$ $(i=1, 2)$ are contained in $\operatorname{Ext} D_T$.

From the assumption $M_{\mu/2}(D_{T^{-1}} \cap E)=0$ there exist a finite number of closed discs U_i $(i=1, \cdots, n)$ satisfying the following conditions:

$$\text{(i)} \qquad D_{T^{-1}} \cap E \subset \bigcup_{i=1}^{n} U_i \subset D_{T^{-1}} \cap T^{-1}(D_0)\,,$$

$$\text{(ii)} \qquad \sum_{i=1}^{n} (r(U_i))^{\mu/2} < \varepsilon \qquad \text{for any small } \varepsilon\ (>0)\,,$$

where $r(U_i)$ denotes the radius of U_i. It is obvious that $\bigcup_{i=1}^{n} U_i$ and $\bigcup_{i=1}^{n} T(U_i)$ are the covering of $D_{T^{-1}} \cap E$ and $E \cap \operatorname{Ext} D_T$, respectively. Then we have from (2.18)

$$\sum_{i=1}^{n} (r(T(U_i)))^{\mu/2} \leqq (K)^{\mu/2} \sum_{i=1}^{n} (r(U_i))^{\mu/2} < K^{\mu/2} \varepsilon\,. \tag{2.19}$$

From the arbitrariness of ε, it holds that $M_{\mu/2}(E \cap \operatorname{Ext} D_T)=0$ from the above condition (ii) and (2.19).

In the same manner as the above it holds that $M_{\mu/2}(E \cap \operatorname{Ext} D_{T^{-1}})=0$ under the assumption $M_{\mu/2}(E \cap D_{T^{-1}})=0$.

Further we can conclude that it follows from (2.19) that $M_{\mu/2}(E \cap D_T)>0$ (or $=\infty$) for any T $(\in \mathfrak{G}^*(\alpha_0))$ under the assumption $M_{\mu/2}(E)>0$ (or $=\infty$) q. e. d.

By using Lemmas 2 and 3 we shall prove the following theorem.

THEOREM 3. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *If $G^*(\alpha_0)$ is the μ-convergent type, then it holds $M_{\mu/2}(E)=0$.*

Proof. From the assumption, it holds $\lim_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z_0)=0$ for some T $(\in \mathfrak{G}^*(\alpha_0))$ and some point $z_0 \in D_T$. Then it is easily seen from Lemma 1 that

$$\lim_{n\to\infty} \chi_{n,\infty}^{(\mu;T^*)}(z)=0 \tag{2.20}$$

for any T^* $(\in \mathfrak{G}^*(\alpha_0))$ and any point $z \in D_{T^*}$.

For an arbitrarily fixed element T^* $(\in \mathfrak{G}^*(\alpha_0))$ we consider all closed discs $D_{S_{(n)}}(T_j)$ $(j=1, 2, \cdots)$ of grade n for $D_T \cap E$, where $S_{(n)}=T_{i_n} \cdots T_{i_1}$ $(T_j \neq T_{i_1}^{-1})$ has the form $T^* S_{(n-1)}$. Then we have from from (2.17) of Lemma 2

(2.21) $$M_{\mu/2}(E\cap D_{T^*})\leqq \sum_{S_{(n)}\in G^*(\alpha_0)}\Big(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}' (r_{S_{(n)}}(T_j))^{\mu/2}\Big),$$

for all radii $r_{S_{(n)}}(T_j)$ ($S_{(n)}\in G^*(\alpha_0)$) of such closed discs $D_{S_{(n)}}(T_j)$. We have from Proposition 3 the following inequality:

(2.22) $$\sum_{S_{(n)}\in G^*(\alpha_0)}\Big(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}' (r_{S_{(n)}}(T_j))^{\mu/2}\Big)\leqq K(G^*(\alpha_0),\mu)\sum_{S_{(n)}\in G^*(\alpha_0)}(R_{S_{(n)}})^{\mu},\ S_{(n)}=T^*S_{(n-1)}.$$

Since $R^{\mu}_{S_{(n)}}=R^{\mu}_{S^{-1}_{(n)}}=(R_{S^{-1}_{(n-1)}T^{*-1}})^{\mu}$, it holds

$$\sum_{S_{(n)}\in G^*(\alpha_0)}\Big(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}' (r_{S_{(n)}}(T_j))^{\mu/2}\Big)$$

(2.23) $$\leqq K(G^*(\alpha_0),\mu)(R_{T^{*-1}})^{\mu}\sum_{S_{(n)}\in G^*(\alpha_0)}(R_{S^{-1}_{(n-1)}T^{*-1}})^{\mu}/(R_{T^{*-1}})^{\mu}$$

$$=K(G^*(\alpha_0),\mu)(R_{T^{*-1}})^{\mu}\chi^{(\mu;T^{*-1})}_{n-1,\infty}(T^{*-1}(\infty)).$$

Hence from (2.20), (2.21), (2.23) and Lemma 2, we have $M_{\mu/2}(E\cap D_{T^*})=0$. Thus it implies from Lemma 3 that $M_{\mu/2}(E)=0$. q. e. d.

10. Next we want to show that $M_{\mu/2}(E)=\infty$ is equivalent to the proposition that $G^*(\alpha_0)$ is the μ-divergent type. For this purpose it suffices to prove the converse of Theorem 2.

THEOREM 4. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *If $M_{\mu/2}(E)=\infty$, then it holds $\lim_{n\to\infty}\chi^{(\mu;T)}_{n,\infty}(z)=\infty$ for any element T ($\in\mathfrak{G}^*(\alpha_0)$) and any point $z\in D_T$. Hence $G^*(\alpha_0)$ is the μ-divergent type.*

Proof. We shall easily see from Lemma 3 that $M_{\mu/2}(E\cap D_T)=\infty$ for any element $T\in\mathfrak{G}^*(\alpha_0)$. Hence we have from Lemma 2

(2.24) $$\lim_{n\to\infty}\sum_{S_{(n)}\in G^*(\alpha_0)}\Big(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}' (r_{S_{(n)}}(T_j))^{\mu/2}\Big)=\infty,\quad S_{(n)}=TS_{(n-1)}.$$

Then from (1.16) of Proposition 3 we have

(2.25) $$\sum_{S_{(n)}\in G^*(\alpha_0)}\Big(\sum_{T_j\in\mathfrak{G}^*(\alpha_0)}{}' (r_{S_{(n)}}(T_j))^{\mu/2}\leqq K(G^*(\alpha_0),\mu)\sum_{S_{(n)}\in G^*(\alpha_0)}(R_{S_{(n)}})^{\mu}$$
$$=K(G^*(\alpha_0),\mu)(R_{T^{-1}})^{\mu}\chi^{(\mu;T^{-1})}_{n-1,\infty}(T^{-1}(\infty)).$$

Since T is any element of $\mathfrak{G}^*(\alpha_0)$, it holds from (2.24), (2.25) and Lemma 1

$$\lim_{n\to\infty}\chi^{(\mu;T)}_{n,\infty}(z)=\infty$$

for any element T ($\in\mathfrak{G}^*(\alpha_0)$) and any point $z\in D_T$. q. e. d.

11. Now the Hausdorff dimension $d(E)$ of the singular set E of $G^*(\alpha_0)$ is defined in the following:

$$d(E)=\sup\Big\{\frac{\mu}{2};\ M_{\frac{\mu}{2}}(E)=\infty\Big\}=\inf\Big\{\frac{\mu}{2};\ M_{\frac{\mu}{2}}(E)=0\Big\}.$$

In the former paper [2], we had the result that the Hausdorff dimension increases strictly according to increment of the number of boundary circles of the fundamental domain. From this result, if we denote the Hausdorff dimension of E_N by $\mu_N/2$, it is obvious that $\mu_N/2<\mu_{N+1}/2<\cdots$ and $\lim_{N\to\infty}\mu_N/2=\mu_0/2=d(E)$. In the finite case, we had in [1] the following result: $0<M_{\mu_N/2}(E_N)<\infty$. But in our case we can presently show only half of this result.

THEOREM 5. *Let $d(E)=\mu_0/2$ be the Hausdorff dimension of the singular set E of $G^*(\alpha_0)$. If $\mu_0/2>\alpha_0$, then it holds that $M_{\mu_0/2}(E)<\infty$.*

Proof. Assume that $M_{\mu_0/2}(E)=\infty$. Then it is easily seen from Theorem 4 that $G^*(\alpha_0)$ is the μ_0-divergent type, that is, $\lim_{n\to\infty}\chi_{n,\infty}^{(\mu_0;T^*)}(z_0)=\infty$ for some element T^* $(\in\mathfrak{G}^*(\alpha_0))$ and some point $z_0\in D_{T^*}$. Then from Lemma 1 it holds $\lim_{n\to\infty}\chi_{n,\infty}^{(\mu_0;T)}(z)=\infty$ for any element T $(\in\mathfrak{G})$ and any point $z\in D_T$. Hence there exists the positive integer N_0 depending only on any large number M (>1) such that it holds

$$\chi_{n,N_0}^{(\mu_0;T)}(z)>M$$

for any element T $(\in\mathfrak{G}_{N_0})$ and any point $z\in D_T$. Since $\chi_{n,N_0}^{(\mu_0;T)}(z)$ is a continuous function of μ for the fixed number n, N_0 and $z\in D_T$, we can take a small number δ (>0) such that it holds

$$\chi_{n,N_0}^{(\mu_0+\delta;T)}(z)>M.$$

Then we find that $M_{(\mu_0+\delta)/2}(E_{N_0})=\infty$ for E_{N_0}, and hence $M_{(\mu_0+\delta)/2}(E)=\infty$. This contradicts that $\mu_0/2$ is the Hausdorff dimension of E. Thus it holds that $M_{\mu_0/2}(E)<\infty$. q. e. d.

Problem. Let $G^*(\alpha_0)$ be a Kleinian group with properties (A) and (B).

1. Does it hold that $\lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z)=0$ for any element T $(\in\mathfrak{G}^*(\alpha_0))$ and any point $z\in D_T$, if $M_{\mu/2}(E)=0$? If it is true, then $G^*(\alpha_0)$ is the μ-convergent type and from Theorem 3 $M_{\mu/2}(E)=0$ is equivalent to the proposition that $G^*(\alpha_0)$ is the μ-convergent type. 2. Let $d(E)=\mu_0/2$ be the Hausdorff dimension of the singular set E of $G^*(\alpha_0)$. Does it hold $0<M_{\mu_0/2}(E)$ for $\mu_0/2>\alpha_0$? If so, is $G^*(\alpha_0)$ the μ_0-finite type, that is, $0<\varliminf_{n\to\infty}\chi_{n,\infty}^{(\mu_0;T)}(z)\leqq\varlimsup_{n\to\infty}\chi_{n,\infty}^{(\mu_0;T)}(z)<\infty$ for some element T $(\in\mathfrak{G}^*(\alpha_0))$ and some point $z\in D_T$? In the finite case G_N is the μ_N-finite type. 3. Does it always hold $\mu_0/2\geqq\alpha_0$ for $G^*(\alpha_0)$?

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