G. TSAGAS
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ON THE SINGULARITIES OF HARMONIC 1-FORMS ON A RIEMANNIAN MANIFOLD

Dedicated to Professor K. Yano on this sixtieth Birthday

BY GRIGORIOS TSAGAS

1. Introduction. Let M be a compact Riemannian manifold of diminsion n. We consider a vector field X on the manifold M. Then the point P is called a singular point for the vector field X, if $X(P)=0$. It is known that if every vector field X on M does not have singularities, then the index of X is zero and therefore the Euler characteristic of the manifold $k(M)=0$, ([1], p. 549) ([2], p. 203).

A harmonic 1-form ξ on the manifold M is a special covariant vector field on M. The purpose of the present paper is to show that if the manifold is compact of even dimension and admits a metric whose sectional curvature is negative δ-pinched, then every harmonic 1-form on M has a singularity.

2. We assume that the Riemannian manifold M is compact and even dimension. If ξ is a harmonic 1-form, to this harmonic 1-form we associate by the duality of the metric a contravariant vector field X.

Let P be a point of the manifold M. We consider a normal coordinate neighborhood U of P with normal coordinate system $(x^1, \cdots, x^n)$ at P. The Riemannian metric g, the harmonic 1-form ξ and the vector field X have, in the neighborhood U, components $g=(g_{ij})$, $\xi=(\xi_i)$ and $X=(X^j=g^{ji}\xi_i)$, respectively.

If α, β are two vector fields on the manifold M their local inner product is defined

$$(\alpha, \beta)=\alpha^i\beta_i=\alpha_i\beta^i$$

and the norm of a vector field α is defined by

$$|\alpha|^2=\alpha^i\alpha_i$$

and for the harmonic 1-form ξ we have

$$|\xi|^2=\xi^i\xi_i\,. \tag{2.1}$$

On the manifold M we consider a function defined as follows

$$\begin{aligned} f&: M \longrightarrow IR \\ f&: P \longrightarrow f(P)=|\xi|^2{}_P=(\xi_i\xi^i)_P \end{aligned} \tag{2.2}$$

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The function f is continuous on the manifold M. Since this manifold is compact and the function f is continuous, on this compact manifold, there exist a point P at which f attains the minimum at P, that is

$$f(Q) \geqq f(P)$$

For every Q in a neighborhood of P.

Since the function $f=|\xi|^2$ has a minimum at the point P, then we have

$$(d(|\xi|^2))_P = \langle \nabla\xi, X\rangle_P = (\nabla\xi, X)_P = 0 .$$

It is known, that the harmonic 1-form ξ, in local condinates, statisfies the relations

$$\nabla_i\xi_j - \nabla_j\xi_i = 0 \qquad \nabla_i\xi^i = 0 . \tag{2.4}$$

Let $T_P(M)$ be the tangent space of M at the point P. From the first relation of (2.4) we conclude that $(\nabla\xi)_P$ can be considered as a bilinear symmetric form on the vector space $T_P(M)$.

From the linear Algebra the following theorem is known.

THEOREM (1) *Let F be a bilinear symmetric form on the vector space $T_P(M)$. Then there exists a basse $\{E_1, \cdots, E_n\}$ of $T_P(M)$ such that we have $F(E_i, E_j)=0$, for $i \neq j$, $F(E_i, E_j)=1$ for $1 \leqq i \leqq p$, $F(E_i, E_j)=-1$ for $p+1<k<r$ and $F(E_i, E_j)=0$ for $r+1 \leqq j \leqq n$. The number r is the rank of F and p is an integer $0 \leqq p \leqq r$, which is uniquely determined by F.*

From the relation (2.3) and the second relation of (2.4) the fact that the dimension of the vector space $T_p(M)$ is even dimension from the above theorem we conclude that the null space of $(\nabla\xi)_p$ is at least dimension 2.

Therefore there is another unit vector t perpendicular to X for which we have

$$\langle(\nabla\xi)_P t\rangle = ((\nabla\xi)_P t) = 0 \tag{2.5}$$

We obtain the covariant derivative of the function $f=|\xi|^2=\xi_i\xi^i$ in the direction of the vector t, then we have

$$\nabla_t\nabla_t(|\xi|^2) = 2\langle\nabla_t X, \nabla_t X\rangle + 2\langle X, \nabla_t\nabla_t\xi\rangle \tag{2.6}$$

or

$$(\nabla_t\nabla_t(|\xi|^2)_P = 2(\langle(\nabla\xi)_p, t\rangle)^2 + (2\langle X, \nabla_t\nabla_t\xi\rangle)_P . \tag{2.7}$$

The relation (2.7) by means of (2.5) takes the form

$$(\nabla_t\nabla_t(|\xi|^2))_P = 2(\langle X, \nabla_t\nabla_t\xi\rangle)_P . \tag{2.8}$$

Since the exterior 1-form ξ is harmonic, then it satisfies the relation ([4], p. 42)

$$g^{bc}\nabla_b\nabla_c\xi_i = R_{ij}\xi^j , \qquad i=1, \cdots, n . \tag{2.9}$$

The relation (2.9) for the point P, which is the origin of the normal coordinate system $(x^1, \cdots, x^n)$, gives

$$(\nabla_1\nabla_1\xi_i+\cdots+\nabla_n\nabla_n\xi_i)_P=(\sum_{j=1}^{n}R_{ij}\xi^j)_P\,,\qquad i=1,\cdots,n \tag{2.10}$$

which can by simply written

$$\nabla_1\nabla_1\xi_i+\cdots+\nabla_n\nabla_n\xi_i=\sum_{j=1}^{n}R_{ij}\xi_j\,,\qquad i=1,\cdots,n \tag{2.11}$$

the above notation will be used below, that means, we evaluate all the relations at the point P.

From (2.4) we obtain

$$\nabla_k\nabla_i\xi_j-\nabla_k\nabla_j\xi_i=0\,,\qquad k=1,\cdots,n\,,\qquad i\neq j=1,\cdots,n\,, \tag{2.12}$$

$$\nabla_k\nabla_1\xi_1+\nabla_k\nabla_2\xi_2+\cdots+\nabla_k\nabla_n\xi_n=0\,,\qquad k=1,\cdots,n\,. \tag{2.13}$$

We also have the formula

$$\nabla_k\nabla_j\xi_i-\nabla_j\nabla_k\xi_i=-\sum_{l=1}^{n}R_{lijk}\xi_l\,,\qquad k\neq j\,,\qquad i=1,\cdots,n\,. \tag{2.14}$$

The relations (2.11), (2.12), (2.13) and (2.14) form a system of n^3-n^2+2n equations with $\nabla_k\nabla_j\xi_i$ unknowns $i, j, k=1,\cdots,n$.

If we put

$$\nabla_i\nabla_i\xi_k=0\,,\qquad i,\, k=1,\cdots,n\,,\qquad i\neq k \tag{2.15}$$

then from (2.11) we obtain

$$\nabla_i\nabla_i\xi_i=\sum_{j=1}^{n}R_{ij}\xi_j\,. \tag{2.16}$$

From (2.13) and for $k=1$ we have

$$\nabla_1\nabla_1\xi_1+\nabla_1\nabla_2\xi_2+\cdots+\nabla_1\nabla_n\xi_n=0 \tag{2.17}$$

which by means of (2.14) and (2.16) takes the form

$$\sum_{j=1}^{n}R_{1j}\xi_j+\nabla_2\nabla_1\xi_2-\sum_{l=1}^{n}R_{l22l}\xi_l+\cdots+\nabla_n\nabla_1\xi_n-\sum_{l=1}^{n}R_{lnn1}\xi_l=0\,. \tag{2.18}$$

The relation (2.18) by virtue of (2.12) becomes

$$\sum_{i=1}^{n}R_{1j}\xi_j+\nabla_2\nabla_2\xi_1-\sum_{l=1}^{n}R_{l221}\xi_l+\cdots+\nabla_n\nabla_n\xi_1-\sum_{l=1}^{n}R_{lnn1}\xi_l=0$$

which by means of (2.15) takes the form

$$\sum_{j=1}^{n}R_{1j}\xi_j-\sum_{l=1}^{n}R_{l221}\xi_l-\cdots-\sum_{l=1}^{n}R_{lnn1}\xi_l=0$$

from which we obtain

$$\begin{aligned}&(R_{11}-R_{1221}-\cdots-R_{1nn1})\xi_1\\&\qquad+(R_{12}-R_{2331}-\cdots-R_{2nn1})\xi_2\\&\qquad\qquad+\cdots+(R_{1n}-R_{n221}-\cdots-R_{nn-1n-11})\xi_n=0\,.\end{aligned} \tag{2.19}$$

We also have the formula

$$R_{jk}=\sum_{s=1}^{n} R_{sjks}$$

from which we obtain

$$\begin{aligned}
&R_{11}=R_{2112}+R_{3113}+\cdots+R_{n11n}\\
&R_{12}=R_{3123}+R_{4124}+\cdots+R_{n12n}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&R_{1n}=R_{21n2}+R_{31n3}+\cdots+R_{n-11n\,n-1}\,.
\end{aligned} \tag{2.20}$$

The relation (2.19), by means of (2.20), is true and therefore (2.17) and from this all the relations (2.13).

From the above we conclude that we have determined n^2 unknowns $\nabla_i\nabla_i\xi_j$ $i=1,\cdots,n$ $j=1,\cdots,n$. Their values are given by (2.15) and (2.16). Therefore we have to determine n^3-n^2 unknowns. These unknowns satisfy n^3-n^2 equations, which are (2.12) and (2.14)

From (2.12) and (2.15) we obtain

$$\nabla_i\nabla_k\xi_i=0\,,\qquad i\neq k\,,\qquad i,\,k=1,\cdots,n\,. \tag{2.21}$$

Similarly from (2.14) and (2.21) we have

$$\nabla_k\nabla_i\xi_i=-\sum_{l=1}^{n} R_{liik}\xi_l\,,\qquad i\neq k\,,\qquad i,\,k=1,\cdots,n\,. \tag{2.22}$$

The other unknowns are determined by virtue of the following system

$$\nabla_i\nabla_j\xi_k=\nabla_i\nabla_k\xi_j\,,\qquad \nabla_j\nabla_k\xi_i=\nabla_j\nabla_i\xi_k\,,\qquad \nabla_k\nabla_i\xi_j=\nabla_k\nabla_j\xi_i\,, \tag{2.23}$$

$$\nabla_i\nabla_j\xi_k-\nabla_j\nabla_i\xi_k=-\sum_{l=1}^{n} R_{lkji}\xi_l\,, \tag{2.34}$$

$$\nabla_j\nabla_k\xi_i-\nabla_k\nabla_j\xi_i=-\sum_{l=1}^{n} R_{likj}\xi_l\,, \tag{2.35}$$

$$\nabla_i\nabla_k\xi_j-\nabla_k\nabla_i\xi_j=-\sum_{l=1}^{n} R_{ljki}\xi_l\,, \tag{2.26}$$

$1\leqq i<j<k\leqq n$, in which we have six equations with six unknowns. One solution of this system is

$$\nabla_i\nabla_j\xi_k=\nabla_i\nabla_k\xi_j=0\,, \tag{2.27}$$

$$\nabla_j\nabla_i\xi_k=\nabla_j\nabla_k\xi_i=-\sum_{l=1}^{n} R_{lkij}\xi_l\,, \tag{2.28}$$

$$\nabla_k\nabla_i\xi_j=\nabla_k\nabla_j\xi_i=-\sum_{l=1}^{n} R_{ljik}\xi_l\,. \tag{2.29}$$

Let $\{e_n,\cdots,e_n\}$ be an orthonormal base of $T_P(M)$. We assume that the vector t has components $(t^1,\cdots t^n)$ with respect to this base. Then we have

$$(\nabla_t\nabla_t\xi_\iota)_P=\sum_{\lambda=1}^n\sum_{\mu=1}^n t^\lambda t^\mu\nabla_{e_\lambda}\nabla_{e_\mu}\xi_i=\sum_{\lambda=1}^n\sum_{\mu=1}^n t^\lambda t^\mu\nabla_\lambda\nabla_\mu\xi_\iota\,. \tag{2.30}$$

The relation (2.8) by means of (2.30) takes the form

$$\begin{aligned}(\nabla_t\nabla_t(|\xi|^2))_P&=2(\xi_1\nabla_t\nabla_t\xi_1+\cdots+\xi_n\nabla_t\nabla_t\xi_n)\\&=2\xi_1\sum_{\lambda=1}^n\sum_{\mu=1}^n t^\lambda t^\mu\nabla_\lambda\nabla_\mu\xi_1+\cdots+2\xi_n\sum_{\lambda=1}^n\sum_{\mu=1}^n t^\lambda t^\mu\nabla_\lambda\nabla_\mu\xi_n\,.\end{aligned} \tag{2.31}$$

From the relation (2.31) and by means of (2.15), (2.16), (2.20), (2.21), (2.22), (2.27), (2.28) and (2.29) we obtain

$$\begin{aligned}1/2(\nabla_t\nabla_t(|\xi|^2))_P&=-R_{1212}(t^1\xi_1-t^2\xi_2)^2\\&-t^1t^2\{\sum_{l=3}^n(R_{l112}\xi_1+R_{l221}\xi_2)\xi_l\}+t^1t^2(\sum_{l=3}^n(\nabla_2\nabla_1\xi_l)\xi_l)-\cdots-R_{1n1n}(t^1\xi_1-t^n\xi_n)^2\\&-t^1t^n\{\sum_{l=2}^{n-1}(R_{l11n}\xi_1+R_{lnn1}\xi_n)\xi_l\}+t^1t^n(\sum_{l=2}^{n-1}(\nabla_n\nabla_1\xi_l)\xi_l-R_{2323}(t^2\xi_2-t^3\xi_3)^2\\&-t^2t^3\{\sum_{l\neq1,\neq2,3}^n(R_{l223}\xi_2+R_{l332}\xi_3)\xi_l\}\\&+t^2t^3\{(\nabla_2\nabla_3\xi_1)\xi_1+\sum_{l=1,\neq2,3}^n(\nabla_3\nabla_2\xi_l)\xi_l\}-\cdots-R_{2n2n}(t^2\xi_2-t^n\xi_n)^2\\&-t^2t^n\{\sum_{l=l,\neq2}^{n-1}(R_{l22n}\xi_2+R_{lnn2}\xi_n)\xi_l\}+t^2t^n\{(\nabla_2\nabla_n\xi_1)\xi_1+\sum_{l=1,\neq2}^{n-1}(\nabla_n\nabla_2\xi_l)\xi_l\}\\&-R_{3434}(t^3\xi_3-t^4\xi_4)^2-t^3t^4\{\sum_{l=1\neq3,4}^n R_{l334}\xi_3+R_{l443}\xi_4)\xi_l\}\\&+t^3t^4\{(\nabla_3\nabla_4\xi_1)\xi_1+(\nabla_3\nabla_4\xi_2)\xi_2+\sum_{l=1,\neq3,4}^n(\nabla_4\nabla_3\xi_l)\xi_l\}-\cdots-R_{n-1n-1n}(t^{n-1}\xi_{n-1}-t^n\xi_n)^2\\&-t^{n-1}t^n\{\sum_{l=1}^{n-2}(R_{ln-1n-1n}\xi_{n-1}+R_{lnnn-1}\xi_n)\xi_l\}+t^{n-1}t^n\{\sum_{l=1}^{n-2}(\nabla_{n-1}\nabla_n\xi_l)\xi_l+\sum_{l=1}^{n-2}(\nabla_n\nabla_{n-1}\xi_l)\xi_l\}\\&+(t^1)^2(R_{12}\xi_1\xi_2+\cdots+R_{1n}\xi_1\xi_n)\\&+(t^2)^2(R_{21}\xi_2\xi_1+\cdots+R_{2n}\xi_2\xi_n)+\cdots+(t^n)^2(R_{n1}\xi_n\xi_1+\cdots+R_{n-1n}\xi_{n-1}\xi_n)\,.\end{aligned}$$

The relation (2.32) by virtue of

$$\begin{aligned}R_{\alpha\beta}=\sum R_{s\alpha\beta s}&=R_{1\alpha\beta1}+\cdots+R_{\alpha-1\alpha\beta\alpha-1}+R_{\alpha+1\alpha\beta\alpha+1}\\&+\cdots+R_{\beta-1\alpha\beta\beta-1}+R_{\beta+1\alpha\beta\beta+1}+\cdots+R_{n\alpha\beta n}\,,\\&\alpha,\beta=1,\cdots,n\,,\qquad\alpha<\beta\,,\end{aligned}$$

(2.28) and (2.29) takes the form

$$\begin{aligned}&(\nabla_t\nabla_t(|\xi|^2)_P\\&=-R_{1212}(t^1\xi_1-t^2\xi_2)^2-2t^1t^2(\sum_{l=3}^n R_{l221}\xi_2\xi_l)\\&\qquad-\cdots-R_{1n1n}(t^1\xi_1-t^n\xi_n)^2-2t^1t^n(\sum_{l=2}^n R_{lnn1}\xi_n\xi_l)\end{aligned}$$

$$-R_{2323}(t^2\xi_2-t^3\xi_3)^2-2t^2t^3(\sum_{l=1,\neq 2,3}^{n} R_{l332}\xi_l\xi_3+\sum_{l=1,\neq 3}^{n} R_{12l3}\xi_1\xi_l)$$

$$-\cdots-R_{2n2n}(t^2\xi_2-t^n\xi_n)^2-2t^2t^n(\sum_{l=1\neq 2}^{n-1} R_{lnn2}\xi_l\xi_n+\sum_{l=1}^{n-1} R_{12ln}\xi_1\xi_l)$$

$$-R_{3434}(t^3\xi_3-t^4\xi_4)^2-2t^3t^4(\sum_{l=1,\neq 3,4}^{n} R_{l443}\xi_4\xi_l+\sum_{l=1\neq 4}^{n} R_{13l4}\xi_1\xi_l+\sum_{l=1\neq 4}^{n} R_{23l4}\xi_2\xi_l)$$

$$-\cdots-R_{n-1nn-1n}(t^{n-1}\xi_{n-1}-t^n\xi^n)^2$$

$$-2t^{n-1}t^n(\sum_{l=1}^{n-2} R_{lnnn-1}\xi_l\xi_n+\sum_{l=1}^{n-2} R_{1n-1ln}\xi_1\xi_l+\sum_{l=1}^{n-1} R_{2n-1ln}\xi_2\xi_l$$

$$+\cdots+\sum_{l=1}^{n-1} R_{n-2n-1ln}\xi_{n-2}\xi_1)$$

$$+(t^1)^2\sum_{s=1,\neq l}^{n}\sum_{1=2}^{n} R_{s1ls}\xi_1\xi_l+\cdots+(t^n)^2\sum_{s=1,\neq l}^{n-1}\sum_{l=1}^{n-1} R_{snls}\xi_l\xi_n\,. \tag{2.33}$$

We assume that the Riemannian manifold M is negative δ-pinched, that means its sectional curvature $\sigma(\lambda)$ satisfies the inequalities

$$-1\leqq\sigma(\lambda)\leqq-\delta \tag{2.34}$$

for every $\lambda\in T_P(M)$ and $P\in M$.

It is known, that the following formulas hold ([3], p. 477)

$$\langle R(e_i, e_j)e_k, e_l\rangle=R_{ijkl}\,,\qquad \sigma(e_i, e_j)=\sigma_{ij}=-R_{ijij}\,. \tag{2.35}$$

Where R_{ijkl} are the components of the Riemannian curvature.

The components of the Riemannian curvature satisfy the inequalities

$$|R_{ijik}|\leqq(1-\delta)/2\,,\qquad |R_{ijkl}|\leqq 2(1-\delta)/3\,,\qquad i\neq j\neq k\neq l\,. \tag{2.36}$$

From (2.36) we obtain the inequalities

$$R_{ijik}(t^j)^2\xi_j\xi_k\leqq\frac{\varepsilon(1-\delta)}{2}(t^j)^2\xi_j\xi_k\,, \tag{2.37}$$

$$-R_{ijik}(\xi_i)^2t^jt^k\leqq\frac{\varepsilon(1-\delta)}{2}(\xi_i)^2t^jt^k\,, \tag{2.38}$$

$$-R_{ijik}\xi_i\xi_jt^it^k\leqq\frac{\varepsilon(1-\delta)}{2}\xi_i\xi_jt^it^k\,, \tag{2.39}$$

$$-R_{ijkl}\xi_i\xi_kt^jt^l\leqq\frac{2\varepsilon(1-\delta)}{3}\xi_i\xi_kt^jt^l\,, \tag{2.40}$$

where $\varepsilon=+1$, or -1.

We also have the inequalities

$$|\xi_it^j|\leqq C\,,\qquad |\xi_it^i-\xi_jt^j|\leqq 2C\,,\qquad i, j=1, \cdots, n\,. \tag{2.41}$$

The inequalities (2.37), (2.38), (2.39) and (2.40) by virtue of the first of (2.40), can be written

$$-R_{ijik}(t^j)^2\xi_j\xi_k \leqq \frac{C^2(1-\delta)}{2}\,, \qquad -R_{ijik}(\xi_i)^2 t^j t^k \leqq \frac{C^2(1-\delta)}{3}, \tag{2.42}$$

$$-R_{ijik}\xi_i\xi_j t^i t^k \leqq \frac{C^2(1-\delta)}{2}\,, \qquad -R_{ijkl}\xi_i\xi_k t^i t^l \leqq \frac{2c^2(1-\delta)}{3}\,. \tag{2.43}$$

The relation (2.33) by means of (2.34), the second of (2.35), (2.42) and (2.43) becomes

$$(\nabla_t\nabla_t(|\xi|^2))_P \leqq \frac{2C^2 n(n-1)}{9}\{n^2+n-6-\delta(n^2+n+3)\}\,. \tag{2.44}$$

If the number δ satisfies the inequality

$$\delta > \frac{n^2+n-6}{n^2+n+3} \tag{2.45}$$

then

$$(\nabla_t\nabla_t(|\xi|^2))_P < 0\,. \tag{2.46}$$

Since the function $f=|\xi|^2$ has a minimum at the point P and the minimum value is different from zero, then we have $(\nabla_t\nabla_t(|\xi|^2))_P>0$. If the sectional curvature of the manifold is negative δ-pinched $\delta>(n^2+n-6)/(n^2+n+3)$ then in order the inequality (2.45) is valid we must have $|\xi|^2{}_P=0$ and therefore the harmonic 1-form ξ has a singularity at the point P.

Therefore we have the theorem

THEOREM (II). *Let M be a compact negative δ-pinched Riemannian manifold of even dimension. If $\delta>(n^2+n-6)/(n^2+n+3)$, then every harmonic 1-form on M has a singularity.*

From the above theorem we obtain the corollary.

COROLLARY (III). *Let M be a compact Riemannian manifold of even dimension. If the sectional curvature of M is constant negative, then every harmonic 1-form on M has a singularity.*

3. If the dimension of the manifold M is 2, then the formula (2.32) takes the form

$$(\nabla_t\nabla_t(|\xi|^2))_P = R_{1212}(t^1\xi_1 - t^2\xi_2)^2\,. \tag{3.1}$$

From the formula (3.1) we have the theorem.

THEOREM (IV). *Let M be a compact Riemannian manifold of two dimension. If the sectional curvature of the manifold is strictly negative, then every harmonic 1-form on M has a singularity.*

We assume that M is a compact surface with h handles where $h \geqq 2$, that means the genus of the surface M is $\geqq 2$. Let $\xi \in H^1(M, IR)$ be a harmonic 1-form. The vector space $H^1(M, IR)$ is independent of the Riemannian metric on the surface. It is known that a compact surface of genus greater or equal to two admits a metric with a constant negative Gaussian curvature.

From this we obtain the following theorem.

THEOREM (V). *Let M be a compact surface of genus $g \geqq 2$. Then every harmonic* 1*-form on M has a singularity.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PATRAS
PATRAS, GREECE.