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ON THE FAMILY OF ANALYTIC MAPPINGS AMONG ULTRAHYPERELLIPTIC SURFACES

Dedicated to Professor Yûsaku Komatu on his sixtieth birthday

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§ **1.** Let R(resp. S) be an ultrahyperelliptic surface defined by an equation $y^2=G(z)$(resp. $u^2=g(w)$), where G(resp. g) is an entire function having no zero other than an infinite number of simple zeros.

Let φ be a non-trivial analytic mapping of R into S. Then

$$h(z)=\mathcal{P}_S\circ\varphi\circ\mathcal{P}_R^{-1}(z)$$

is an entire function, where $\mathcal{P}_R$(resp. $\mathcal{P}_S$) is the projection map $(z, y)\to z$(resp. $(w, u)\to w$) [4]. This entire function $h(z)$ is called the projection of the analytic mapping φ.

In this paper we shall prove the following theorem.

THEOREM. *Let R and S be two ultrahyperelliptic surfaces. Suppose that there exists a non-trivial analytic mapping φ of R into S such that the projection of φ is a transcendental entire function. Then there is no non-trivial analytic mapping ψ of R into S such that the projection of ψ is a polynomial.*

Under some restrictions on R and S, Niino proved the above fact [3] (cf. [2]).

§ **2.** To prove our theorem we need the following two lemmas. The standard symbols of the Nevanlinna theory are used throughout the paper.

LEMMA 1 [4]. *If there exists a non-trivial analytic mapping φ of R into S, then the projection $h(z)$ of φ satisfies an equation*

$$f(z)^2G(z)=g\circ h(z) \tag{1}$$

with a suitable entire function $f(z)$. Conversely, if a non-constant entire function $h(z)$ satisfies the equation (1) with a suitable entire function $f(z)$, then there exists a non-trivial analytic mapping φ of R into S such that the projection of φ is $h(z)$.

LEMMA 2 (cf. [1]). *Let $h(z)$ be a transcendental entire function. For given three numbers A, B and C there is a number R_0 (>0) and an increasing sequence*

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$\{R_n\}_{n=1}^{\infty}$ with $R_n \to \infty$ $(n\to\infty)$ such that for all n $(\geqq 1)$ and all r in $[R_n, R_n{}^A]$ and all ω satisfying $R_0 \leqq |\omega| \leqq r^B$ we have

$$n\left(r, \frac{1}{h-\omega}\right) > C. \tag{2}$$

Proof. We can prove this lemma along the same line as in [1]. Suppose at first that there is a number $\tilde{R}$ such that for all $r(\geqq \tilde{R})$ and all ω satisfying $|\omega| = r^B$ we have

$$n\left(r, \frac{1}{h-\omega}\right) > C.$$

In this case our assertion holds for every increasing sequence $\{R_n\}_{n=1}^{\infty}$ with $R_1 \geqq \tilde{R}$ and $R_0 \geqq \tilde{R}^B$.

Suppose next that the above is false, that is, for arbitrary large r there exists an ω satisfying $|\omega| = r^B$ such that

$$n\left(r, \frac{1}{h-\omega}\right) \leqq C.$$

We choose δ so that $|\delta| > |h(0)|$ and

$$N\left(r, \frac{1}{h-\delta}\right) \sim T(r, h) \qquad (r\to\infty). \tag{3}$$

Now put $R_0 = |h(0)| + |\delta| + 1$. Let $\{R_n\}_{n=1}^{\infty}$ be an increasing sequence with $R_1 > R_0$ and $R_n \to \infty$ $(n\to\infty)$ such that for all $n(\geqq 1)$ there is an ω satisfying $|\omega| = R_n{}^{2AB}$ and

$$n\left(R_n{}^{2A}, \frac{1}{h-\omega}\right) \leqq C. \tag{4}$$

Assume that for arbitrary large n the statement of our Lemma does not hold where R_0 and $\{R_n\}_{n=1}^{\infty}$ are defined above. Then for such n there is an Ω, depending on n, such that $R_0 \leqq |\Omega| \leqq R_n{}^{AB}$ and

$$n\left(\rho, \frac{1}{h-\Omega}\right) \leqq C \qquad (\rho \leqq R_n). \tag{5}$$

Choose ρ to satisfy $R_n/2 \leqq \rho \leqq R_n$ such that

$$m\left(\rho, \frac{h'}{h-\omega}\right) = o(T(\rho, h)), \qquad (n\to\infty), \tag{6}$$

$$m\left(\rho, \frac{h'}{h-\Omega}\right) = o(T(\rho, h)), \qquad (n\to\infty). \tag{7}$$

The relations (6) and (7) can be derived from the choice of ω and Ω, since $h(z)$ is transcendental. Hence by (5), (6) and (7) we have

$$T\left(\rho, \frac{h'}{h-\omega}\right) = o(T(\rho, h)), \qquad (n\to\infty), \tag{8}$$

$$T\left(\rho, \frac{h'}{h-\Omega}\right) = o(T(\rho, h)), \qquad (n\to\infty). \tag{9}$$

Put $k=(\delta-\Omega)/(\omega-\delta)$ and consider

$$H(z)=\frac{h'(z)}{h(z)-\omega}+k\frac{h'(z)}{h(z)-\Omega}=\frac{(\omega-\Omega)h'(z)(h(z)-\delta)}{(\omega-\delta)(h(z)-\omega)(h(z)-\Omega)}\,.$$

Then

$$N\Big(\rho, \frac{1}{H}\Big)\geqq N\Big(\rho, \frac{1}{h-\delta}\Big)=(1+o(1))T(\rho, h)\,. \tag{10}$$

By (10) and the choice of δ, ω and Ω yield

$$T(\rho, H)\geqq(1+o(1))(T(\rho, h))\,. \tag{11}$$

On the other hand, by (8) and (9)

$$T(\rho, H)=o(T(\rho, h))\,. \tag{12}$$

The relations (11) and (12) are mutually incompatible for large n. Consequently we can see that there is a number $R_0(>0)$ and a sequence $\{R_n\}_{n=1}^{\infty}$ with the properties given in the statement of the Lemma.

§ **3.** We shall prove our theorem.

Proof of theorem. Suppose that there exists a pair of two ultrahyperelliptic surfaces R and S such that there exist two non-trivial analytic mappings φ_1 and φ_2 with the projections $p(z)$ and $h(z)$, respectively, where $p(z)$ is a polynomial and $h(z)$ is a transcendental entire function. Then by Lemma 1 we have

$$f_1(z)^2G(z)=g\circ p(z)\,, \tag{13}$$

$$f_2(z)^2G(z)=g\circ h(z)\,, \tag{14}$$

where f_1 and f_2 are suitable entire functions.

Put $p(z)=\alpha z^{\nu}+\beta z^{\nu-1}+\cdots+\gamma$ $(\alpha\neq 0)$. Then for given ε $(0<\varepsilon<1)$

$$n\Big(r, \frac{1}{g\circ p}\Big)\leqq \nu n\Big(|\alpha|r^{\nu}(1+\varepsilon), \frac{1}{g}\Big)+O(1)\,.$$

Hence

$$N\Big(r, \frac{1}{g\circ p}\Big)\leqq N\Big(|\alpha|r^{\nu}(1+\varepsilon), \frac{1}{g}\Big)+O(\log r)\,.$$

Since g is transcendental, by (13)

$$N\Big(r, \frac{1}{G}\Big)\leqq N\Big(r, \frac{1}{g\circ p}\Big)\leqq(1+\varepsilon)N\Big(|\alpha|r^{\nu}(1+\varepsilon), \frac{1}{g}\Big)\,. \tag{15}$$

This inequality holds for all large r.

By (14) we have

$$N\Big(r, \frac{1}{f_2}\Big)\leqq N\Big(r, \frac{1}{h'}\Big)$$
$$\leqq T(r, h')+O(1)\leqq T(r, h)+O(\log rT(r, h))\leqq 2T(r, h)$$

outside a set E of finite measure, since $h(z)$ is a transcendental entire function.

On the other hand, by the second fundamental theorem, we have

$$\tilde{K}T(r, h) \leqq N\left(r, \frac{1}{g \circ h}\right)$$

for arbitrary but fixed constant $\tilde{K}$, if $r \notin E$. Hence we have

$$N\left(r, \frac{1}{G}\right) \geqq (1-\varepsilon) N\left(r, \frac{1}{g \circ h}\right) \tag{16}$$

outside the set E. By (15) and (16) we get

$$N\left(|\alpha| r^{\nu}(1+\varepsilon), \frac{1}{g}\right) \geqq \frac{1-\varepsilon}{1+\varepsilon} N\left(r, \frac{1}{g \circ h}\right), \tag{17}$$

which holds outside the set E.

Now we apply our Lemma 2 for $A=3$, $B=\nu+1$, $C=4(\nu+1)$ and $h(z)$. Let $\{R_n\}_{n=1}^{\infty}$ be a sequence satisfying the statement of the Lemma 2.

Let $\{w_\nu\}_{\nu=1}^{\infty}$ be the zeros of $g(w)$. Choose r_n satisfying $R_n^2 \leqq r_n \leqq R_n^3$ and $r_n \notin E$. Then, for large n,

$$\begin{aligned} N\left(r_n, \frac{1}{g \circ h}\right) &\geqq \int_{R_n}^{r_n} \frac{n(t, 1/g \circ h)}{t} dt \\ &\geqq \int_{R_n}^{r_n} \frac{1}{t} \left\{ \sum_{w_\nu, R_0 \leqq |w_\nu| \leqq M(r_n, h)} n\left(t, \frac{1}{h-w_\nu}\right) \right\} dt \\ &\geqq 4(\nu+1) \int_{R_n}^{r_n} \frac{n(t^{\nu+1}, 1/g) - n(R_0, 1/g)}{t} dt \\ &\geqq 4 \int_{R_n^{\nu+1}}^{r_n^{\nu+1}} \frac{n(t, 1/g)}{t} dt - O(\log r_n) \\ &\geqq 4N\left(r_n^{\nu+1}, \frac{1}{g}\right) - N\left(R_n^{\nu+1}, \frac{1}{g}\right) - O(\log r_n) \\ &\geqq 2N\left(r_n^{\nu+1}, \frac{1}{g}\right). \end{aligned} \tag{18}$$

By (17) and (18), as $n \to \infty$,

$$\begin{aligned} N\left(r_n, \frac{1}{g \circ h}\right) &\geqq 2N\left(r_n^{\nu+1}, \frac{1}{g}\right) \\ &\geqq 2N\left(|\alpha| r_n^{\nu}(1+\varepsilon), \frac{1}{g}\right) \geqq 2\frac{1-\varepsilon}{1+\varepsilon} N\left(r_n, \frac{1}{g \circ h}\right). \end{aligned}$$

It is untenable. This completes the proof of our theorem.

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