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NOTES ON A *K*-SPACE OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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§ 1. Introduction. Let M be an almost Hermitian manifold of dimension n with an almost Hermitian structure (F, g), i. e. with an almost complex structure tensor F and a positive definite Riemannian metric tensor g satisfying $F^2=-I$ and $g(FX, FY)=g(X, Y)$ for any vector fields X and Y on M where I denotes the identity transformation. By R we denote the Riemannian curvature tensor; $R(X, Y)Z=[\nabla_X, \nabla_Y]Z-\nabla_{[X,Y]}Z$ where ∇ is the operator of covariant differentiation with respect to g.

If M satisfies

$$(1.1) \qquad (\nabla_X F)Y+(\nabla_Y F)X=0 \qquad \text{(or equivalently } (\nabla_X F)X=0)$$

for any vector fields X and Y on M, then M is called a K-space (or Tachibana space or nearly Kähler manifold). A Kähler manifold is a K-space but a K-space is not necessarily a Kähler manifold [2].

Now, let M be an almost Hermitian manifold and $T_m(M)$ a tangent space of M at a point $m\in M$. Then the holomorphic sectional curvature $H(X)$ with respect to a unit vector $X\in T_m(M)$ is defined by $H(X)=-g(R(X, FX)X, FX)$. If $H(X)$ is always constant with respect to any unit vector $X\in T_m(M)$, then M is said to be of constant holomorphic sectional curvature at a point $m\in M$. Moreover, if $H(X)$ is of constant holomorphic sectional curvature at every point $m\in M$, then M is said to be of constant holomorphic sectional curvature. Since the constant $H(X)$ here depends on the point $m\in M$, we shall write $c(m)$ instead of $H(X)$.

It is well known that if a Kähler manifold M is of constant holomorphic sectional curvature $c(m)$ at every point $m\in M$, then the Riemannian curvature tensoı of M, $R(X, Y, Z, W)=g(R(X, Y)Z, W)$ is of the form

$$(1.2) \qquad R(X, Y, Z, W) = \frac{c(m)}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W) + g(X, FW)g(Y, FZ)-g(X, FZ)g(Y, FW)-2g(X, FY)g(Z, FW)\}$$

for any vectors $X, Y, Z, W\in T_m(M)$ and the scalar $c(m)$ is an absolute constant [9].

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The purpose of this note is to prove the following theorem which is a generalization of the above result to a K-space and its some applications will be stated in § 4.

THEOREM. *If a K-space M is of constant holomorphic sectional curvature $c(m)$ at every point $m \in M$, then the Riemannian curvature tensor of M is of the form*

$$
\begin{aligned}
(1.3)\qquad & R(X, Y, Z, W) \\
&= \frac{c(m)}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W)+g(X, FW)g(Y, FZ) \\
&\qquad -g(X, FZ)g(Y, FW)-2g(X, FY)g(Z, FW)\} \\
&\quad +\frac{1}{4}\{g((\nabla_X F)W, (\nabla_Y F)Z)-g((\nabla_X F)Z, (\nabla_Y F)W) \\
&\qquad -2g((\nabla_X F)Y, (\nabla_Z F)W)\}
\end{aligned}
$$

for any vectors $X, Y, Z, W \in T_m(M)$ and the scalar $c(m)$ is an absolute constant.

§ 2. **Preliminaries.** Let M be a K-space and $R_{kji}{}^{h}$[1], R_{ji}, $F_j{}^i$ and g_{ji} the local components of the Riemannian curvature tensor, the Ricci tensor, the almost complex structure tensor and the metric tensor respectively and put $R_{kjih}=g_{ah}R_{kji}{}^a$, $F_{ji}=g_{ai}F_j{}^a$ etc..

The following identities in a K-space [4], [5] are well known.

$$(2.1)\qquad \nabla_j F_{ih}+\nabla_i F_{jh}=0\,,$$

$$(2.2)\qquad \nabla_j F_{ih}+F_j{}^a F_i{}^b \nabla_a F_{bh}=0\,,$$

$$(2.3)\qquad R^*{}_{ji}=R^*{}_{ij}$$

where $R^*{}_{ji}=\frac{1}{2}F^{ab}R_{abti}F_j{}^t$.

$$(2.4)\qquad R_{ji}=F_j{}^a F_i{}^b R_{ab}\,,\qquad R^*{}_{ji}=F_j{}^a R_i{}^b R^*{}_{ab}\,,$$

$$(2.5)\qquad (\nabla_j F_{ts})\nabla_i F^{ts}=R_{ji}-R^*{}_{ji}\,,$$

$$(2.6)\qquad (\nabla_j F_{ts})\nabla^j F^{ts}=S-S^*=\text{constant}\geqq 0$$

where $S=g^{ji}R_{ji}$ and $S^*=g^{ji}R^*{}_{ji}$.

$$(2.7)\qquad \nabla_j R^*{}_i{}^j=\frac{1}{2}\nabla_i S^*\,,\qquad \nabla_j R_i{}^j-\nabla_j R^*{}_i{}^j=\frac{1}{2}(\nabla_i S-\nabla_i S^*)=0 \text{ [3]},$$

$$(2.8)\qquad R_{jikh}-F_j{}^a F_i{}^b R_{abkh}=-(\nabla_j F_i{}^r)\nabla_k F_{hr} \text{ [1]}.$$

In this place, multiplying (2.8) by $\nabla_s F^{ji}$ and making use of (2.2) and (2.5), we have

1) The Latin indices run over the range $1, 2, \cdots, n$.

$$2(\nabla_s F^{ji})R_{jikh}=-(R_s{}^r-R^*{}_s{}^r)\nabla_k F_{hr}$$

and then, moreover multiplying the last equation by $\nabla^s F^{kh}$, we have

$$2\nabla_s F^{ji}(\nabla^s F^{kh})R_{jikh}=-(R_s{}^r-R^*{}_s{}^r)(\nabla_k F_{hr})\nabla^s F^{kh}$$

or

$$\nabla^j F^i{}_s(\nabla^k F^{hs})R_{jikh}=-\frac{1}{2}(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji}). \tag{2.9}$$

Next, in (2.9), by the first Bianchi's identity, we have

$$(\nabla^j F^i{}_s)\nabla^k F^{hs}(R_{jkhi}+R_{jhik})=\frac{1}{2}(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})$$

from which we have

$$\nabla^j F^i{}_s(\nabla^k F^{hs})R_{jkhi}=\frac{1}{4}(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji}). \tag{2.10}$$

Moreover, we know the following

LEMMA 2.1. (Gray [1]) *Let M be a K-space. Then we have*

$$\begin{aligned} R(X, Y, X, Y)=\frac{1}{32}\{&3Q(X+FY)+3Q(X-FY)-Q(X+Y) \\ &-Q(X-Y)-4Q(X)-Q(Y)\}-\frac{3}{4}\|(\nabla_X F)Y\|^2 \end{aligned} \tag{2.11}$$

for any vectors $X, Y\in T_m(M)$, where $Q(X)=R(X, FX, X, FX)$.[2)]

LEMMA 2.2. (Watanabe and Takamatsu [8]) *Let M be a non-Kähler K-space of constant holomorphic sectional curvature. Then we have*

$$R_{ji}=\frac{S}{n}g_{ji}, \quad R^*{}_{ji}=\frac{S^*}{n}g_{ji}, \quad S=5S^*. \tag{2.12}$$

§ 3. Proof of theorem.

First of all, we have

$$\begin{aligned} &\|X\pm FY\|^4 \\ &\quad=g(X\pm FY, X\pm FY)^2 \\ &\quad=(g(X, X)+g(Y, Y))^2\pm 4(g(X, X)+g(Y, Y))g(X, FY)+4g(X, FY)^2, \\ &\|X\pm Y\|^4 \\ &\quad=(g(X, X)+g(Y, Y))^2\pm 4(g(X, X)+g(Y, Y))g(X, Y)+4g(X, Y)^2. \end{aligned}$$

Thus, if we put $Q(X)=R(X, FX, X, FX)=-H(X)\|X\|^4$, i.e. for convenient we write $H(X)$ instead of $H\left(\frac{X}{\|X\|}\right)$, then we have

2) Our curvature tensor is different from Gray's in sign.

$$\begin{aligned} Q(X+FY) &= -H(X+FY)\|X+YF\|^4 \\ &= -H(X+FY)\{(g(X, X)+g(Y, Y))^2 \\ &\qquad +4(g(X, X)+g(Y, Y))g(X, FY)+4g(X, FY)^2\} . \end{aligned}$$

Similarly, calculating $Q(X-FY)$, $Q(X+Y)$ and $Q(X-Y)$, and substituting these results into (2.11), we have

$$\begin{aligned} R(X, Y, X, Y) \\ = -\frac{1}{32}[& 3H(X+FY)\{(g(X, X)+g(Y, Y))^2 \\ & \quad +4(g(X, X)+g(Y, Y))g(X, FY)+4g(X, FY)^2\} \\ & +3H(X-FY)\{g(X, X)+g(Y, Y))^2 \\ & \quad -4(g(X, X)+g(Y, Y))g(X, FY)+4g(X, FY)^2\} \\ & -H(X+Y)\{(g(X, X)+g(Y, Y))^2 \\ & \quad +4(g(X, X)+g(Y, Y))g(X, Y)+4g(X, Y)^2\} \\ & -H(X-Y)\{(g(X, X)+g(Y, Y))^2 \\ & \quad -4(g(X, X)+g(Y, Y))g(X, Y)+4g(X, Y)^2\} \\ & -4H(X)g(X, X)^2-4H(Y)g(Y, Y)^2] \\ -\frac{3}{4}\|(\nabla_X F)Y\|^2 . \end{aligned} \tag{3.1}$$

Hence, for the constant holomorphic sectional curvature $c(m)$, from (3.1) we have

$$\begin{aligned} R(X, Y, X, Y) = \frac{c(m)}{4}\{g(X, Y)^2-g(X, X)g(Y, Y)-3g(X, FY)^2\} \\ -\frac{3}{4}\|(\nabla_X F)Y\|^2 . \end{aligned} \tag{3.2}$$

Replacing Y by $Y+W$ in (3.2), we have

$$\begin{aligned} & R(X, Y, X, W) \\ & \quad = \frac{c(m)}{4}\{g(X, Y)g(X, W)-g(X, X)g(Y, W)-3g(X, FY)g(X, FW)\} \\ & \quad -\frac{3}{4}g((\nabla_X F)Y, (\nabla_X F)W) \end{aligned}$$

and replacing X by $X+Z$ in the last equation, we have

$$\begin{aligned} & R(X, Y, Z, W)+R(Z, Y, X, W) \\ & \quad = \frac{c(m)}{4}\{g(X, W)g(Y, Z)+g(X, Y)g(Z, W)-2g(X, Z)g(Y, W) \\ & \qquad -3g(X, FW)g(Z, FY)-3g(X, FY)g(Z, FW)\} \end{aligned} \tag{3.3}$$

$$+\frac{3}{4}\{g((\nabla_X F)W, (\nabla_Y F)Z)-g((\nabla_X F)Y, (\nabla_Z F)W)\}.$$

Interchanging X and Y in (3.3) and subtracting the equation thus obtained from (3.3), we have

$$\begin{aligned}(3.4)\qquad & R(X, Y, Z, W)+R(Z, Y, X, W)-R(Y, X, Z, W)-R(Z, X, Y, W)\\ &=\frac{c(m)}{4}\{g(X, W)g(Y, Z)-g(Y, W)g(X, Z)+g(X, Y)g(Z, W)\\ &\qquad -g(Y, X)g(Z, W)-2g(X, Z)g(Y, W)+2g(Y, Z)g(X, W)\\ &\qquad -3g(X, FW)g(Z, FY)+3g(Y, FW)g(Z, FX)\\ &\qquad -3g(X, FY)g(Z, FW)+3g(Y, FX)g(Z, FW)\}\\ &+\frac{3}{4}\{g((\nabla_X F)W, (\nabla_Y F)Z)-g((\nabla_X F)Z, (\nabla_Y F)W)-2g((\nabla_X F)Y, \nabla_Z F)W)\}\end{aligned}$$

from which (1.3) follows by virtue of the first Bianchi's identity.

Now, with local components (1.3) can be written as

$$\begin{aligned}(3.5)\qquad R_{kjih}=&\frac{c(m)}{4}(g_{kh}g_{ji}-g_{ki}g_{jh}+F_{kh}F_{ji}-F_{ki}F_{jh}-2F_{kj}F_{ih})\\ &-\frac{1}{4}\{2(\nabla_k F_j{}^a)\nabla_i F_{ha}+(\nabla_i F_j{}^a)\nabla_k F_{ha}+(\nabla_h F_j{}^a)\nabla_i F_{ka}\}.\end{aligned}$$

Contracting (3.5) by g^{kh}, we have

$$(3.6)\qquad 4R_{ji}-(n+2)c(m)g_{ji}-3(R_{ji}-R^*{}_{ji})=0.$$

Applying ∇^j to (3.6) and making use of (2.7), we have

$$(3.7)\qquad 2\nabla_i S-(n+2)\nabla_i c(m)=0.$$

On the other hand, contracting (3.6) by g^{ji}, we have

$$(3.8)\qquad 4S-n(n+2)c(m)-3(S-S^*)=0.$$

Applying ∇_i to (3.8) and making use of (2.6), we have

$$(3.9)\qquad 4\nabla_i S-n(n+2)\nabla_i c(m)=0.$$

Thus, eliminating $\nabla_i S$ from (3.7) and (3.9), we have

$$(n+2)(n-2)\nabla_i c(m)=0$$

from which it follows that $\nabla_i c(m)=0$ i.e. $c(m)$ is an absolute constant. Q.E.D.

Remark. From (3.8), by (2.12) we have

$$(3.10)\qquad c=c(m)=\frac{S+3S^*}{n(n+2)}=\frac{8S}{5n(n+2)}$$

and the constant>0 [6].

§ **4. Some applications.** In this section, making use of the main theorem we prove the following

THEOREM 4.1. *Let M be a non-Kähler K-space of constant holomorphic sectional curvature. Then we have*

$$R_{kjih}R^{kjih}=\frac{6n+44}{25n(n+2)}S^2=constant. \tag{4.1}$$

Proof. Multiplying (3.5) by R^{kjih}, we have

$$\begin{aligned} R_{kjih}R^{kjih} &= \frac{c}{4}(S+S+R^{kjih}F_{kh}F_{ji}-R^{kjih}F_{ki}F_{jh}-2R^{kjih}F_{kj}F_{ih}) \\ &\quad -\frac{1}{4}\{2R^{kjih}(\nabla_k F_j{}^s)\nabla_i F_{hs}+R^{kjih}(\nabla_i F_j{}^s)\nabla_k F_{hs}+R^{kjih}(\nabla_h F_j{}^s)\nabla_i F_{ks}\}. \end{aligned} \tag{4.2}$$

Since $R^{kjih}F_{kh}F_{ji}=S^*$ and $R^{kjih}F_{kj}F_{ih}=-2S^*$, by (2.9) and (2.10), (4.2) turns out to be

$$R_{kjih}R^{kjih}=\frac{c}{2}(S+3S^*)+\frac{3}{8}(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji}). \tag{4.3}$$

On the other hand, by (2.12), we have

$$R_{ji}-R^*{}_{ji}=\frac{S-S^*}{n}g_{ji}=\frac{4S}{5n}g_{ji}.$$

Then substituting the last equation and (3.10) into (4.3), we have (4.1). Q. E. D.

For a 6-dimensional K-space, (4.1) becomes

$$R_{kjih}R^{kjih}=\frac{1}{15}S^2.$$

This is equivalent to

$$\left\{R_{kjih}-\frac{S}{30}(g_{kh}g_{ji}-g_{ki}g_{jh})\right\}\left\{R^{kjih}-\frac{S}{30}(g^{kh}g^{ji}-g^{ki}g^{jh})\right\}=0$$

from which it follows that

$$R_{kjih}-\frac{S}{30}(g_{kh}g_{ji}-g_{ki}g_{jh})=0.$$

Thus, we have the following

THEOREM 4.2. (Tanno [7]). *Let M be a* 6*-dimensional non-Kähler K-Space of constant holomorphic sectional curvature. Then M is a space of constant curvature.*

THEOREM 4.3. *Let M be a K-space satisfying*

$$R_{kjih}R^{kjih}\leqq\frac{3(n+2)(S-S^*)^2+4(S+3S^*)^2}{8n(n+2)}.$$

Then M is of constant holomorphic sectional curvature.

Proof. We put

$$T_{kjih}=4R_{kjih}+\alpha(g_{kh}g_{ji}-g_{ki}g_{jh}+F_{kh}F_{ji}-F_{ki}F_{jh}-2F_{kj}F_{ih})$$
$$+(\nabla_i F_j{}^s)\nabla_k F_{hs}+(\nabla_h F_j{}^s)\nabla_i F_{ks}+2(\nabla_k F_j{}^s)\nabla_i F_{hs}$$
$$=4R_{kjih}+\alpha A_{kjih}+B_{kjih}$$

where

$$A_{kjih}=g_{kh}g_{ji}-g_{ki}g_{jh}+F_{kh}F_{ji}-F_{ki}F_{jh}-2F_{kj}F_{ih}\,,$$
$$B_{kjih}=(\nabla_i F_j{}^s)\nabla_k F_{hs}+(\nabla_h F_j{}^s)\nabla_i F_{ks}+2(\nabla_k F_j{}^s)\nabla_i F_{hs}\,,$$
$$\alpha=-\frac{S+3S^*}{n(n+2)}\quad\text{(scalar)}$$

and calculate the square of T_{kjih}.

First of all, we easily have

$$A_{kjih}A^{kjih}=8n^2+16n\,. \tag{4.4}$$

Next, by (2.8), we have

$$(\nabla^i F^{js})\nabla^k F^h{}_s(\nabla_h F_j{}^t)\nabla_i F_{kt}=-(\nabla^i F^{js})\nabla^k F^h{}_s(R_{hjik}-F_h{}^a F_j{}^b R_{abik})$$
$$=-\nabla^i F^{js}(\nabla^k F^h{}_s)R_{hjik}+F_h{}^a F_j{}^b\nabla^i F^{js}(\nabla^k F^h{}_s)R_{abik}$$
$$=-\nabla^i F^{js}(\nabla^k F^h{}_s)R_{hjik}+F^{js}F^h{}_s\nabla^i F_j{}^b(\nabla^k F_h{}^a)R_{abik}$$
$$=-\nabla^i F^{js}(\nabla^k F^h{}_s)R_{hjik}+\nabla^i F^{hb}(\nabla^k F_h{}^a)R_{abik}$$
$$=0\,^{3)}\,.$$

Similarly, we have

$$(\nabla^h F^{js})\nabla^i F^k{}_s(\nabla_k F_j{}^t)\nabla_i F_{ht}=(\nabla^k F^{js})\nabla^i F^h{}_s(\nabla_i F_j{}^t)\nabla_k F_{ht}=0\,.$$

Hence, by (2.5), we have

$$B_{kjih}B^{kjih}=6(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})\,. \tag{4.5}$$

Moreover, we easily have

$$R^{kjih}A_{kjih}=2S+6S^*\,, \tag{4.6}$$
$$A^{kjih}B_{kjih}=0 \tag{4.7}$$

and by (2.9) and (2.10), we have

$$R^{kjih}B_{kjih}=-\frac{3}{2}(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})\,. \tag{4.8}$$

Consequently, by (4.4), (4.5), (4.6), (4.7) and (4.8), we have

$$T_{kjih}T^{kjih}=16R_{kjih}R^{kjih}+\alpha^2 A_{kjih}A^{kjih}+B_{kjih}B^{kjih}$$
$$+8\alpha A_{kjih}R^{kjih}+2\alpha A_{kjih}B^{kjih}+8R_{kjih}B^{kjih}$$
$$=16R_{kjih}R^{kjih}+(8n^2+16n)\alpha^2+6(R_{ji}-R^*{}_{ji})(R^{ji}-R^{*ji})$$

3) This is verified too by purity and hybridity of ∇F.

$$+8(2S+6S^*)\alpha-12(R_{ji}-R^*_{ji})(R^{ji}-R^{*ji})$$

$$=16R_{kjih}R^{kjih}-6(R_{ji}-R^*_{ji})(R^{ji}-R^{*ji})-\frac{8(S+3S^*)^2}{n(n+2)}.$$

On the other hand, since $(R_{ji}-R^*_{ji})(R^{ji}-R^{*ji})\geqq\frac{(S-S^*)^2}{n}$, from the last equation, we have

$$T_{kjih}T^{kjih}\leqq 16R_{kjih}R^{kjih}-\frac{6(n+2)(S-S^*)^2+8(S+3S^*)^2}{n(n+2)}.$$

Thus, by the assumption, we have $T_{kjih}T^{kjih}\leqq 0$ i.e. $T_{kjih}=0$. But, as we have seen in the proof of the main theorem, from $T_{kjih}=0$ it follows that the scalar α is an absolute constant and therefore M is of constant holomorphic sectional curvature. Q.E.D.

THEOREM 4.4. *Let M be a non-Kähler K-space satisfying*

$$R_{kjih}R^{kjih}\leqq\frac{6n+44}{25n(n+2)}S^2 \quad \text{and} \quad S=5S^*.$$

Then M is of constant holomorphic sectional curvature.

Proof. By $S=5S^*$, we easily have

$$\frac{6n+44}{25n(n+2)}S^2=\frac{3(n+2)(S-S^*)^2+4(S+3S^*)^2}{8n(n+2)}.$$

Hence, the theorem follows from Theorem 4.3.

Remark. By (2.6) and $S=5S^*$, we see that S=constant.

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