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ON HYPERSURFACES WITH NORMAL $(f, g, u_{(k)}, \alpha_{(k)})$-STRUCTURE IN AN EVEN-DIMENSIONAL SPHERE

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§ 0. Introduction.

Yano and Okumura [8] have studied hypersurfaces of a manifold with (f, g, u, v, λ)-structure. These submanifolds admit an $(f, g, u_{(k)}, \alpha_{(k)})$-structure, that is, a set of a tensor field f of type (1, 1), a Riemannian metric g, three 1-forms u, v and w and functions α, β and λ satisfying certain algebraic conditions [4]. In particular, a hypersurface of an even-dimensional sphere carries an $(f, g, u_{(k)}, \alpha_{(k)})$-structure (see also [4]).

The submanifolds of codimension 2 in an almost contact metric manifold also admit the same kind of structure (see [5]).

Let M be an m-dimensional differentiable manifold with $(f, g, u_{(k)}, \alpha_{(k)})$-structure. We define on $M\times R^3$, R^3 being a 3-dimensional Euclidean space, a tensor field F of type (1, 1) with local components $F_B{}^A$ given by

$$(0.1)\qquad (F_B{}^A)=\begin{pmatrix} f_j{}^h & u^h & v^h & w^h \\ -u_j & 0 & -\lambda & \beta \\ -v_j & \lambda & 0 & \alpha \\ -w_j & -\beta & -\alpha & 0 \end{pmatrix}$$

in $\{N\times R^3; x^A\}$, $\{N; x^h\}$ being a coordinate neighborhood of M and $x^{\bar 1}$, $x^{\bar 2}$, $x^{\bar 3}$ being cartesian coordinates in R^3, where $f_j{}^h$, u_j, v_j and w_j are respectively local components of f, u, v and w, $u^h=u_i g^{ih}$, $v^h=v_i g^{ih}$ and $w^h=w_i g^{ih}$ in $\{N; x^h\}$, and, where g^{ih} are entries of the inverse matrix of the matrix (g_{ih}) whose entries are components of a Riemannian metric on M. (The indices $A, B, C, \cdots$ run over the range $\{1, 2, \cdots, m+3\}$ and $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, m\}$.) We denote $m+1$, $m+2$ and $m+3$ respectively by $\bar 1$, $\bar 2$ and $\bar 3$.

Denoting $\partial/\partial x^A$ by ∂_A, the Nijenhuis tensor $[F, F]$ of F has local components

$$S_{CB}{}^A=F_C{}^E\partial_E F_B{}^A-F_B{}^E\partial_E F_C{}^A-(\partial_C F_B{}^E-\partial_B F_C{}^E)F_E{}^A$$

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in $M\times R^3$. Thus, denoting ∇_j by the operator of covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ji} \right\}$ formed with g_{ji} of M and using (0.1), we can write down local components of the tensor $S_{CB}{}^A$ as follows;

$$
\begin{aligned}
S_{ji}{}^h &= f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h \\
&\quad + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h \\
&\quad + (\nabla_j w_i - \nabla_i w_j) w^h , \\
S_{ji}{}^{\bar{3}} &= -f_j{}^t \nabla_t w_i + f_i{}^t \nabla_t w_j + w_t (\nabla_j f_i{}^t - \nabla_i f_j{}^t) \\
&\quad - \beta(\nabla_j u_i - \nabla_i u_j) - \alpha(\nabla_j v_i - \nabla_i v_j) ,
\end{aligned} \tag{0.2}
$$

etc.

Specially, if $S_{ji}{}^h=0$, then we say that the $(f, g, u_{(k)}, \alpha_{(k)})$-structure is *normal* [4].

In the previous paper [4], Pak and the present authors proved the following theorem:

THEOREM A. *Let M be a complete and connected hypersurface of an even-dimensional sphere S^{2n}. If the induced $(f, g, u_{(k)}, \alpha_{(k)})$-structure is normal, $S_{ji}{}^{\bar{3}}=0$, the vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly independent and the function λ is almost everywhere non-zero on M, then M is congruent to S^{2n-1} or $S^p\times S^{2n-1-p}$ $(p=1, 2, \cdots, 2n-2)$ naturally embedded in S^{2n}.*

The main purpose of the present paper is to neglect the condition $S_{ji}{}^{\bar{3}}=0$ as an extension of Theorem A.

In § 1, we recall the definition of $(f, g, u_{(k)}, \alpha_{(k)})$-structure and give structure equations on M.

In § 2, we study hypersurfaces with normal $(f, g, u_{(k)}, \alpha_{(k)})$-structure in an even-dimensional sphere S^{2n} by using the following theorem proved by Ishihara and Ki one of the present authors [3]:

THEOREM B. *Let (M, g) be a complete and connected hypersurface immersed in a sphere $S^{m+1}(1)$ with induced metric g_{ji} and assume that there is in (M, g) an almost product structure $P_i{}^h$ of rank p such that $\nabla_j P_i{}^h=0$. If the second fundamental tensor h_{ji} of the hypersurface (M, g) has the form $h_{ji}=aP_{ji}+bQ_{ji}$, a and b being mutually different non-zero constants, where $P_{ji}=P_j{}^t g_{ti}$ and $Q_{ji}=g_{ji}-P_{ji}$, and if $m-1\geqq p\geqq 1$, then the hypersurface (M, g) is congruent to $S^p(r_1)\times S^{m-p}(r_2)$ naturally embedded in $S^{m+1}(1)$, where $1/r_1{}^2=1+a^2$ and $1/r_2{}^2=1+b^2$.*

§ 1. Hypersurfaces of an even-dimensional sphere.

Let E be a $(2n+1)$-dimensional Euclidean space and X the position vector starting from the origin of E and ending at a point of E. The E being odd-dimensional, it can be regared as a manifold with cosymplectic structure, that is, an aggregation (F, ξ, η, G) of a tensor field F of type (1,1), a vector field ξ, a 1-form η and a Riemannian metric G satisfying

$$
\begin{aligned}
&F^2=-I+\eta\otimes\xi\,,\\
&F\xi=0\,,\qquad \eta\circ F=0\,,\qquad \eta(\xi)=1\,,\\
&G(FY,\,FZ)=G(Y,\,Z)-\eta(Y)\eta(Z)\,,\\
&G(\xi,\,Y)=\eta(Y)
\end{aligned}
\tag{1.1}
$$

for arbitrary vector fields Y and Z and

$$
\tilde{\nabla} F=0\,,\qquad \tilde{\nabla}\xi=0\,,
\tag{1.2}
$$

where I denotes the unit tensor and $\tilde{\nabla}$ the Riemannian connection of E.

Let S^{2n} be a $2n$-dimensional sphere which is covered by a system of coordinate neighborhoods $\{U\,;y^b\}$, where here and in this section the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$, then S^{2n} is naturally immersed in E as a hypersurface by $X: S^{2n}\to E$.

We put $X_b=\partial_b X$ $(\partial_b=\partial/\partial y^b)$, then X_b are $2n$ linearly independent local vector fields tangent to $X(S^{2n})$ and $g_{cb}=X_c\cdot X_b$ is the Riemannian metric induced on S^{2n} from that of E, the dot denoting the inner product of vectors of $X(S^{2n})$. In the sequel, $X(S^{2n})$ is identified with S^{2n} itself.

We choose $-X$ as a unit normal C to S^{2n} in such a way that $X_1, X_2, \cdots, X_{2n}$, C give the positive orientation of E.

The transforms FX_b and FC of X_b and C respectively by F, and the vector field ξ can be expressed as

$$
\begin{aligned}
&FX_b=f_b{}^e X_e+v_b C\,,\\
&FC=-v^e X_e\,,\\
&\xi=u^e X_e-\lambda C\,,
\end{aligned}
\tag{1.3}
$$

where $f_b{}^e$ is a tensor field of type (1,1), v_b is of 1-form, $v^e=v_a g^{ae}$, u^e is a vector field and λ is a function, all globally defined on S^{2n}.

Transvecting each of (1.3) with F respectively and using (1.1) and (1.3) itself, we find

$$
\begin{aligned}
&f_e{}^b f_c{}^e=-\delta_c^b+u_c u^b+v_c v^b\,,\\
&g_{ea} f_c{}^e f_b{}^a=g_{cb}-u_c u_b-v_c v_b\,,\\
&f_e{}^b u^e=-\lambda v^b\,,\qquad f_e{}^b v^e=\lambda u^b\,,\\
&u_e u^e=v_e v^e=1-\lambda^2\,,\qquad u_e v^e=0\,,\qquad u_e=u^a g_{ae}\,,
\end{aligned}
\tag{1.4}
$$

that is, S^{2n} admits an (f, g, u, v, λ)-structure (cf. [9]).

We denote ∇_c by the operator of covariant differentiation with respect to the

Christoffel symbols $\left\{{a \atop c\ b}\right\}$ formed with g_{cb}. Then equation of Gauss and Weingarten are

$$\nabla_c X_b = g_{cb} C, \qquad \nabla_c C = -X_c \tag{1.5}$$

because the second fundamental tensor with respect to unit normal C is equal to g_{cb}.

Differentiating each equation of (1.3) covariantly and using (1.2), (1.3) and (1.5), we have

$$\begin{gathered} \nabla_c f_b{}^e = -g_{cb} v^e + \delta_c^e v_b, \\ \nabla_c u_b = -\lambda g_{cb}, \qquad \nabla_c v_b = f_{cb}, \\ \nabla_c \lambda = u_c. \end{gathered} \tag{1.6}$$

We now compute

$$S_{cb}{}^a = [f, f]_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a, \tag{1.7}$$

where $[f, f]_{cb}{}^a$ is the Nijenhuis tensor formed with $f_b{}^a$.

Substituting (1.6) into (1.7), we get $S_{cb}{}^a = 0$, which means that the (f, g, u, v, λ)-structure is normal.

Hence, S^{2n} admits a normal (f, g, u, v, λ)-structure.

Consider a $(2n-1)$-dimensional manifold M covered by a system of coordinate neighborhoods $\{V; x^h\}$, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n-1\}$, and assume that M is differentiably immersed in S^{2n} by the immersion $i: M \to S^{2n}$ which is expressed locally by $y^b = y^b(x^h)$.

We put $B_h{}^b = \partial_h y^b$ $(\partial_h = \partial/\partial x^h)$. We assume that we can choose a unit vector N^b of S^{2n} normal to M in such a way that $2n$ vectors $B_h{}^b$, N^b give the positive orientation of S^{2n}. The transforms $f_e{}^b B_j{}^e$ and $f_e{}^b N^e$ of $B_j{}^e$ and N^e respectively by $f_e{}^b$ can be written in the forms

$$f_e{}^b B_j{}^e = f_j{}^i B_i{}^b + w_j N^b, \qquad f_e{}^b N^e = -w^i B_i{}^b, \tag{1.8}$$

where $f_j{}^i$ is a tensor field of type (1, 1), w_j is of 1-form and $w^i = w_t g^{ti}$, g_{ji} being the Riemannian metric on M induced from that of S^{2n}, and the vectors u^b, v^b can be expressed as

$$u^b = u^i B_i{}^b + \beta N^b, \qquad v^b = v^i B_i{}^b + \alpha N^b, \tag{1.9}$$

where u^i, v^i are vectors and α, β are functions on M.

Applying $f_b{}^a$ to (1.8) and (1.9) respectively and taking account of (1.4), (1.8) and (1.9), we can find

$$\begin{gathered} f_j{}^t f_t{}^i = -\delta_j^i + u_j u^i + v_j v^i + w_j w^i, \\ g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i - w_j w_i, \\ f_t{}^i u^t = -\lambda v^i + \beta w^i, \qquad f_t{}^i v^t = \lambda u^i + \alpha w^i, \end{gathered}$$

(1.10)
$$f_t{}^i w^t = -\beta u^i - \alpha v^i ,$$
$$u_t u^t = 1-\beta^2-\lambda^2 , \qquad v_t v^t = 1-\alpha^2-\lambda^2 ,$$
$$w_t w^t = 1-\alpha^2-\beta^2 ,$$
$$u_t v^t = -\alpha\beta , \qquad u_t w^t = -\alpha\lambda , \qquad v_t w^t = \beta\lambda ,$$

where $u_i = u^t g_{ti}$ and $v_i = v^t g_{ti}$, that is, M admits an $(f, g, u_{(k)}, \alpha_{(k)})$-structure ([1], [4], [8]).

If we put $f_{ji} = f_j{}^t g_{ti}$, we can easily verify that f_{ji} is skew-symmetric because of (1.10).

Denoting ∇_j by the operator of covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop j\ i} \right\}$ formed with g_{ji}, equations of Gauss and Weingarten of M are

(1.11)
$$\nabla_j B_i{}^a = h_{ji} N^a , \qquad \nabla_j N^a = -h_j{}^i B_i{}^a ,$$

where h_{ji} is the second fundamental tensor and $h_j{}^i$ is defined by $h_j{}^i = h_{jt} g^{ti}$.

Differentiating (1.8) and (1.9) covariantly along M respectively and making use of (1.6), (1.8), (1.9) and (1.11), we have

(1.12)
$$\nabla_k f_j{}^i = -g_{kj} v^i + \delta_k^i v_j - h_{kj} w^i + h_k{}^i w_j ,$$

(1.13)
$$\begin{cases} \nabla_k u_j = -\lambda g_{kj} + \beta h_{kj} , \qquad \nabla_k v_j = \alpha h_{kj} + f_{kj} , \\ \nabla_k w_j = -\alpha g_{kj} - h_{kt} f_j{}^t , \end{cases}$$

(1.14)
$$\nabla_k \alpha = -h_{kt} v^t + w_k , \qquad \nabla_k \beta = -h_{kt} u^t .$$

Transvecting the last equation of (1.6) with $B_k{}^c$ and using (1.9), we obtain

(1.15)
$$\nabla_k \lambda = u_k .$$

Since an even-dimensional sphere S^{2n} is a space of constant curvature, the Codazzi equation of M is given by

(1.16)
$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0 .$$

Substituting (1.12) and (1.13) into (0.2), we get

(1.17)
$$S_{ji}{}^h = (f_j{}^t h_t{}^h - h_j{}^t f_t{}^h) w_i - (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h) w_j .$$

We prove the following two propositions.

PROPOSITION 1.1. *In a manifold with $(f, g, u_{(k)}, \alpha_{(k)})$-structure, the vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly independednt if and only if $1-\alpha^2-\beta^2-\lambda^2 \neq 0$.*

Moreover, if vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly dependent, then $h_{ji} = (\lambda/\beta) g_{ji}$ in M.

Proof. See [4].

PROPOSITION 1.2. *Let M be a hypersurface of a $2n$-dimensional sphere S^{2n}. Then the necessary and sufficient condition that the induced $(f, g, u_{(k)}, \alpha_{(k)})$-structure on M is normal is*

$$f_j{}^t h_t{}^h - h_j{}^t f_t{}^h = 0 ,$$

which is equivalent to

$$h_{jt} f_i{}^t + h_{it} f_j{}^t = 0 . \tag{1.18}$$

Proof. From (1.17) the sufficiency is trivial.

Assume that $(f, g, u_{(k)}, \alpha_{(k)})$-structure is normal, that is, $S_{ji}{}^h = 0$. Putting $T_j{}^h = f_j{}^t h_t{}^h - h_j{}^t f_t{}^h$, (1.17) becomes

$$T_j{}^h w_i - T_i{}^h w_j = 0 , \tag{1.19}$$

from which, contracting with respect to h and i,

$$T_j{}^t w_t = 0 \tag{1.20}$$

by virtue of the symmetry of $T_i{}^h$.

Transvecting (1.19) with w^i and using (1.20), we find

$$(1-\alpha^2-\beta^2) T_j{}^h = 0 .$$

On $N_0 = \{P \in M : T_j{}^h(P) \neq 0\}$ we have $1-\alpha^2-\beta^2=0$, from which, $w_j=0$, it follows that $\beta u_j + \alpha v_j = 0$ on N_0 by the definition of $w_t f_j{}^t$. Since the last equation means that u_j and v_j are linearly dependent, we get $1-\alpha^2-\beta^2-\lambda^2=0$ and consequently $h_{ji} = (\lambda/\beta) g_{ji}$ on this set by virtue of Proposition 1.1. Thus we find $h_{ji}=0$, which implies $T_j{}^h = 0$ on N_0, that is, $T_j{}^h = 0$ on the whole space M. Therefore the necessity is also proved.

§ 2. Hypersurfaces with normal $(f, g, u_{(k)}, \alpha_{(k)})$-structure.

In this section, we assume that the $(f, g, u_{(k)}, \alpha_{(k)})$-structure induced in a hypersurface M of an even-dimensional sphere S^{2n} is normal, the vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly independent and functions β, λ are almost everywhere non-zero on M.

Now, transvecting (1.18) with $v^j v^i$, $w^j w^i$, $u^j v^i$ and $u^j w^i$ respectively, and using the definition of $(f, g, u_{(k)}, \alpha_{(k)})$-structure, we have

$$\lambda h(u, v) = -\alpha h(v, w) , \tag{2.1}$$

$$\beta h(u, w) = -\alpha h(v, w) , \tag{2.2}$$

$$\lambda h(u, u) + \alpha h(u, w) - \lambda h(v, v) + \beta h(v, w) = 0 , \tag{2.3}$$

$$-\beta h(u, u) - \alpha h(u, v) - \lambda h(v, w) + \beta h(w, w) = 0 , \tag{2.4}$$

$h(u, v)$, $h(v, w)$, $\cdots$ and $h(w, w)$ being denoted by respectively $h(u, v)=h_{ts}u^t v^s$, $h(v, w)=h_{ts}v^t w^s$, $\cdots$ and $h(w, w)=h_{ts}w^t w^s$.

Multiplying (2.4) by λ and substituting (2.1) into the equation obtained, we get

$$\beta\lambda h(u, u)=(\alpha^2-\lambda^2)h(v, w)+\beta\lambda h(w, w)\,, \tag{2.5}$$

from which, combining (2.2) and (2.3),

$$\beta\lambda h(v, v)=(\beta^2-\lambda^2)h(v, w)+\beta\lambda h(w, w)\,. \tag{2.6}$$

LEMMA 2.1. *Let M be a hypersurface of an even-dimensional sphere S^{2n}. If the induced $(f, g, u_{(k)}, \alpha_{(k)})$-structure on M is normal, the vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly independent and functions β and λ are almost everywhere non-zero on M, then*

$$h_{jt}u^t=(\alpha^2 x+y)u_j-\alpha\beta x v_j-\alpha\lambda x w_j\,, \tag{2.7}$$

$$h_{jt}v^t=-\alpha\beta x u_j+(\beta^2 x+y)v_j+\beta\lambda x w_j\,, \tag{2.8}$$

$$h_{jt}w^t=-\alpha\lambda x u_j+\beta\lambda x v_j+(\lambda^2 x+y)w_j\,, \tag{2.9}$$

x and y being given by respectively

$$\begin{aligned} D\beta\lambda x&=(1-\alpha^2-\beta^2)h(v, w)-\beta\lambda h(w, w)\,,\\ D\beta\lambda y&=-\lambda^2 h(v, w)+\beta\lambda h(w, w) \end{aligned} \tag{2.10}$$

and $D=1-\alpha^2-\beta^2-\lambda^2$.

Proof. Transvecting (1.18) with $f_k{}^i$, we obtain

$$h_{jt}(-\delta_k^t+u_k u^t+v_k v^t+w_k w^t)+h_{it}f_k{}^i f_j{}^t=0\,,$$

from which, taking skew-symmetric parts,

$$(h_{jt}u^t)u_k+(h_{jt}v^t)v_k+(h_{jt}w^t)w_k=(h_{kt}u^t)u_j+(h_{kt}v^t)v_j+(h_{kt}w^t)w_j\,. \tag{2.11}$$

Transvecting (2.11) with u^k, v^k and w^k respectively, and using (1.10), we have

$$(1-\beta^2-\lambda^2)h_{jt}u^t-\alpha\beta h_{jt}v^t-\alpha\lambda h_{jt}w^t=h(u, u)u_j+h(u, v)v_j+h(u, w)w_j\,, \tag{2.12}$$

$$-\alpha\beta h_{jt}u^t+(1-\alpha^2-\lambda^2)h_{jt}v^t+\beta\lambda h_{jt}w^t=h(u, v)u_j+h(v, v)v_j+h(v, w)w_j\,, \tag{2.13}$$

$$-\alpha\lambda h_{jt}u^t+\beta\lambda h_{jt}v^t+(1-\alpha^2-\beta^2)h_{jt}w^t=h(u, w)u_j+h(v, w)v_j+h(w, w)w_j\,, \tag{2.14}$$

from which, computing coefficient determinant with respect to $h_{jt}u^t$, $h_{jt}v^t$, $h_{jt}w^t$,

$$\begin{vmatrix} 1-\beta^2-\lambda^2 & -\alpha\beta & -\alpha\lambda \\ -\alpha\beta & 1-\alpha^2-\lambda^2 & \beta\lambda \\ -\alpha\lambda & \beta\lambda & 1-\alpha^2-\beta^2 \end{vmatrix}=D^2\,.$$

Since u^h, v^h and w^h are linearly independent, D is not zero by virtue of Proposition 1.1.

Therefore, we find from (2.12), (2.13) and (2.14)

$$\begin{aligned} h_{jt}u^t=&\frac{1}{D}\{(1-\alpha^2)h(u, u)+\alpha\beta h(u, v)+\alpha\lambda h(u, w)\}u_j \\ &+\frac{1}{D}\{(1-\alpha^2)h(u, v)+\alpha\beta h(v, v)+\alpha\lambda h(v, w)\}v_j \\ &+\frac{1}{D}\{(1-\alpha^2)h(u, w)+\alpha\beta h(v, w)+\alpha\lambda h(w, w)\}w_j\,, \end{aligned}$$

from which, multiplying by $\beta\lambda$ and substituting (2.1), (2.2), (2.5) and (2.6),

$$\begin{aligned} \beta\lambda h_{jt}u^t=&\frac{1}{D}[\alpha^2\{(1-\alpha^2-\beta^2)h(v, w)-\beta\lambda h(w, w)\}-\lambda^2 h(v, w)+\beta\lambda h(w, w)]u_j \\ &-\frac{1}{D}\alpha\beta\{(1-\alpha^2-\beta^2)h(v, w)-\beta\lambda h(w, w)\}v_j \\ &-\frac{1}{D}\alpha\lambda\{(1-\alpha^2-\beta^2)h(v, w)-\beta\lambda h(w, w)\}w_j\,, \end{aligned}$$

which implies (2.7) because of (2.10).

In the same way, we can verify (2.8) and (2.9).

LEMMA 2.2. *Under the same assumptions as those stated in Lemma* 2.1, *we have*

$$h_{jt}h_i{}^t=\Big\{(1-D)x+y-\frac{\beta}{\lambda}\Big\}h_{ji}+\frac{\beta}{\lambda}\{(1-D)x+y\}g_{ji}\,. \tag{2.15}$$

Proof. Differentiating (1.18) covariantly and using (1.12), we find

$$\begin{aligned} (\nabla_k h_{jt})f_i{}^t+(\nabla_k h_{it})f_j{}^t=&-(h_{kt}h_j{}^t)w_i-(h_{kt}h_i{}^t)w_j \\ &+h_{kj}(h_{it}w^t-v_i)+h_{ki}(h_{jt}w^t-v_j) \\ &+g_{kj}(h_{it}v^t)+g_{ki}(h_{jt}v^t)\,, \end{aligned} \tag{2.16}$$

from which, taking the skew-symmetric part with respect to k and j

$$\begin{aligned} (\nabla_k h_{it})f_j{}^t-(\nabla_j h_{it})f_k{}^t=&-(h_{kt}h_i{}^t)w_j+(h_{jt}h_i{}^t)w_k \\ &+h_{ki}(h_{jt}w^t-v_j)-h_{ji}(h_{kt}w^t-v_k) \\ &+g_{ki}(h_{jt}v^t)-g_{ji}(h_{kt}v^t)\,, \end{aligned} \tag{2.17}$$

and again skew-symmetric parts with respect to k and i,

$$\begin{aligned} (\nabla_k h_{jt})f_i{}^t-(\nabla_i h_{jt})f_k{}^t=&-(h_{kt}h_j{}^t)w_i+(h_{it}h_j{}^t)w_k \\ &+h_{kj}(h_{it}w^t-v_i)-h_{ij}(h_{kt}w^t-v_k) \\ &+g_{kj}(h_{it}v^t)-g_{ij}(h_{kt}v^t) \end{aligned} \tag{2.18}$$

because of (1.16).

Calculating (2.16)-(2.17)-(2.18) and using (1.16), we obtain

$$(\nabla_j h_{it})f_k{}^t = -(h_{jt}h_i{}^t)w_k + h_{ji}(h_{kt}w^t - v_k) + g_{ji}(h_{kt}v^t),$$

from which, substituting (2.8) and (2.9),

$$(\nabla_j h_{it})f_k{}^t = -(h_{jt}h_i{}^t)w_k + h_{ji}\{-\alpha\lambda x u_k + (\beta\lambda x - 1)v_k + (\lambda^2 x + y)w_k\} + g_{ji}\{-\alpha\beta x u_k + (\beta^2 x + y)v_k + \beta\lambda x w_k\}. \tag{2.19}$$

Transvecting (2.19) with u^k, v^k and w^k respectively, and making use of (1.10), we have

$$(-\lambda v^t + \beta w^t)\nabla_j h_{it} = \alpha\lambda h_{jt}h_i{}^t - \alpha\{\lambda(x+y) - \beta\}h_{ji} - \alpha\beta(x+y)g_{ji}, \tag{2.20}$$

$$(\lambda u^t + \alpha w^t)\nabla_j h_{it} = -\beta\lambda h_{jt}h_i{}^t + \{\beta\lambda(x+y) - (1-\alpha^2-\lambda^2)\}h_{ji} + \{\beta^2 x + (1-\alpha^2-\lambda^2)y\}g_{ji} \tag{2.21}$$

and

$$(-\beta u^t - \alpha v^t)\nabla_j h_{it} = -(1-\alpha^2-\beta^2)h_{jt}h_i{}^t + \{\lambda^2 x + (1-\alpha^2-\beta^2)y - \beta\lambda\}h_{ji} + \beta\lambda(x+y)g_{ji}. \tag{2.22}$$

Multiplying (2.20) and (2.21) by α and $-\beta$ respectively, and adding two equations obtained, we get

$$\lambda(-\alpha v^t - \beta u^t)\nabla_j h_{it} = \lambda(\alpha^2+\beta^2)h_{jt}h_i{}^t + \{-\lambda(\alpha^2+\beta^2)(x+y) + \beta(1-\lambda^2)\}h_{ji} - \beta\{(\alpha^2+\beta^2)x + (1-\lambda^2)y\}g_{ji}. \tag{2.23}$$

Comparing with (2.22) and (2.23), we easily see that

$$\lambda h_{jt}h_i{}^t - [\lambda\{(1-D)x + y\} - \beta]h_{ji} - \beta\{(1-D)x + y\}g_{ji} = 0,$$

which verifies the lemma.

LEMMA 2.3. *Under the same assumptions as those stated in Lemma 2.1, $x=0$ and $h_{ji} = y g_{ji}$ are equivalent on M.*

Proof. Let $x=0$. Then (2.7), (2.8) and (2.9) become respectively

$$h_{it}u^t = y u_j, \qquad h_{jt}v^t = y v_j, \qquad h_{jt}w^t = y w_j. \tag{2.24}$$

Differentiating the second equation of (2.24) covariantly and using (1.13), we have

$$(\nabla_k h_{jt})v^t + h_{jt}(\alpha h_k{}^t + f_k{}^t) = (\nabla_k y)v_j + y(\alpha h_{kj} + f_{kj}),$$

from which, taking skew-symmetric parts and using (1.16) and (1.18),

$$2h_{jt}f_k{}^t=(\nabla_k y)v_j-(\nabla_j y)v_k+2yf_{kj}\,. \tag{2.25}$$

Transvecting (2.25) with w^j and using (2.24), we find $\beta\lambda\nabla_k y=(w^t\nabla_t y)v_k$. So (2.25) can be written as the form

$$h_{jt}f_k{}^t=yf_{kj}\,. \tag{2.26}$$

Transvecting (2.26) with $f_i{}^k$ and using (1.10), we get

$$h_{jt}(-\delta_i^t+u_iu^t+v_iv^t+w_iw^t)=y(-g_{ji}+u_ju_i+v_jv_i+w_jw_i)\,,$$

or, using (2.24), $h_{ji}=yg_{ji}$.

Conversely, if $h_{ji}=yg_{ji}$, then $h_{jt}v^t=yv_j$. From this and (2.8), we find

$$x(-\alpha\beta u_j+\beta^2v_j+\beta\lambda w_j)=0\,,$$

which suggests $x=0$ because u_j, v_j and w_j are linearly independent, and β is almost everywhere non-zero. Therefore Lemma 2.3 is proved.

LEMMA 2.4. *Under the same assumptions as those stated in Lemma* 2.1, *we find*

$$\nabla_k h_{ji}=0\,. \tag{2.27}$$

Proof. Applying (2.15) to u^i and taking account of (2.7)~(2.9), we have

$$\begin{aligned}&\{(1-D)x+2y\}(\alpha^2xu_j-\alpha\beta xv_j-\alpha\lambda xw_j)+y^2u_j\\&=\left\{(1-D)x+y-\frac{\beta}{\lambda}\right\}\{(\alpha^2x+y)u_j-\alpha\beta xv_j-\alpha\lambda xw_j\}\\&\quad+\frac{\beta}{\lambda}\{(1-D)x+y\}u_j\,,\end{aligned}$$

and consequently

$$\left(y+\frac{\beta}{\lambda}\right)x\{(\beta^2+\lambda^2)u_j+\alpha\beta v_j+\alpha\lambda w_j\}=0\,.$$

Since u_j, v_j and w_j are linearly independent and β, λ are almost everywhere non-zero, the last equation implies that

$$\left(y+\frac{\beta}{\lambda}\right)x=0\,. \tag{2.28}$$

We have from (2.7) and (2.8)

$$\beta h_{jt}u^t+\alpha h_{jt}v^t=y(\beta u_j+\alpha v_j)\,. \tag{2.29}$$

Differentiating (2.29) covariantly, we find

$$\begin{aligned}&(\nabla_k\beta)h_{jt}u^t+\beta(\nabla_k h_{jt})u^t+\beta h_{jt}\nabla_k u^t\\&\quad+(\nabla_k\alpha)h_{jt}v^t+\alpha(\nabla_k h_{jt})v^t+\alpha h_{jt}\nabla_k v^t\\&=(\nabla_k y)(\beta u_j+\alpha v_j)+y\{(\nabla_k\beta)u_j+\beta\nabla_k u_j+(\nabla_k\alpha)v_j+\alpha\nabla_k v_j\}\,,\end{aligned}$$

from which, taking the skew-symmetric part and making use of (1.13), (1.14), (1.16) and (1.18),

$$\begin{aligned} & w_k(h_{jt}v^t)-w_j(h_{kt}v^t)+2\alpha h_{jt}f_k{}^t \\ & \quad =(\nabla_k y)(\beta u_j+\alpha v_j)-(\nabla_j y)(\beta u_k+\alpha v_k) \\ & \qquad +y\{(-h_{kt}u^t)u_j-(-h_{jt}u^t)u_k \\ & \qquad\qquad +(-h_{kt}v^t+w_k)v_j-(-h_{jt}v^t+w_j)v_k+2\alpha f_{kj}\}\,, \end{aligned}$$

or, using (2.7), (2.8) and (2.28),

$$2\alpha h_{jt}f_k{}^t=(\nabla_k y)(\beta u_j+\alpha v_j)-(\nabla_j y)(\beta u_k+\alpha v_k)+2\alpha y f_{kj}\,.$$

Transvecting the above equation with u^j and substituting (2.7) into the equation obtained, we get

$$D\beta\nabla_k y-(u^t\nabla_t y)(\beta u_k+\alpha v_k)=0\,. \tag{2.30}$$

In $N_1=\{P\in M: \alpha x(P)\neq 0\}$ $y=-\frac{\beta}{\lambda}$ by virtue of (2.28). Differentiating this equation covariantly and making use of (1.14), (1.15) and (2.7), we have

$$\nabla_j y=\frac{\alpha x}{\lambda}(\alpha u_j-\beta v_j-\lambda w_j) \qquad \text{on} \quad N_1\,,$$

or, comparing the above equation with (2.30), $\alpha x=0$ because u_j, v_j and w_j are linearly independent. This contradicts the construction of the set N_1.

Thereupon, on the whole space M,

$$\alpha x=0\,. \tag{2.31}$$

From (2.7) and (2.31) we have

$$h_{jt}u^t=yu_j\,. \tag{2.32}$$

Differentiating (2.32) covariantly, we find

$$(\nabla_k h_{jt})u^t+h_{jt}\nabla_k u^t=(\nabla_k y)u_j+y\nabla_k u_j\,,$$

which contains

$$x(\nabla_k h_{jt})u^t+xh_{jt}\nabla_k u^t=x(\nabla_k y)u_j+xy\nabla_k u_j\,. \tag{2.33}$$

On the other hand, computing covariant differentiation of $\frac{\beta}{\lambda}$ and taking account of (1.14), (1.15), (2.7) and (2.31), we get

$$\nabla_k\frac{\beta}{\lambda}=-\frac{1}{\lambda}\Big(y+\frac{\beta}{\lambda}\Big)u_k\,. \tag{2.34}$$

Differentiating (2.28) covariantly and using (2.28) itself and (2.34), we have $x\nabla_k y+\big(y+\frac{\beta}{\lambda}\big)\nabla_k x=0$, which implies $x^2\nabla_k y+x\big(y+\frac{\beta}{\lambda}\big)\nabla_k x=0$. This equation shows that

$$x\nabla_k y=0 \tag{2.35}$$

because of (2.28).

From (2.21) and (2.31) we get

$$\begin{aligned} x\lambda(\nabla_j h_{it})u^t = & -x\beta\lambda h_{jt}h_i{}^t + x\{\beta\lambda(x+y)-(1-\lambda^2)\}h_{ji} \\ & + x\{\beta^2 x+(1-\lambda^2)y\}g_{ji}\,. \end{aligned} \tag{2.36}$$

Substituting (2.35) and (2.36) into (2.33) and making use of (1.13), we have

$$\begin{aligned} & -x\beta\lambda h_{kt}h_j{}^t + x\{\beta\lambda(x+y)-(1-\lambda^2)\}h_{kj} \\ & \quad + x\{\beta^2 x+(1-\lambda^2)y\}g_{kj} + \lambda x h_{jt}(-\lambda\delta_k^t + \beta h_k{}^t) \\ & = \lambda x y(-\lambda g_{kj} + \beta h_{kj}) \end{aligned}$$

and consequently $x\{(\beta\lambda x-1)h_{kj}+(\beta^2 x+y)g_{kj}\}=0$, which implies $x(\beta\lambda x-1)(h_{kj}-yg_{kj})=0$ by virtue of (2.28). On a set $N_2=\{P\in M: x(\beta\lambda x-1)(P)\neq 0\}$, $h_{kj}-yg_{kj}=0$. From the result of Lemma 2.3 the last equation shows that $x=0$ on N_2. Thus the set N_2 is void, that is,

$$x(\beta\lambda x-1)=0 \tag{2.37}$$

on M.

We denote the set $\{Q\in M;\ \beta(Q)\lambda(Q)x(Q)\neq 1\}$ by $\tilde{N}$. Then on $\tilde{N}$ $x=0$ and by virtue of Lemma 2.3 $h_{ji}=yg_{ji}$ on $\tilde{N}$. Differentiating the last equation covariantly, we find $\nabla_k h_{ji}=(\nabla_k y)g_{ji}$, from which

$$(\nabla_k y)g_{ji}-(\nabla_j y)g_{ki}=0\,.$$

Thus we have $2(n-1)\nabla_k y=0$, that is, $y=$const. on the connected components of $\tilde{N}$. Hence we have $\nabla_k h_{ji}=0$ on $\tilde{N}$. Now we put $N_3=\{P\in M: (\nabla_k h_{ji})(P)\neq 0\}$. Then $\beta\lambda x=1$ and $x\neq 0$ on N_3.

On the other hand, if we denote by N_4 the set $N_3\cap\tilde{N}^c$ ($\tilde{N}^c$ is the complement on $\tilde{N}$), then

$$y=-\frac{\beta}{\lambda}\,, \qquad \alpha=0\,, \qquad \beta\lambda x-1=0 \tag{2.38}$$

on N_4 by virtue of (2.28), (2.31) and (2.37).

Substituting (2.38) into (2.15), we get

$$h_{jt}h_i{}^t=\frac{\lambda^2-\beta^2}{\beta\lambda}h_{ji}+g_{ji}$$

on N_4. Moreover $\dfrac{\lambda^2-\beta^2}{\beta\lambda}$ is constant because of (2.34) on this set. Therefore, taking account of (1.16) we find $\nabla_k h_{ji}=0$ on N_4. This contradicts the construction of the set N_3. Hence N_3 is empty, that is, $\nabla_k h_{ji}=0$ on the whole space M. And so the proof of Lemma 2.4 is completed (cf. [6]).

From (2.15) and (2.31) we can easily verify that eigenvalues of $(h_j{}^i)$ are $(\beta^2+\lambda^2)x+y$ and $-\dfrac{\beta}{\lambda}$. Putting $A=(\beta^2+\lambda^2)x+y-\dfrac{\beta}{\lambda}$ and $B=\dfrac{\beta}{\lambda}\{(\beta^2+\lambda^2)x+y\}$, (2.15) can be represented in the form

$$h_{jt}h_i^t = Ah_{ji} + Bg_{ji}\,. \tag{2.39}$$

Differentiating (2.39) covariantly and making use of Lemma 2.4, we have

$$(\nabla_k A)h_{ji} + (\nabla_k B)g_{ji} = 0\,, \tag{2.40}$$

from which, transvecting with g^{ji},

$$h_t^t \nabla_k A + (2n-1)\nabla_k B = 0\,. \tag{2.41}$$

Substituting (2.41) into (2.40), we obtain

$$\Big(h_{ji} - \frac{1}{2n-1} h_t^t g_{ji}\Big)\nabla_k A = 0\,,$$

which implies

$$\{h_{ji}h^{ji} - (h_t^t)^2/(2n-1)\}\nabla_k A = 0\,. \tag{2.42}$$

Since

$$\Big(h_{ji} - \frac{1}{2n-1} h_t^t g_{ji}\Big)\Big(h^{ji} - \frac{1}{2n-1} h_t^t g^{ji}\Big) = h_{kj}h^{ji} - (h_t^t)^2/(2n-1)\,,$$

it follows that $h_{ji} - \frac{1}{2n-1} h_t^t g_{ji} = 0$ if and only if $h_{ji}h^{ji} - (h_t^t)^2/(2n-1) = 0$. Moreover $h_{ji}h^{ji} - (h_t^t)^2/(2n-1)$ is constant by virtue of (2.27).

Therefore, from (2.42) we may consider only two cases;

Case (A): $\quad h_{ji}h^{ji} - (h_t^t)^2/(2n-1) = 0.$

Case (B): $\quad \nabla_k A = 0.$

In the Case (A) we see that M is totally umbilical. Moreover, if M is complete, then M is congruent to S^{2n-1}.

The other Case (B) implies $\nabla_k B = 0$ because of (2.41). Hence eigenvalues $-\frac{\beta}{\lambda}$ and $(\beta^2+\lambda^2)x + y$ of (h_j^i) are both constants by virtue of constancy of A and B. Therefore, using (2.34), we find $\Big(y + \frac{\beta}{\lambda}\Big)u_k = 0$, from which, $y = -\frac{\beta}{\lambda}$ because of linearly independency of u_k, v_k and w_k.

So an eigenvalue $(\beta^2+\lambda^2)x + y$ of (h_j^i) becomes $(\beta^2+\lambda^2)x - \frac{\beta}{\lambda}$ and non-zero constant. In fact, we assume $(\beta^2+\lambda^2)x - \frac{\beta}{\lambda} = 0$. Then $x = \frac{\beta}{\lambda(\beta^2+\lambda^2)}$ because β and λ are almost everywhere non-zero, from which, substituting into (2.37), $\beta\lambda^2 = 0$. It contradicts our assumptions.

Denoting $(\beta^2+\lambda^2)x - \frac{\beta}{\lambda}$ and $-\frac{\beta}{\lambda}$ respectively by a and b, and r by multiplicity of a, a and b are both non-zero constants. When $a = b$, $r = 0$ or $r = 2n-1$, it is contained in the Case (A).

Thus we may only consider that $a \neq b$ and $1 \leqq r \leqq 2n-2$. Now we define a (1, 1)-type tensor P_j^i of the from;

$$P_j{}^i=\frac{1}{a-b}(h_j{}^i-b\delta_j^i).$$

Then we can easily see that

$$1\leqq \text{rank of } (P_j{}^i)\leqq 2n-2, \tag{2.43}$$

$$P_j{}^tP_{ti}=P_{ji}, \tag{2.44}$$

that is, $P_j{}^i$ is an almost product structure such that

$$\nabla_k P_j{}^i=0 \tag{2.45}$$

because of Lemma 2.4, where $P_{ji}=P_j{}^tg_{ti}$.

Putting $Q_{ji}=g_{ji}-P_{ji}$, we find

$$h_{ji}=aP_{ji}+bQ_{ji}. \tag{2.46}$$

Moreover, if M is complete and connected, the equations (2.43)~(2.46) mean that assumptions of Theorem B are all satisfied.

Summing up the conclusions obtained in Case (A) and Case (B), we have

THEOREM 2.5. *Let M be a complete and connected hypersurface of an even-dimensional sphere S^{2n}. If the induced $(f, g, u_{(k)}, \alpha_{(k)})$-structure is normal, the vectors u^h, v^h and w^h (or associated 1-forms u_i, v_i and w_i) are linearly independent and functions β, λ are non-zero almost everywhere on M, then M is congruent to S^{2n-1} or $S^p\times S^{2n-1-p}$ $(p=1, 2, \cdots, 2n-2)$ naturally embedded in S^{2n}.*

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