Y. MUTŌ
KŌDAI MATH. SEM. REP
26 (1975), 409—423

CRITICAL RIEMANNIAN METRICS ON PRODUCT MANIFOLDS

BY YOSIO MUTŌ

A critical Riemannian metric of this paper means a critical point of a functional I of a C^∞ Riemannian metric g on a compact orientable C^∞ manifold M, restricted by $\mathrm{Vol}(M, g)=1$ and defined by an integral of the square of the curvature tensor of (M, g). In the present paper a critical Riemannian metric g_{12} such that $(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)$ is studied and relations between critical Riemannian metrics g_{12}, g_1 and g_2 on M, M_1 and M_2 respectively are obtained. Furthermore it is shown that in certain cases the index of I at g_{12} is positive.

In a previous paper [5] the present author considered the space $\mathcal{M}(M)$ of C^∞ Riemannian metrics g on a compact orientable C^∞ manifold M satisfying the condition

$$\int_M dV_g=1 \tag{0.1}$$

where dV_g is the volume element of M measured by g. He studied a mapping $I: \mathcal{M}(M)\to \boldsymbol{R}$ induced by the integral

$$I[g]=\int_M \|K_g\|^2 dV_g \tag{0.2}$$

where K_g is the curvature tensor of (M, g) and $\|K_g\|^2$ is its square.

If η is a diffeomorphism of M and η^* its pull back, then we have $\eta^*(g)\in\mathcal{M}(M)$ and $I[g]=I[\eta^*(g)]$. Let $\mathcal{D}(M)$ be the group of diffeomorphisms of M and $\mathcal{M}(M)/\mathcal{D}(M)$ be the space where each point is an orbit O_g by $\mathcal{D}(M)$ through an element g of $\mathcal{M}(M)$. Then we can deduce a mapping $\tilde{I}: \mathcal{M}(M)/\mathcal{D}(M)\to\boldsymbol{R}$ from the mapping $I: \mathcal{M}(M)\to\boldsymbol{R}$ by $\tilde{I}(O_g)=I[g]$. As O_g is a critical point of $\tilde{I}$ if and only if g is a critical point of I, we adopt the convention to say that g is a critical point of $\tilde{I}$ when O_g is a critical point of $\tilde{I}$. We also say that $\tilde{I}$ has a minimum or a local minimum at g when $\tilde{I}$ has a minimum or a local minimum at O_g. Thus, if we say that $\tilde{I}$ has a local minimum at g, this means that there exists a neighborhood U of O_g in $\mathcal{M}(M)$ such that, if g_1 is a Riemannian metric satisfying $g_1\notin O_g$, $g_1\in U$, then $I[g_1]>I[g]$.

Remark 1. The manifold of C^∞ Riemannian metrics on M, which we denote for the present by $\mathcal{M}^\#(M)$ in order to distinguish from our $\mathcal{M}(M)$, has been

Received Nov 26, 1973.

studied by D. Ebin [4]. He has analysed the action of $\mathcal{D}(M)$ on $\mathcal{M}^{\#}(M)$ and proved the existence of submanifolds S of $\mathcal{M}^{\#}(M)$ with a certain property. It results that to study deformations in $\mathcal{M}^{\#}(M)/\mathcal{D}(M)$ we need only study curves in $\mathcal{M}^{\#}(M)$ whose tangent at g is in $\delta^{-1}(0)$, namely, orthogonal to the orbits (see also M. Berger and D. Ebin [3]). The same is also valid with $\mathcal{M}(M)$ as the latter is a submanifold of $\mathcal{M}^{\#}(M)$ invariant by the action of $\mathcal{D}(M)$.

Remark 2. The mapping I has been studied by M. Berger and the formula for a critical point has been obtained [2].

It was proved in [5] that, when M is diffeomorphic to an S^n, $\tilde{I}$ has a local minimum at a metric g_0 of positive constant curvature.

The purpose of the present paper is to study the mapping I or $\tilde{I}$ when M is a product manifold $M_1 \times M_2$ where M_1 and M_2 are compact orientable C^∞ manifolds.

When we say in the present paper that g is a critical Riemannian metric on M, it always means that g is a critical point of the mapping I or $\tilde{I}$ defined by (0, 2). At that time (M, g) is called a critical Riemannian manifold.

First we get the following theorems.

THEOREM 1. *Let M, M_1, M_2 be compact orientable C^∞ manifolds such that $M=M_1\times M_2$ and $\dim M_1=m_1$, $\dim M_2=m_2$. Let $g_{12}\in\mathcal{M}(M)$ be a C^∞ Riemannian metric such that there exist a Riemannian metric $'g_1$ homothetic to a critical Riemannian metric g_1 on M_1 and a Riemannian metric $'g_2$ homothetic to a critical Riemannian metric g_2 on M_2 satisfying*

$$(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)\,.$$

Then a necessary and sufficient condition that g_{12} be a critical Riemannian metric on M is that the square $\|'K_1\|^2$ of the curvature tensor of $(M_1, 'g_1)$ and the square $\|'K_2\|^2$ of the curvature tensor of $(M_2, 'g_2)$ be constant and

$$\frac{\|'K_1\|^2}{m_1}=\frac{\|'K_2\|^2}{m_2}\,.$$

THEOREM 2. *Let M, M_1, M_2 be compact orientable C^∞ manifolds and let $g_{12}\in\mathcal{M}(M)$ be such that there exist a Riemannian metric $'g_1$ on M_1 and a Riemannian metric $'g_2$ on M_2 satisfying*

$$(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)\,.$$

Then a necessary and sufficient condition that g_{12} be a critical Riemannian metric is that $'g_1$ and $'g_2$ be homothetic to a critical Riemannian metric g_1 on M_1 and a critical Riemannian metric g_2 on M_2 respectively and the squares of the curvature tensors, $\|'K_1\|^2$ and $\|'K_2\|^2$, of the Riemannian manifolds $(M_1, 'g_1)$ and $(M_2, 'g_2)$ respectively be constant satisfying

$$\frac{\|'K_1\|^2}{m_1}=\frac{\|'K_2\|^2}{m_2}\,.$$

In the last part of the present paper, the index of I at such critical Riemannian metric g_{12} is studied and it is proved that this index is positive in certain cases. Especially the mapping $I: \mathcal{M}(S^{m_1}\times S^{m_2})\to \boldsymbol{R}$ has positive index at a critical Riemannian metric g_{12} such that $(S^{m_1}\times S^{m_2}, g_{12})=(S^{m_1}, 'g_1)\times(S^{m_2}, 'g_2)$ where $'g_1$ and $'g_2$ are Riemannian metrics of positive constant curvature, if $m_1\geqq 3$ and $m_2\geqq 3$, or, if $m_1\geqq 4$ and $m_2=2$. This is a remarkable result as A. Avez has obtained the following theorem [1].

THEOREM A. *Let M be a compact orientable C^∞ manifold of dimension* 4. *Then the functional $I[g]$ has an absolute minimum at g if and only if g is an Einstein metric.*

§1. Product manifold and Riemannian metrics.

Let M, M_1, M_2 be compact orientable C^∞ manifold such that $M=M_1\times M_2$ and let $\mathcal{M}(M)$ be the space of all C^∞ Riemannian metrics g on M such that the volume of M measured by g is 1. Similarly we can define $\mathcal{M}(M_1)$ and $\mathcal{M}(M_2)$. Let us consider a Riemannian metric $g_{12}\in\mathcal{M}(M)$ such that there exist Riemannian metrics $'g_1$ and $'g_2$ satisfying $(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)$ where $'g_1$ and $'g_2$ need not satisfy $'g_1\in\mathcal{M}(M_1)$, or $'g_2\in\mathcal{M}(M_2)$. We denote the set of all such Riemannian metrics g_{12} by $\mathcal{M}_{12}(M_1\times M_2)$ or $\mathcal{M}_{12}(M)$.

To begin with we calculate the curvature tensor of (M, g_{12}).

Let U_ξ, $\xi\in\Lambda_1$, and V_η, $\eta\in\Lambda_2$, be coordinate neighborhoods of M_1 and M_2 respectively such that $\{U_\xi, \xi\in\Lambda_1\}$ and $\{V_\eta, \eta\in\Lambda_2\}$ cover M_1 and M_2 respectively. Then $\{U_\xi\times V_\eta, \xi\in\Lambda_1, \eta\in\Lambda_2\}$ covers M and we can use local coordinates

$$(x_{(\xi)}{}^1, \cdots, x_{(\xi)}{}^{m_1}, y_{(\eta)}{}^{m_1+1}, \cdots, y_{(\eta)}{}^{m_1+m_2}),$$

where $m_1=\dim M_1$, $m_2=\dim M_2$, to denote a point $P=P_1\times P_2$ of M if $P\in U_\xi\times V_\eta$.

We let the indices $a, b, c, \cdots, h, i, j, \cdots, p, q, r, \cdots$ run the range $\{1, \cdots, m_1\}$ and the indices $\alpha, \beta, \gamma, \cdots, \kappa, \lambda, \mu, \cdots, \pi, \rho, \sigma, \cdots$ the range $\{m_1+1, \cdots, m_1+m_2\}$. We also let the indices $A, B, C, \cdots, H, I, J, \cdots, P, Q, R, \cdots$ run the range $\{1, \cdots, m_1, m_1+1, \cdots, m_1+m_2\}$ so that a point of M may be denoted by $(x_{(\xi)}{}^h, y_{(\eta)}{}^\kappa)$ or simply by (x^h, y^κ). Moreover, (x^A) stands for (x^h, y^κ). We use natural frame in each coordinate neighborhood $U_\xi\times V_\eta$ so that a tensor is expressed by its components. For example, a (1, 1)-tensor of M is given by $T_B{}^A$ or, if written separately, by $T_b{}^a, T_\beta{}^a, T_b{}^\alpha, T_\beta{}^\alpha$.

Since M is a product manifold and the local coordinates in M are induced by local coordinates in M_1 and those in M_2, a (1, 1)-tensor field $A_b{}^a$ on M_1 and a (1, 1)-tensor field $B_\beta{}^\alpha$ on M_2 induce a (1, 1)-tensor field $C_B{}^A$ on M such that

$$C_b{}^a(P)=A_b{}^a(P_1), \qquad C_\beta{}^\alpha(P)=B_\beta{}^\alpha(P_2), \qquad C_\beta{}^a(P)=C_b{}^\alpha(P)=0$$

where $P=P_1\times P_2$. But in general a (1, 1)-tensor field $T_B{}^A$ on M does not have such a property, for example, $T_b{}^a(P)$ may depend on y^κ and $T_\beta{}^a$ need not vanish.

Now, let $g_{12}\in M_{12}(M)$ be a Riemannian metric on M such that

$$(1.1) \qquad (M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)\,.$$

Denoting the components of g_{12}, $'g_1$, $'g_2$ by g_{JI}, $'g_{ji}$, $'g_{\mu\lambda}$ respectively, we have

$$g_{ji}='g_{ji}\,, \qquad g_{\mu\lambda}='g_{\mu\lambda}\,, \qquad g_{j\lambda}=0\,.$$

Let $\left\{ {H \atop JI} \right\}$, $'\left\{ {h \atop ji} \right\}$, $'\left\{ {\kappa \atop \mu\lambda} \right\}$ be the Christoffel symbols derived from g_{JI}, $'g_{ji}$, $'g_{\mu\lambda}$ respectively and $K_{KJI}{}^H$, $'K_{kji}{}^h$, $'K_{\nu\mu\lambda}{}^\kappa$ be the components of the curvature tensors of (M, g_{12}), $(M_1, 'g_1)$, $(M_2, 'g_2)$ respectively. Then we have

$$\left\{ {h \atop ji} \right\}='\left\{ {h \atop ji} \right\}, \qquad \left\{ {\kappa \atop \mu\lambda} \right\}='\left\{ {\kappa \atop \mu\lambda} \right\}, \qquad \text{all other } \left\{ {H \atop JI} \right\}=0$$

and

$$K_{kji}{}^h='K_{kji}{}^h\,, \qquad K_{\nu\mu\lambda}{}^\kappa='K_{\nu\mu\lambda}{}^\kappa\,, \qquad \text{all other } K_{KJI}{}^H=0\,.$$

The covariant components K_{KJIH}, $'K_{kjih}$, $'K_{\nu\mu\lambda\kappa}$ and the contravariant components K^{KJIH}, $'K^{kjih}$, $'K^{\nu\mu\lambda\kappa}$ also satisfy

$$K_{kjih}='K_{kjih}\,, \qquad K_{\nu\mu\lambda\kappa}='K_{\nu\mu\lambda\kappa}\,, \qquad \text{all other } K_{KJIH}=0\,,$$

$$K^{kjih}='K^{kjih}\,, \qquad K^{\nu\mu\lambda\kappa}='K^{\nu\mu\lambda\kappa}\,, \qquad \text{all other } K^{KJIH}=0\,.$$

We have also

$$K_{ji}='K_{ji}\,, \qquad K_{\mu\lambda}='K_{\mu\lambda}\,, \qquad \text{all other } K_{JI}=0\,,$$

$$K^{ji}='K^{ji}\,, \qquad K^{\mu\lambda}='K^{\mu\lambda}\,, \qquad \text{all other } K^{JI}=0$$

for the components of Ricci tensors of (M, g_{12}), $(M_1, 'g_1)$ and $(M_2, 'g_2)$. The scalar curvature $Sc(K)$ of g_{12} and the scalar curvatures $Sc('K_1)$ of $'g_1$, $Sc('K_2)$ of $'g_2$ satisfy

$$Sc(K)=Sc('K_1)+Sc('K_2)\,.$$

Then we have the following formula for the integral $I[g_{12}]$,

$$(1.2) \qquad I[g_{12}]=\int_M K_{KJIH}K^{KJIH}dV_g$$

$$=\int_M ['K_{kjih}'K^{kjih}+'K_{\nu\mu\lambda\kappa}'K^{\nu\mu\lambda\kappa}]dV_g$$

where

$$dV_g=\{\det('g_{ji})\det('g_{\mu\lambda})\}^{\frac{1}{2}}dx^1\cdots dx^m\,, \qquad m=\dim M\,.$$

§2. A critical Riemannian metric on a product manifold (Proof of Theorem 1).

Now we want to get a critical Riemannian metric g_{12} on $M=M_1\times M_2$ such that $g_{12}\in M_{12}(M)$.

The metric g_{12} in (1.1) satisfies $g_{12} \in \mathcal{M}(M)$, while $'g_1$ and $'g_2$ need not satisfy $'g_1 \in \mathcal{M}(M_1)$, $'g_2 \in \mathcal{M}(M_2)$. But it is easy to see that there exist some positive numbers α_1 and α_2 such that $g_1=(\alpha_1)^2 {'g_1}$ and $g_2=(\alpha_2)^2 {'g_2}$ satisfy $g_1 \in \mathcal{M}(M_1)$ and $g_2 \in \mathcal{M}(M_2)$. Let us find a relation between α_1 and α_2. If we denote for the present the volume element of M measured by g_{12} by dV and the volume elements of M_1 and M_2 measured by $'g_1$ and $'g_2$ respectively by $d'V_1$ and $d'V_2$, then we have $dV=d'V_1 d'V_2$, hence

$$\left[\int_{M_1} d'V_1\right] \cdot \left[\int_{M_2} d'V_2\right]=1 .$$

On the other hand, if we denote the volume elements of M_1 and M_2 measured by g_1 and g_2 respectively by dV_1 and dV_2, then we have

$$(\alpha_1)^{m_1}\int_{M_1} d'V_1=\int_{M_1} dV_1=1 ,$$

$$(\alpha_2)^{m_2}\int_{M_2} d'V_2=\int_{M_2} dV_2=1 .$$

Hence we have

$$(\alpha_1)^{m_1}(\alpha_2)^{m_2}=1 . \tag{2.1}$$

A necessary and sufficient condition that a Riemannian metric g be a critical Riemannian metric was obtained by M. Berger [2] as a system of differential equations involving the curvature tensor, the Ricci tensor, covariant derivatives of the scalar curvature and the Ricci tensor. Let us examine the equations for a moment.

For that purpose let M be for the present any compact orientable C^∞ manifold. If, using local coordinates $x^1, \cdots, x^n$ and the natural frame, we denote tensors by their components, so that the curvature tensor and the Ricci tensor by $K_{kji}{}^h$ and K_{ji}, and raise or lower indices by the components g^{ji} or g_{ji} of the fundamental tensor, the equations in question are as in [5]

$$\begin{aligned} 2\nabla_j \nabla_i Sc(K) &- 4\nabla_p \nabla^p K_{ji} \\ &+4K_{jp}K^p{}_i - 4K_{jqpi}K^{qp} \\ &-2K^{rqp}{}_j K_{rqpi} + \frac{1}{2} K_{dcba}K^{dcba} g_{ji} = c g_{ji} \end{aligned} \tag{2.2}$$

where $Sc(K)$ is the scalar curvature, ∇_i means the covariant differentiation with the use of the Christoffel symbols of g, and c is a number which is chosen suitably so that a solution may exist.

Let us assume g is a critical Riemannian metric and $'g$ is a Riemannian metric homothetic to g, namely, there exists a positive number α such that $g=\alpha^2 {'g}$. Let the components of $'g$ be denoted by $'g_{ji}$ and the components of the curvature tensor and the Ricci tensor of $(M, 'g)$ by $'K_{kji}{}^h$ and $'K_{ji}$. Let the indices of these tensors be raised and lowered by the components $'g^{ji}$ and $'g_{ji}$

of the fundamental tensor $'g$ and the scalar curvature of $(M, 'g)$ be denoted by $Sc('K)$. As g and $'g$ have the same Christoffel symbols, covariant differentiation is the same in $(M, 'g)$ as in (M, g) and we have $'K_{kji}{}^{h}=K_{kji}{}^{h}$, $'K_{ji}=K_{ji}$, $Sc('K)=\alpha^2 Sc(K)$, $'\nabla_i=\nabla_i$, $'\nabla^i=\alpha^2\nabla^i$, $'K_i{}^h=\alpha^2 K_i{}^h$, $'K_{jqpi}'K^{qp}=\alpha^2 K_{jqpi}K^{qp}$, $'K^{rqp}{}_j{}'K_{rqpi}=\alpha^2 K^{rqp}{}_j K_{rqpi}$, $'K_{dcba}'K^{dcba}=\alpha^4 K_{dcba}K^{dcba}$. Hence we get

$$
\begin{aligned}
(2.3)\qquad & 2'\nabla_j{}'\nabla_i Sc('K)-4'\nabla_p{}'\nabla^p{}'K_{ji} \\
& +4'K_{jp}{}'K^p{}_i-4'K_{jqpi}{}'K^{qp} \\
& -2'K^{rqp}{}_j{}'K_{rqpi}+\frac{1}{2}'K_{dcba}{}'K^{dcba}{}'g_{ji}=c\alpha^4{}'g_{ji}\,,
\end{aligned}
$$

where c is the same number as in (2.2).

As c is not given beforehand, we get the following lemma.

LEMMA 2.1. *Let M be a compact orientable C^∞ manifold and $'g$ be a C^∞ Riemannian metric on M. A necessary and sufficient condition that there exist a critical Riemannian metric g homothetic to $'g$ is that there exist a constant c_1 such that*

$$
\begin{aligned}
(2.4)\qquad & 2'\nabla_j{}'\nabla_i Sc('K)-4'\nabla_p{}'\nabla^p{}'K_{ji} \\
& +4'K_{jp}{}'K^p{}_i-4'K_{jqpi}{}'K^{qp} \\
& -2'K^{rqp}{}_j{}'K_{rqpi}+\frac{1}{2}'K_{dcba}{}'K^{dcba}{}'g_{ji}=c_1{}'g_{ji}\,.
\end{aligned}
$$

Now let us return to the subject and prove Theorem 1.

A necessary and sufficient condition that there exist a critical Riemannian metric g_1 on M_1 such that $g_1=(\alpha_1)^2{}'g_1$, where α_1 is a positive number, is, as we see immediately from Lemma 2.1, that there exist a constant c_1 such that

$$
\begin{aligned}
(2.5)\qquad & 2'\nabla_j{}'\nabla_i Sc('K_1)-4'\nabla_p{}'\nabla^p{}'K_{ji} \\
& +4'K_{jp}{}'K^p{}_i-4'K_{jqpi}{}'K^{qp} \\
& -2'K^{rqp}{}_j{}'K_{rqpi}+\frac{1}{2}'K_{dcba}{}'K^{dcba}{}'g_{ji}=c_1{}'g_{ji}
\end{aligned}
$$

where all tensors, the scalar curvature $Sc('K_1)$ and covariant differentiation are those of the Riemannian structure in $(M_1, 'g_1)$. Similarly, a necessary and sufficient condition that there exist a critical Riemannian metric g_2 on M_2 such that $g_2=(\alpha_2)^2{}'g_2$ is that there exist a constant c_2 such that

$$
\begin{aligned}
(2.6)\qquad & 2'\nabla_\mu{}'\nabla_\lambda Sc('K_2)-4'\nabla_\rho{}'\nabla^\rho{}'K_{\mu\lambda} \\
& +4'K_{\mu\rho}{}'K^\rho{}_\lambda-4'K_{\mu\sigma\rho\lambda}{}'K^{\sigma\rho} \\
& -2'K^{\tau\sigma\rho}{}_\mu{}'K_{\tau\sigma\rho\lambda}+\frac{1}{2}'K_{\tau\sigma\rho\pi}{}'K^{\tau\sigma\rho\pi}{}'g_{\mu\lambda}=c_2{}'g_{\mu\lambda}
\end{aligned}
$$

where all tensors, the scalar curvature $Sc('K_2)$ and covariant differentiation are those of $(M_2, 'g_2)$.

On the other hand, a necessary and sufficient condition that g_{12} with components g_{JI} be a critical Riemannian metric on M is that there exist a constant c such that

$$
\begin{aligned}
(2.7)\qquad & 2\nabla_J\nabla_I Sc(K)-4\nabla_P\nabla^P K_{JI}\\
& +4K_{JP}K^P{}_I-4K_{JQPI}K^{QP}\\
& -2K^{RQP}{}_J K_{RQPI}+\frac{1}{2}K_{DCBA}K^{DCBA}g_{JI}=cg_{JI}
\end{aligned}
$$

where all tensors, the scalar curvature $Sc(K)$ and covariant differentiation are those of (M, g_{12}).

As we have (1.1), all genuine quantities of $(M_1, 'g_1)$ do not depend on x^κ and all genuine quantities of $(M_2, 'g_2)$ do not depend on x^h. From (1.1) and all the formulas following (1.1) we thus obtain following relations between quantities in (M, g_{12}) and quantities in $(M_1, 'g_1)$ or $(M_2, 'g_2)$,

$$
\begin{aligned}
& \nabla_i Sc(K)='\nabla_i Sc('K_1)\,, \quad \nabla_\lambda Sc(K)='\nabla_\lambda Sc('K_2)\,,\\
& \nabla_j\nabla_i Sc(K)='\nabla_j'\nabla_i Sc('K_1)\,, \quad \nabla_\mu\nabla_\lambda Sc(K)='\nabla_\mu'\nabla_\lambda Sc('K_2)\,, \quad \nabla_j\nabla_\lambda Sc(K)=0\,,\\
& \nabla_P\nabla^P K_{ji}='\nabla_p'\nabla^p{}'K_{ji}\,, \quad \nabla_P\nabla^P K_{\mu\lambda}='\nabla_\rho'\nabla^\rho{}'K_{\mu\lambda}\,, \quad \nabla_P\nabla^P K_{j\lambda}=0\,,\\
& K_{jP}K^P{}_i='K_{jp}'K^p{}_i\,, \quad K_{\mu P}K^P{}_\lambda='K_{\mu\rho}'K^\rho{}_\lambda\,, \quad K_{jP}K^P{}_\lambda=0\,,\\
& K_{jQPi}K^{QP}='K_{jqpi}'K^{qp}\,, \quad K_{\mu QP\lambda}K^{QP}='K_{\mu\sigma\rho\lambda}'K^{\sigma\rho}\,, \quad K_{jQP\lambda}K^{QP}=0\,,\\
& K^{RQP}{}_j K_{RQPi}='K^{rqp}{}_j'K_{rqpi}\,, \quad K^{RQP}{}_\mu K_{RQP\lambda}='K^{\tau\sigma\rho}{}_\mu'K_{\tau\sigma\rho\lambda}\,, \quad K^{RQP}{}_j K_{RQP\lambda}=0\,,\\
& K_{DCBA}K^{DCBA}='K_{dcba}'K^{dcba}+'K_{\tau\sigma\rho\pi}'K^{\tau\sigma\rho\pi}\,.
\end{aligned}
$$

Thus (2.7) is equivalent in this case to the following set of equations (2.8) and (2.9),

$$
\begin{aligned}
(2.8)\qquad & 2'\nabla_j'\nabla_i Sc('K_1)-4'\nabla_p'\nabla^p{}'K_{ji}\\
& +4'K_{jp}'K^p{}_i-4'K_{jqpi}'K^{qp}\\
& -2'K^{rqp}{}_j'K_{rqpi}+\frac{1}{2}('K_{dcba}'K^{dcba}+'K_{\tau\sigma\rho\pi}'K^{\tau\sigma\rho\pi})'g_{ji}=c'g_{ji}\,,
\end{aligned}
$$

$$
\begin{aligned}
(2.9)\qquad & 2'\nabla_\mu'\nabla_\lambda Sc('K_2)-4'\nabla_\rho'\nabla^\rho{}'K_{\mu\lambda}\\
& +4'K_{\mu\rho}'K^\rho{}_\lambda-4'K_{\mu\sigma\rho\lambda}'K^{\sigma\rho}\\
& -2'K^{\tau\sigma\rho}{}_\mu'K_{\tau\sigma\rho\lambda}+\frac{1}{2}('K_{dcba}'K^{dcba}+'K_{\tau\sigma\rho\pi}'K^{\tau\sigma\rho\pi})'g_{\mu\lambda}=c'g_{\mu\lambda}\,.
\end{aligned}
$$

Now let us assume that $g_1=(\alpha_1)^2{}'g_1$ and $g_2=(\alpha_2)^2{}'g_2$ are critical Riemannian metrics on M_1 and M_2 respectively. Then we get from (2.5) and (2.8) or from (2.6) and (2.9)

$$c_1+\frac{1}{2}\|'K_2\|^2=c\,, \qquad c_2+\frac{1}{2}\|'K_1\|^2=c \tag{2.10}$$

where $'K_1$ and $'K_2$ are the curvature tensors of $(M_1, 'g_1)$ and $(M_2, 'g_2)$ respectively. Thus we have

$$\|'K_1\|^2-\|'K_2\|^2=2(c_1-c_2)\,, \tag{2.11}$$

which proves that, if g_{12}, g_1 and g_2 are critical Riemannian metrics on M, M_1 and M_2 respectively, then $\|'K_1\|^2$ and $\|'K_2\|^2$ are constant on M.

Furthermore, we get from (2.5) and (2.6), by transvecting with $'g^{ji}$ and $'g^{\mu\lambda}$,

$$\begin{aligned} c_1&=-\frac{2}{m_1}{}'\nabla_p{}'\nabla^p Sc('K_1)+\Big(\frac{1}{2}-\frac{2}{m_1}\Big)\|'K_1\|^2\,,\\ c_2&=-\frac{2}{m_2}{}'\nabla_\rho{}'\nabla^\rho Sc('K_2)+\Big(\frac{1}{2}-\frac{2}{m_2}\Big)\|'K_2\|^2\,. \end{aligned} \tag{2.12}$$

Hence $'\nabla_p{}'\nabla^p Sc('K_1)$ must be constant. But we have

$$\int_{M_1}{}'\nabla_p{}'\nabla^p Sc('K_1)dV_{g_1}=0\,.$$

Thus we get $Sc('K_1)=\text{const}$. Similarly we get $Sc('K_2)=\text{const}$. At the same time we get

$$c_1=\Big(\frac{1}{2}-\frac{2}{m_1}\Big)\|'K_1\|^2\,, \qquad c_2=\Big(\frac{1}{2}-\frac{2}{m_2}\Big)\|'K_2\|^2\,. \tag{2.13}$$

From this and (2.11) we get

$$\frac{\|'K_1\|^2}{m_1}=\frac{\|'K_2\|^2}{m_2} \tag{2.14}$$

and

$$\frac{(\alpha_1)^4\|K_1\|^2}{m_1}=\frac{(\alpha_2)^4\|K_2\|^2}{m_2}\,.$$

Conversely, if we have (2.14) where $\|'K_1\|^2$ and $\|'K_2\|^2$ are constant, then we have $'\nabla_p{}'\nabla^p Sc('K_1)=\text{const}$ and $'\nabla_\rho{}'\nabla^\rho Sc('K_2)=\text{const}$ from (2.12), hence $Sc('K_1)=\text{const}$ and $Sc('K_2)=\text{const}$. Thus we get (2.13). Furthermore we can determine c by (2.10). As we have (2.5) and (2.6), we get (2.8) and (2.9).

Thus we have proved Theorem 1.

From this theorem we get

THEOREM 2.2. *Let M, M_1, M_2 be compact orientable C^∞ manifolds such that $M=M_1\times M_2$. Assume that g_1 and g_2 are non-flat critical Riemannian metrics on M_1 and M_2 respectively. Then a necessary and sufficient condition that there exist a critical Riemannian metric g_{12} on M and Riemannian metrics $'g_1$ and $'g_2$*

satisfying

$$(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)$$

and such that $'g_1$ and $'g_2$ are homothetic to g_1 and g_2 respectively is that the squares of the curvature tensors, $\|K_1\|^2$ and $\|K_2\|^2$, of (M_1, g_1) and (M_2, g_2) be constant.

Proof. If such a critical Riemannian metric g_{12} exists and if we put $'g_1=\alpha_1^{-2}g_1$ and $'g_2=\alpha_2^{-2}g_2$, we get $\|'K_1\|^2=\alpha_1^4\|K_1\|^2$, $\|'K_2\|^2=\alpha_2^4\|K_2\|^2$. Thus $\|K_1\|^2$ and $\|K_2\|^2$ are constant because of Theorem 1. Conversely let us assume $\|K_1\|^2$ and $\|K_2\|^2$ are constant. If we put

$$m_2\|K_1\|^2=m_1\|K_2\|^2A^{4(m_1+m_2)},$$

A is constant and does not vanish as g_1 and g_2 are non flat. Then

$$'g_1=\alpha_1^{-2}g_1, \qquad 'g_2=\alpha_2^{-2}g_2$$

where

$$\alpha_1=A^{-m_2}, \qquad \alpha_2=A^{m_1}$$

are Riemannian metrics such that

$$\|'K_1\|^2=A^{-4m_2}\|K_1\|^2, \qquad \|'K_2\|^2=A^{4m_1}\|K_2\|^2,$$

hence

$$\frac{\|'K_1\|^2}{m_1}=\frac{\|'K_2\|^2}{m_2}.$$

Moreover $g_{12}\in\mathcal{M}(M)$ because of $\alpha_1^{m_1}\alpha_2^{m_2}=1$. Thus g_{12} is a critical Riemannian metric because of Theorem 1.

§ 3. Proof of Theorem 2.

Next let us consider the case in which g_{12}, $'g_1$ and $'g_2$ satisfy (1.1) and g_{12} is a critical Riemannian metric on M. Then we get from (2.8) and (2.9)

$$-2'\nabla_p'\nabla^p Sc('K_1)-2\|'K_1\|^2+\frac{m_1}{2}(\|'K_1\|^2+\|'K_2\|^2)=m_1c,$$

$$-2'\nabla_\rho'\nabla^\rho Sc('K_2)-2\|'K_2\|^2+\frac{m_2}{2}(\|'K_1\|^2+\|'K_2\|^2)=m_2c.$$

Hence

$$\frac{1}{m_1}\{'\nabla_p'\nabla^p Sc('K_1)+\|'K_1\|^2\}=\frac{1}{m_2}\{'\nabla_\rho'\nabla^\rho Sc('K_2)+\|'K_2\|^2\}$$

$$=\frac{1}{2}\Big[\frac{1}{2}(\|'K_1\|^2+\|'K_2\|^2)-c\Big]$$

is a constant which we shall write C. Then we have

$$-2C+\frac{1}{2}(\|'K_1\|^2+\|'K_2\|^2)=c$$

and consequently $\|'K_1\|^2$, $\|'K_2\|^2$, $'\nabla_p{}'\nabla^p Sc('K_1)$, $'\nabla_\rho{}'\nabla^\rho Sc('K_2)$ are constants on M. Thus $Sc('K_1)$ and $Sc('K_2)$ are again constants.

On the other hand we get from (2.8)

$$2'\nabla_j{}'\nabla_i Sc('K_1)-4'\nabla_p{}'\nabla^p{}'K_{ji}+4'K_{jp}{}'K^p{}_i-4'K_{jqpi}{}'K^{qp}$$
$$-2'K^{rqp}{}_j{}'K_{rqpi}+\frac{1}{2}\|'K_1\|^2{}'g_{ji}=\left\{c-\frac{1}{2}\|'K_2\|^2\right\}'g_{ji}$$

which is equivalent to (2.6) if we put

$$c_1=c-\frac{1}{2}\|'K_2\|^2 .$$

Taking Lemma 2.1 into account, we see that $'g_1$ is homothetic to a critical Riemannian metric on M_1. Similarly $'g_2$ is homothetic to a critical Riemannian metric on M_2. Thus we have proved Theorem 2 in view of Theorem 1.

If M_1 admits a locally flat Riemannian metric $'g_1$, then we have $\|'K_1\|^2=0$. Hence (2.14) is not satisfied if $\|'K_2\|^2>0$. Thus we obtain

THEOREM 3.1. *A Riemannian manifold $(M, g)=(M_1, 'g_1)\times(M_2, 'g_2)$ can not be a critical Riemannian manifold if $(M_1, 'g_1)$ is locally flat and $(M_2, 'g_2)$ is not locally flat.*

§ 4. The index of a critical Riemannian manifold $(M_1, 'g_1)\times(M_2, 'g_2)$.

Let M, M_1, M_2 be the same as before and g_{12} in $(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)$ be a critical Riemannian metric. By Theorem 2 $'g_1$ and $'g_2$ are homothetic to g_1 and g_2 respectively which are critical Riemannian metrics on M_1 and M_2 respectively with constant $Sc(K_1)$, $Sc(K_2)$, $\|K_1\|^2$ and $\|K_2\|^2$. We examine now the index of $I:\mathcal{M}(M)\to\boldsymbol{R}$ at the critical point g_{12}, which we call the index of the critical Riemannian manifold $(M_1, 'g_1)\times(M_2, 'g_2)$.

We do not calculate the exact value of this index, but intend to show that in certain cases the index is positive.

Let us take Riemannian metrics g on $M=M_1\times M_2$ such that the components g_{JI} of g are given by

$$g_{ji}=e^{2a(y)}{}'g_{ji}\,,\qquad g_{\mu\lambda}='g_{\mu\lambda}\,,\qquad g_{j\lambda}=0 \tag{4.1}$$

where $'g_{ji}$ and $'g_{\mu\lambda}$ are respectively components of $'g_1$ and $'g_2$ again and $a(y)$ is a function of x^κ only ($\kappa=m_1+1, \cdots, m_1+m_2$).

As $g_{12}\in\mathcal{M}(M)$, in order to maintain the relation $g\in\mathcal{M}(M)$, we take $a(y)$ such that

$$\int_M e^{m_1 a}dV_{g_{12}}=1\,.$$

Hence, if dV_2 is the volume element of M_2 measured by $'g_2$, or g_2 homothetic to $'g_2$, we have

$$\int_{M_2} e^{m_1 a} dV_2 = \int_{M_2} dV_2 . \tag{4.2}$$

We denote in § 4 the Christoffel symbols obtained from g by $\left\{ {H \atop JI} \right\}$, while $'\left\{ {h \atop ji} \right\}$ and $'\left\{ {\kappa \atop \mu\lambda} \right\}$ are the same as defined in § 1. Then the Christoffel symbols $\left\{ {H \atop JI} \right\}$, written separately as $\left\{ {h \atop ji} \right\}$, $\left\{ {\kappa \atop ji} \right\}$ and so on, satisfy the following equations,

$$\left\{ {h \atop ji} \right\} = '\left\{ {h \atop ji} \right\},$$

$$\left\{ {\kappa \atop ji} \right\} = \frac{1}{2} g^{\kappa\tau}(-\partial_\tau g_{ji}) = -g_{ji}{'\nabla^\kappa} a ,$$

$$\left\{ {h \atop j\lambda} \right\} = \frac{1}{2} g^{hp}\partial_\lambda g_{jp} = \delta_j^h \partial_\lambda a = \delta_j^h {'\nabla_\lambda} a ,$$

$$\left\{ {\kappa \atop j\lambda} \right\} = 0 , \qquad \left\{ {h \atop \mu\lambda} \right\} = 0 ,$$

$$\left\{ {\kappa \atop \mu\lambda} \right\} = '\left\{ {\kappa \atop \mu\lambda} \right\},$$

where $'\nabla_\lambda$ means covariant differentiation in $(M_2, 'g_2)$ and $'\nabla^\kappa = 'g^{\kappa\lambda}{'\nabla_\lambda} = g^{\kappa\lambda}{'\nabla_\lambda}$.

From these equations we can calculate the components of the curvature tensor of (M, g) and get[1)]

$$K_{kji}{}^h = 'K_{kji}{}^h - {'\nabla_\rho} a {'\nabla^\rho} a (\delta_k^h g_{ji} - \delta_j^h g_{ki}) ,$$

$$K_{\nu ji}{}^\kappa = -({'\nabla_\nu}{'\nabla^\kappa} a + {'\nabla_\nu} a {'\nabla^\kappa} a) g_{ji} ,$$

$$K_{\nu\mu\lambda}{}^\kappa = 'K_{\nu\mu\lambda}{}^\kappa ,$$

$$K_{kji}{}^\kappa = 0 , \qquad K_{kj\lambda}{}^\kappa = 0 , \qquad K_{\nu\mu i}{}^\kappa = 0 .$$

Furthermore we get

$$\begin{aligned} K_{KJIH}K^{KJIH} &= K_{kjih}K^{kjih} + 4K_{\nu ji\kappa}K^{\nu ji\kappa} + K_{\nu\mu\lambda\kappa}K^{\nu\mu\lambda\kappa} \\ &= e^{-4a}\, 'K_{kjih}\, 'K^{kjih} - 4e^{-2a} Sc('K_1) {'\nabla_\rho} a {'\nabla^\rho} a \\ &\quad + 2m_1(m_1-1)({'\nabla_\rho} a {'\nabla^\rho} a)^2 \\ &\quad + 4m_1({'\nabla_\mu}{'\nabla_\lambda} a + {'\nabla_\mu} a {'\nabla_\lambda} a)({'\nabla^\mu}{'\nabla^\lambda} a + {'\nabla^\mu} a {'\nabla^\lambda} a) \\ &\quad + 'K_{\nu\mu\lambda\kappa}\, 'K^{\nu\mu\lambda\kappa} . \end{aligned}$$

1) In these formulas $\nabla A \nabla B$ always means $(\nabla A)(\nabla B)$.

Let us assume $|a|$ to be so small that we can neglect a^3. Then we get from (4.2)

$$\int_{M_2} a\,dV_2 = -\frac{m_1}{2}\int_{M_2} a^2\,dV_2\,,$$

hence

$$\int_M a\,dV_{g_{12}} = -\frac{m_1}{2}\int_M a^2\,dV_{g_{12}}\,.$$

Consequently we have

$$\int_M K_{KJIH}K^{KJIH}dV_g = \int_M\Big[{}'K_{kjih}{}'K^{kjih}\Big\{1+(m_1-4)a+\frac{(m_1-4)^2}{2}a^2\Big\}$$
$$-4Sc('K_1)'\nabla_\rho a'\nabla^\rho a + 4m_1{}'\nabla_\mu{}'\nabla_\lambda a'\nabla^\mu{}'\nabla^\lambda a$$
$$+{}'K_{\nu\mu\lambda\kappa}{}'K^{\nu\mu\lambda\kappa}e^{m_1 a}\Big]dV_{g_{12}}$$

where we have neglected a^3.

Let us denote this integral by $J[a]$. Then, as $Sc('K_1)$, $\|'K_1\|^2$ and $\|'K_2\|^2$ are constant, we get

$$J[a]-J[0] = -2(m_1-4)\|'K_1\|^2\int_M a^2\,dV$$
$$-4Sc('K_1)\int_M {}'\nabla_\rho a'\nabla^\rho a\,dV + 4m_1\int_M {}'\nabla_\mu{}'\nabla_\lambda a'\nabla^\mu{}'\nabla^\lambda a\,dV$$

where dV is the volume element of M measured by g_{12}.

Let $f(y)$ be a function on M_2 satisfying

$${}'\nabla_\mu{}'\nabla^\mu f = -\lambda_1 f\,, \qquad \int_{M_2} f^2\,dV_2 = 1$$

where λ_1 is the smallest positive eigenvalue of the Laplacian. If α is a small positive number and

$$a(y) = \alpha f(y) - \frac{m_1}{2}\alpha^2(f(y))^2\,,$$

we get

$$\int_{M_2} a\,dV_2 = -\frac{m_1}{2}\alpha^2 = -\frac{m_1}{2}\int_{M_2} a^2\,dV_2$$

neglecting a^3. In this case we have

$$J[a]-J[0] = \Big[\{-2(m_1-4)\|'K_1\|^2 - 4\lambda_1 Sc('K_1)\}$$
$$+4m_1\int_M {}'\nabla_\mu{}'\nabla_\lambda f'\nabla^\mu{}'\nabla^\lambda f\,dV\Big]\alpha^2\,,$$

or, if we use

$$\int_M {}'\nabla_\mu{}'\nabla_\lambda a'\nabla^\mu{}'\nabla^\lambda a\,dV = \int_M ({}'\nabla_\mu{}'\nabla^\mu a)^2 dV - \int_M {}'K^{\mu\lambda}{}'\nabla_\mu a'\nabla_\lambda a\,dV\,,$$

then

$$J[a]-J[0]=\Big[\{-2(m_1-4)\|'K_1\|^2-4\lambda_1 Sc('K_1)\}$$
$$+4m_1\lambda_1^2-4m_1\int_M 'K^{\mu\lambda}\,'\nabla_\mu f\,'\nabla_\lambda f\,dV\Big]\alpha^2 .$$

Thus we have the following lemma.

LEMMA 4.1. *Let* $(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)$ *be a critical Riemannian manifold and let* λ_1 *be the smallest positive eigenvalue of the Laplacian on* $(M_2, 'g_2)$. *Let* f *be an eigenfunction satisfying*

$$\int_{M_2} f^2 dV_2=1 .$$

If, in this case,

$$-2(m_1-4)\|'K_1\|^2-4\lambda_1 Sc('K_1)$$
$$+4m_1\lambda_1^2-4m_1\int_M 'K^{\mu\lambda}\,'\nabla_\mu f\,'\nabla_\lambda f\,dV$$

is negative, the index of the Riemannian manifold (M, g_{12}) *is positive.*

COROLLARY 4.2. *Let the Riemannian manifolds* (M, g_{12}), $(M_1, 'g_1)$, $(M_2, 'g_2)$, *the number* λ_1 *and the function* f *be as in Lemma* 4.1. *Furthermore let* $(M_2, 'g_2)$ *be an Einstein manifold. If, in this case,*

$$-2(m_1-4)\|'K_1\|^2-4\lambda_1 Sc('K_1)+4m_1\lambda_1^2-4m_1\lambda_1\frac{Sc('K_2)}{m_2}$$

is negative, the index of the Riemannian manifold (M, g_{12}) *is positive.*

§ 5. The index of a critical Riemannian manifold $(M, g_{12})=(M_1, 'g_1)\times(M_2, 'g_2)$ where M_2 is a sphere.

Let us consider a critical Riemannian manifold

$$(M, g_{12})=(M_1, 'g_1)\times(S, 'g_2)$$

where S is an m_2-sphere and $'g_2$ is a Riemannian metric of constant curvature with $Sc('K_2)>0$. Then we have

$$\lambda_1=\frac{Sc('K_2)}{m_2-1}$$

and there exists a function f on S satisfying

$$'\nabla_\mu\,'\nabla_\lambda f=-\frac{Sc('K_2)}{m_2(m_2-1)}f\,'g_{\mu\lambda} .$$

In this case we get

$$J[a]-J[0]=\Big[-2(m_1-4)\|'K_1\|^2-4\frac{Sc('K_1)Sc('K_2)}{m_2-1}$$
$$+4\frac{m_1(Sc('K_2))^2}{(m_2-1)^2}-4\frac{m_1(Sc('K_2))^2}{m_2(m_2-1)}\Big]\alpha^2$$

because of

$$'K^{\mu\lambda}=\frac{1}{m_2}Sc('K_2)'g^{\mu\lambda}\,.$$

On the other hand we have (2.14) where we can put

$$\|'K_2\|^2=\frac{2(Sc('K_2))^2}{m_2(m_2-1)}\,.$$

Consequently we get

(5.1) $J[a]-J[0]$

$$=\Big[4m_1\frac{-(m_1-4)(m_2-1)+m_2}{m_2{}^2(m_2-1)^2}(Sc('K_2))^2-4\frac{Sc('K_1)Sc('K_2)}{m_2-1}\Big]\alpha^2\,.$$

Thus we have proved the following theorem.

THEOREM 5.1. *Let (M, g_{12}) be a critical Riemannian manifold such that*

$$(M, g_{12})=(M_1, 'g_1)\times(S, 'g_2)$$

where S is an m_2-sphere and $'g_2$ is a Riemannian metric of constant curvature with $Sc('K_2)>0$. If $Sc('K_1)\geqq 0$ and m_1 and m_2 are such that

$$(m_1-4)(m_2-1)-m_2>0\,,$$

then the index of this critical Riemannian manifold is positive. If $m_1=4$ and

$$Sc('K_1)>\frac{4Sc('K_2)}{m_2(m_2-1)}\,,$$

then the index of (M, g_{12}) is also positive.

We can also prove the following theorem.

THEOREM 5.2. *Let g_{12} be a critical Riemannian metric on $S_1\times S_2$ such that*

$$(S_1\times S_2, g_{12})=(S_1, 'g_1)\times(S_2, 'g_2)$$

where S_1 is an m_1-sphere and S_2 is an m_2-sphere and each of $'g_1$ and $'g_2$ is a Riemannian metric of positive constant curvature. If $m_1\geqq 3$ and $m_2\geqq 3$, or, if $m_1\geqq 4$ and $m_2=2$, the index of $(S_1\times S_2, g_{12})$ is positive.

Proof. As we have

$$\|'K_1\|^2=\frac{2(Sc('K_1))^2}{m_1(m_1-1)}\,,\qquad \|'K_2\|^2=\frac{2(Sc('K_2))^2}{m_2(m_2-1)}\,,$$

we get

$$\frac{(Sc('K_1))^2}{m_1^2(m_1-1)}=\frac{(Sc('K_2))^2}{m_2^2(m_2-1)}.$$

Substituting this into (5.1), we get

$$J[a]-J[0]=-\frac{4m_1}{m_2^2(m_2-1)^2}[(m_1-4)(m_2-1)$$
$$+m_2(\sqrt{m_1-1}\sqrt{m_2-1}-1)](\alpha Sc('K_2))^2.$$

From this equation we immediately obtain Theorem 5.2.

Remark 3. If

$$(S^2\times S^2, g_{12})=(S^2, 'g_1)\times(S^2, 'g_2)$$

is a critical Riemannian manifold where $'g_1='g_2$ is a Riemannian metric of positive constant curvature, $(S^2\times S^2, g_{12})$ is an Einstein manifold. This exists and by Avez's theorem [1] this is a critical Riemannian manifold with index null.

The author wishes to express his hearty thanks to Prof. M. Berger who kindly informed him of Avez's theorem.

REFERENCES

[1] AVEZ, A., Applications de la formule de Gauss-Bonnet-Chern aux variétés à quatre dimensions, C. R. Acad. Sc., **256** (1963), 5488-5490.
[2] BERGER, M., Quelques formules de variation pour une structure riemannienne, Ann. scient. Éc. Norm. Sup., 4^e^ série, **3** (1970), 285-294.
[3] BERGER, M. AND D. EBIN, Some decompositions of the space of symmetric tensors on a Riemannian manifold, J. of Diff. Geometry, **3** (1969), 379-392.
[4] EBIN, D., The manifolds of Riemannian metrics, Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society, **15**, 11-40, American Mathematical Society, Providence, 1970.
[5] MUTŌ, Y., Curvature and critical Riemannian metric, J. Math. Soc. Japan, **26** (1974), 686-697

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING
YOKOHAMA NATIONAL UNIVERSITY
YOKOHAMA, JAPAN