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ON THE RIEMANNIAN MANIFOLDS OF THE FORM $B\times_f F$

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1. Introduction.

R. L. Bishop and B. O'Neill [1] constructed a wide class of Riemannian manifolds of negative curvature by warped product using convex functions. For two Riemannian manifolds B and F, a warped product is denoted by $B\times_f F$, where f is a positive C^∞-function on B. Concerning with the Riemannian manifolds satisfying $R(X, Y)\cdot R=0$, S. Tanno (cf. [6]) proved

THEOREM A. *Let F be a Riemannian manifold of constant curvature $K\leqq 0$. Let E^n be an n-dimensional Euclidean space and let f be a positive C^ω-function on E^n. On a warped product $E^n\times_f F$, assume that*

(i) *the condition (*) $R(X, Y)\cdot R=0$ is satisfied, and*

(ii) *the scalar curvature is constant.*

Then $E^n\times_f F$ is locally symmetric.

H. Takagi (cf. [4]) proved

THEOREM B. *Let F be a Riemannian manifold of constant curvature $K\leqq 0$. Let E^n be an n-dimensional Euclidean space. On a warped product $E\times_f F$, assume that*

(i) *it is homogeneous, or*

(ii) $\nabla R_1=0$,

where R_1 denotes the Ricci tensor of $E^n\times_f F$. Then $E^n\times_f F$ is locally symmetric.

In this note, we shall construct more wider class of Riemannian manifolds than those of R. L. Bishop and B. O'Neill by considering f as a positive C^∞-function on $B\times F$ formally in their definition. And we shall give some examples of Riemannian manifolds of the form $E^n\times_f E^1$ such that

(i) *complete and irreducible,*

(ii) *the condition (*) $R(X, Y)\cdot R=0$ is satisfied and $\nabla R\neq 0$,*

(iii) *the scalar curvature is constant and negative.*

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2. Preliminaries.

Let (B, ϕ) and (F, ω) be Riemannian manifolds and f be a positive C^∞-function on $B\times F$. Consider the product manifold $M=B\times F$ with its projections $\pi: B\times F\to B$ and $\eta: B\times F\to F$. Let (M, g) be the Riemannian manifold furnished with the Riemannian structure such that

$$g(X, Y)=\phi(\pi_*X, \pi_*Y)+f^2\omega(\eta_*X, \eta_*Y), \tag{2.1}$$

for any tangent vectors X and Y on M.

We shall prove

PROPOSITION 2.1. *In $B\times_f F$, if there exist continuous functions $C(x)$ and $D(x)$ on B such that*

$$0<C(x)\leqq f(x, p)\leqq D(x)<\infty, \qquad \text{for any } p\in F, \tag{2.2}$$

then, $B\times_f F$ is complete if and only if (B, ϕ) and (F, ω) are complete.

Proof. We assume that $B\times_f F$ is complete. Then a Cauchy sequence in (B, ϕ) imbeds in a horizontal leaf, $B\times\{p\}$, $p\in F$, as a Cauchy sequence, and hence converges. Next, let $\{q_i\}$ be a Cauchy sequence in (F, ω). Then by (2.2) we have $d(q_i, q_j)\leqq D(x)d_2(q_i, q_j)$, for each $x\in B$. Where by d, d_1, d_2 we denote the distance functions on $B\times_f F$, (B, ϕ), (F, ω), respectively. Thus, it is also a Cauchy sequence in (M, g) and hence converges in M. Thus, it converges also in F. Conversely, we assume that (B, ϕ) and (F, ω) are both complete. Let $\{m_i\}$ be a Cauchy sequence in (M, g), where $m_i=(x_i, p_i)$, $i=1, 2, \cdots$. Let α_{ij} be a curve from m_i to m_j in M having length at most $2d(m_i, m_j)$, where $\alpha_{ij}(0)=m_i$, $\alpha_{ij}(1)=m_j$. We can assume that all projections $\pi\circ\alpha_{ij}$ lie in a compact region Ω in B. By (2.1) we get

$$\begin{aligned} L(\alpha_{ij}) &= \int_0^1 \sqrt{g(d\alpha_{ij}/dt, d\alpha_{ij}/dt)}\, dt \\ &= \int_0^1 \sqrt{\phi(d\pi\circ\alpha_{ij}/dt, d\pi\circ\alpha_{ij}/dt)+f^2\omega(d\eta\circ\alpha_{ij}/dt, d\eta\circ\alpha_{ij}/dt)}\, dt \end{aligned} \tag{2.3}$$

And we have $C(x)\geqq c>0$ on Ω where c is constant. Thus, from (2.2), we have

$$f(x, p)\geqq c>0 \qquad \text{on } \Omega\times F. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned} L(\alpha_{ij}) &\geqq c\int_0^1 \sqrt{\omega(d\eta\circ\alpha_{ij}/dt, d\eta\circ\alpha_{ij}/dt)}\, dt \\ &\geqq c d_2(p_i, p_j). \end{aligned}$$

Thus, we have

$$d_2(p_i, p_j)\leqq(2/c)d(m_i, m_j)\,. \tag{2.5}$$

Thus, $\{p_i\}$ is a Cauchy sequence in (F, ω) and hence converges. Of course, $\{x_i\}$ converges in B. Thus $\{m_i=(x_i, p_i)\}$ converges in M.

3. Some examples.

Example 1. $(M^3, g)=E^2\times_f E^1$; $f=c_1\exp(\sqrt{-S/2}\,t)+c_2\exp(-\sqrt{-S/2}\,t)$, $t=(\cos w)u+(-\sin w)v$, where (u, v, w) is a canonical coordinate system on $E^2\times E^1$, and c_1, c_2, S are certain real numbers, $c_1\geqq 0$, $c_2\geqq 0$, $S<0$.

For the above Riemannian manifold (M^3, g), we put

$$\begin{aligned} E_1^*&=(1/f)(\partial/\partial w)\,,\\ E_2^*&=(\cos w)(\partial/\partial u)+(-\sin w)(\partial/\partial v)\,,\\ E_3^*&=(\sin w)(\partial/\partial u)+(\cos w)(\partial/\partial v)\,. \end{aligned} \tag{3.1}$$

Then, (E^*) is a global orthonormal frame field on M^3, and, with respect to this frame, we get

$$\begin{aligned} &B_3^*{}_{ij}=0\,,\qquad B_2^*{}_{ij}=0\,,\qquad B_1^*{}_{32}=1/f\,,\\ &B_1^*{}_{21}=(\sqrt{-S/2}/f)(c_1\exp(\sqrt{-S/2}\,t)-c_2\exp(-\sqrt{-S/2}\,t)) \end{aligned} \tag{3.2}$$

where $\nabla_{E_i^*}E_j^*=\sum_{k=1}^{3}B_i^*{}_{jk}E_k^*$.

Moreover, we get

$$\begin{aligned} &R(E_1^*, E_2^*)=(S/2)E_1^*\wedge E_2^*\,,\\ &R(E_1^*, E_3^*)=0\,,\qquad R(E_2^*, E_3^*)=0\,. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we see that (M^3, g) is irreducible and satisfies $R(X, Y)\cdot R=0$, $\nabla R\neq 0$.

By the definition of f, we get

$$\begin{aligned} &(c_1+c_2)\exp(-\sqrt{-S/2}\;\sqrt{u^2+v^2}\,)\\ &\qquad\leqq f\leqq(c_1+c_2)\exp(\sqrt{-S/2}\sqrt{u^2+v^2}\,)\,, \end{aligned} \tag{3.4}$$

on M^3. From (3.4) and proposition 2.1, we see that (M^3, g) is complete. By (3.3), the scalar curvature of (M^3, g) is S.

Example 2.

$$(M^4, g)=E^3\times_f E^1\,;$$

$$\begin{aligned} f=\exp(\sqrt{-S/2}\,(((2/3)\cos z+1/3)u+((-1/3)\cos z+(1/\sqrt{3})\sin z+1/3)v\\ +((-1/3)\cos z+(-1/\sqrt{3})\sin z+1/3)w))\,, \end{aligned}$$

where (u, v, w, z) is a canonical coordinate system on $E^3\times E^1$, and S is a negative number.

For the above Riemannian manifold (M^4, g), we put

$$
\begin{aligned}
(3.5)\qquad E_1^* &= (1/f)(\partial/\partial z)\,,\\
E_2^* &= ((2/3)\cos z+1/3)(\partial/\partial u)\\
&\quad +((-1/3)\cos z+(1/\sqrt{3})\sin z+1/3)(\partial/\partial v)\\
&\quad +((-1/3)\cos z+(-1/\sqrt{3})\sin z+1/3)(\partial/\partial w)\,,\\
E_3^* &= ((-1/3)\cos z+(-1/\sqrt{3})\sin z+1/3)(\partial/\partial u)\\
&\quad +((2/3)\cos z+1/3)(\partial/\partial v)\\
&\quad +((-1/3)\cos z+(1/\sqrt{3})\sin z+1/3)(\partial/\partial w)\,,\\
E_4^* &= ((-1/3)\cos z+(1/\sqrt{3})\sin z+1/3)(\partial/\partial u)\\
&\quad +((-1/3)\cos z+(-1/\sqrt{3})\sin z+1/3)(\partial/\partial v)\\
&\quad +((2/3)\cos z+1/3)(\partial/\partial w)\,.
\end{aligned}
$$

Then, (E^*) is a global orthonormal frame field on M^4, and with respect to this frame, we get

$$
\begin{aligned}
(3.6)\qquad & B_3^*{}_{ij}=0\,,\quad B_4^*{}_{ij}=0\,,\quad B_2^*{}_{ij}=0\,,\quad B_1^*{}_{32}=-(1/\sqrt{3}f)\,,\\
& B_1^*{}_{34}=(1/\sqrt{3}f)\,,\quad B_1^*{}_{42}=(1/\sqrt{3}f)\,,\quad B_1^*{}_{21}=\sqrt{-S/2}\,,\\
& B_1^*{}_{31}=B_1^*{}_{41}=0\,,\qquad \text{where } \nabla_{E_i^*}E_j^*=\sum_{k=1}^{4}B_i^*{}_{jk}E_k^*\,.
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
(3.7)\qquad & R(E_1^*, E_2^*)=(S/2)E_1^*\wedge E_2^*\,,\\
& R(E_1^*, E_3^*)=0\,,\quad R(E_2^*, E_3^*)=0\,,\quad R(E_1^*, E_4^*)=0\,,\\
& R(E_2^*, E_4^*)=0\,,\quad R(E_3^*, E_4^*)=0\,.
\end{aligned}
$$

From (3.6) and (3.7), we see that (M^4, g) is irreducible and satisfies $R(X, Y)\cdot R=0$, $\nabla R\neq 0$.

By the definition of f, we get

$$
\begin{aligned}
(3.8)\qquad & \exp(-(2/3)(\sqrt{-S/2})(\sqrt{u^2+v^2+w^2-uv-vw-wu}+(1/3)(u+v+w)))\\
& \leqq f\leqq \exp((2/3)(\sqrt{-S/2})(\sqrt{u^2+v^2+w^2-uv-vw-wu}+(1/3)(u+v+w)))\,,
\end{aligned}
$$

on M^4.

From (3.8) and proposition 2.1, we see that (M^4, g) is complete. By (3.7), the scalar curvature of (M^4, g) is S.

Remark. Recently, H. Takagi [5] has showed that these Riemannian manifolds are curvature-homogenous but non-homogenous.

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