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ALMOST COQUATERNION METRIC STRUCTURES ON 3-DIMENSIONAL MANIFOLDS

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We give explicitly almost coquaternion metric structures on 3-dimensional parallelizable manifolds and some conditions under which a 3-dimensional manifold admits a Sasakian 3-structure.

1. We suppose that all the used differentiable manifolds and maps are of class C^∞ and we denote by $\mathfrak{X}(M)$ the Lie algebra of all vector fields on the manifold M.

Let M be a $(4n+3)$-dimensional manifold. An *almost coquaternion metric structure**) on M is an aggregate consisting of three almost cocomplex metric structures**) $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, which satisfy

$$\phi_a\circ\phi_b-\xi_a\otimes\eta_b=-\phi_a\circ\phi_a+\xi_b\otimes\eta_a=\phi_c\,,$$

$$\phi_a\xi_b=-\phi_b\xi_a=\xi_c\,,$$

$$\eta_a\circ\phi_b=-\eta_b\circ\phi_a=\eta_c\,,$$

$$\eta_a(\xi_b)=\eta_b(\xi_a)=0\,,$$

for any cyclic permutation $\{a, b, c\}$ of $\{1, 2, 3\}$. M is said to be an *almost coquaternion Riemannian manifold.*

An almost coquaternion metric structure can be described by means of 1-forms η_a and 2-forms $\Theta_a(X, Y)=g(\phi_a X, Y)$, $a=1, 2, 3$, $\forall X, Y\in\mathfrak{X}(M)$.

THEOREM 1.1. *If* $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, *is an almost coquaternion metric structure, then,* $\forall\alpha: M\to(0, \infty)$, $\forall(A_d^a)\in SO(3)$,

$$\left(A_d^a\phi_a, \frac{1}{\alpha}A_d^a\xi_a, \alpha A_d^a\eta_a, \alpha g+(\alpha^2-\alpha)\sum_a\eta_a\otimes\eta_a\right), \qquad d=1, 2, 3\,,$$

is again an almost coquaternion metric structure on M [10].

An almost coquaternion metric structure on M whose tensor

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*) Or almost contact metric 3-structure [3].

**) Or almost contact metric structures [5].

$$T^1(X, Y)=\frac{2}{3}\sum_a([\phi_a X, \phi_a Y]-\phi_a[\phi_a X, Y]$$
$$-\phi_a[X, \phi_a Y]+\phi_a^2[X, Y]+2d\eta_a(X, Y)\xi_a)$$

vanishes is called a *pseudo-coquaternion metric structure* and the manifold with such a structure a *pseudo-coquaternion Riemannian manifold.* A pseudo-coquaternion metric structure consists of three normal almost cocomplex metric structures and corresponds to the pseudo-quaternion metric structure on $M\times R$, where R is the real line [10], [11].

If

(1) $$\Theta_a=d\eta_a\,, \qquad a=1, 2, 3\,,$$

then $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is a pseudo-coquaternion metric structure iff

(2) $$\nabla_X(\nabla\xi_a)Y=\eta_a(Y)X-g(X, Y)\xi_a \text{ or } -R(X, \xi_a)Y=\eta_a(Y)X-g(X, Y)\xi_a\,,$$

where ∇ is the Riemannian connection and R is the Riemannian curvature tensor $R(X, Y)=\nabla_{[X,Y]}-[\nabla_X, \nabla_Y]$.

An almost coquaternion metric structure whith satisfies the conditions (1) and (2) is said to be a *Sasakian* 3*-structure.* For a Sasakian 3-structure, ξ_a, $a=1, 2, 3$, are unit Killing vector fields (determine a Lie group of translations [1]) with respect to g and we have $\phi_a=\nabla\xi_a$ [7].

THEOREM 1.2. *If* $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, *is a Sasakian* 3*-structure and* (A_d^a) *is an orthogonal matrix whose entries are constants, then*

$$(A_d^a\phi_a, A_d^a\xi_a, A_d^a\eta_a, g)\,, \qquad d=1, 2, 3\,,$$

is again a Sasakian 3*-structure on* M.

2. Let M be a 3-dimensional manifold. We have

THEOREM 2.1. *A* 3*-dimensional manifold* M *has an almost coquaternion metric structure iff it is parallelizable* [9].

Proof. Obviously, every almost coquaternion Riemannian 3-dimensional manifold is parallelizable.

Conversely, the hypothesis that M is parallelizable is equivalent to the fact that it possesses three vector fields ξ_a, $a=1, 2, 3$, which are linearly independent at every point of M. Let η_a be the dual 1-forms, that is,

$$\eta_a(\xi_a)=\delta_{ab}\,, \qquad \sum_a \eta_a\otimes\xi_a=id\,.$$

We define

$$\phi_a=\xi_c\otimes\eta_b-\xi_b\otimes\eta_c\,,$$

where $\{a, b, c\}$ is an even permutation of $\{1, 2, 3\}$, and $g=\sum_a \eta_a\otimes\eta_a$. We can verify without difficulty that $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is an almost coquaternion metric structure on M. Evidently, $\Theta_a=2\eta_b\wedge\eta_c$.

As any orientable 3-dimensional manifold is parallelizable, we have

THEOREM 2.2. *Every 3-dimensional orientable manifold can be endowed with an almost coquaternion metric structure* [9].

Remark. Suppose ξ_a, $a=1, 2, 3$, generate a simply transitive Lie group of transformations G on M and ζ_a, $a=1, 2, 3$, generate the reciprocal group $\bar{G}$ of G [1]. As each transformation of G commutes with each transformation of $\bar{G}$, the almost coquaternion metric structure determined by $\xi_a(\zeta_a)$ is invariant by $\bar{G}(G)$.

3. Let M be a 3-dimensional manifold and $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, an almost coquaternion metric structure on M.

THEOREM 3.1. *Suppose ξ_a, $a=1, 2, 3$, determine a Lie group of motions G with respect to g whose structure constants are C^a_{bc}.*

(i) *If $C^1_{23}=0$, then G is isomorphic to an Abelian group, $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is an integrable almost coquaternion metric structure and M is locally Euclidean.*

(ii) *If $C^1_{23}\neq 0$, then G is isomorphic to a unitary, semi-simple group, $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is a Sasakian 3-structure and M is a space of constant positive curvature.*

Proof. As ξ_a generate a group of motions with respect to g, we have

$$L_{\xi_a}\xi_b=C^c_{ab}\xi_c\,, \qquad a, b, c=1, 2, 3\,, \tag{3}$$

$$L_{\xi_a}g=0 \quad \text{or} \quad (\nabla_Y\eta_a)(X)+(\nabla_X\eta_a)(Y)=0, \quad \forall X, Y\in\mathcal{X}(M)\,, \tag{4}$$

where ∇ is the Riemannian connection. On the other hand, from $g(\xi_b, \xi_c)=\delta_{bc}$, it follows

$$g(L_{\xi_a}\xi_b, \xi_c)+g(\xi_a, L_{\xi_a}\xi_c)=0\,,$$

that is,

$$C^c_{ab}+C^b_{ac}=0\,.$$

From these relations and from the fact that the structure constants C^c_{ab} of the group G are skew-symmetric in the indices a and b it results that all the structure constants are zero besides C^1_{23} (and those which proceed from C^1_{23}) which can be zero or not.

(i) If $C^1_{23}=0$, then G is isomorphic to an Abelian group. In this case we can choose the local coordinates so that $\xi_a=\partial/\partial x^a$ and hence

$$\eta_a=dx^a\,, \qquad \phi_a=\frac{\partial}{\partial x^c}\otimes dx^b-\frac{\partial}{\partial x^b}\otimes dx^c\,, \qquad g=\sum_a dx^a\otimes dx^a\,,$$

So our first statement is true.

(ii) If $C^1_{23}\neq 0$, then the comitant $C_{ab}=C^d_{ac}C^c_{bd}$ has the components $C_{11}=C_{22}=C_{33}=-2(C^1_{23})^2$, $C_{ab}=0$, $a\neq b$. Consequently G is isomorphic to a unitary, semi-simple group.

Without loss of generality, we may assume that $C^1_{23}=-2$. Really, if not so we may work out the change

$$\bar{\xi}_a=-\frac{2}{C^1_{23}}\xi_a$$

and putting

$$[\bar{\xi}_2, \bar{\xi}_3]=\bar{C}^1_{23}\bar{\xi}_1$$

we get $\bar{C}^1_{23}=-2$.

From (4) and

$$d\eta_a(X, Y)=\frac{1}{2}((\nabla_X\eta_a)(Y)-(\nabla_Y\eta_a)(X)), \quad \forall X, Y\in\mathscr{X}(M),$$

we obtain

(5) $$d\eta_a(X, Y)=(\nabla_X\eta_a)(Y).$$

Since $g(\xi_a, \xi_a)=1$, we have $g(\nabla_X\xi_a, \xi_a)=0$, that is,

(6) $$(\nabla_X\eta_a)(\xi_a)=0.$$

From (6) and (4) we get

$$(\nabla_{\xi_a}\eta_a)(Y)=0$$

and hence

$$d\eta_a(\xi_a, Y)=0, \qquad \forall Y\in\mathscr{X}(M).$$

From $[\xi_a, \xi_b]=-2\xi_c=\nabla_{\xi_a}\xi_b-\nabla_{\xi_b}\xi_c$, where $\{a, b, c\}$ is a cyclic permutation of $\{1, 2, 3\}$, it results

(7) $$(\nabla_{\xi_a}\eta_b)(X)-(\nabla_{\xi_b}\eta_a)(X)=-2\eta_c(X).$$

On the other hand, from (4) we obtain

$$(\nabla_{\xi_a}\eta_b)(X)=-(\nabla_X\eta_a)(\xi_b)$$

and $g(\xi_a, \xi_b)=0$ give

$$g(\nabla_X\xi_a, \xi_b)+g(\xi_a, \nabla_X\xi_b)=0 \quad \text{or} \quad (\nabla_X\eta_a)(\xi_b)+(\nabla_X\eta_b)(\xi_a)=0.$$

Thus

(8) $$(\nabla_{\xi_a}\eta_b)(X)+(\nabla_{\xi_b}\eta_a)(X)=0,$$

which together with (7) give

(9) $$\nabla_{\xi_b}\eta_a=-\nabla_{\xi_a}\eta_b=\eta_c.$$

By virtue of (9) and (5) we have

(10) $$d\eta_a(\xi_b, Y)=-d\eta_b(\xi_a, Y)=\eta_c(Y) \quad \text{or} \quad d\eta_a=\Theta_a=2\eta_b\wedge\eta_c.$$

From (5) and (10) we get

$$(11)\qquad (\nabla_X\eta_a)(Y)=\eta_b(X)\eta_c(Y)-\eta_c(X)\eta_b(Y) \quad \text{or}$$

$$\nabla_X\xi_a=\eta_b(X)\xi_c-\eta_c(X)\xi_b\,,\qquad \forall X,\, Y\in\mathcal{X}(M)\,,$$

where $\{a, b, c\}$ is a cyclic permutation of $\{1, 2, 3\}$.

From (11) we obtain

$$(12)\qquad \nabla_X(\nabla\xi_a)(Y)=\eta_a(Y)X-g(X, Y)\xi_a\,,$$

which shows that $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is a Sasakian 3-structure.

As (12) is equivalent to

$$R(X, \xi_a)Y=g(X, Y)\xi_a-g(\xi_a, Y)X\,,$$

multiplying by $\eta_a(Z)$ and summing for a, we obtain

$$R(X, Y)Z=g(X, Y)Z-g(Y, Z)X\,.$$

So M has constant curvature 1.

THEOREM 3.2. *A 3-dimensional manifold M admits a Sasakian 3-structure iff it possesses three independent vector fields which determine a unitary semi-simple Lie group of transformations.*

Proof. We first assume that M possesses a Sasakian 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$. From

$$\Theta_a(X, Y)=d\eta_a(X, Y)=(\nabla_X\eta_a)(Y)\,,\qquad \forall X,\, Y\in\mathcal{X}(M)\,,$$

it follows that ξ_a are Killing vector fields of the Riemannian metric g for which

$$[\xi_a, \xi_b]=\nabla_{\xi_a}\xi_b-\nabla_{\xi_b}\xi_a=-2\xi_c\,.$$

So ξ_a generate a unitary semi-simple Lie group of transformations.

Conversely, let ξ_a, $a=1, 2, 3$, be three independent vector fields on M which determine a unitary semi-simple Lie group of transformations. Without loss of generality, we can suppose

$$[\xi_a, \xi_b]=-2\xi_c \quad \text{or} \quad L_{\xi_a}\xi_b=-2\xi_c\,.$$

From $\eta_a(\xi_b)=\delta_{ab}$ we find

$$(L_{\xi_a}\eta_a)(\xi_b)+\eta_a(L_{\xi_a}\xi_b)=0$$

and hence

$$(L_{\xi_a}\eta_a)(\xi_b)=0\,,\quad \text{that is,}\quad L_{\xi_a}\eta_a=0\,.$$

Analogously, we have

$$(L_{\xi_c}\eta_a)(\xi_b)+\eta_a(L_{\xi_c}\xi_a)=0$$

and hence

$$L_{\xi_a}\eta_b=-L_{\xi_b}\eta_a=-2\eta_c\,.$$

From these relations we obtain

$$L_{\xi_a} g = L_{\xi_a}(\sum_b \eta_b \otimes \eta_b) = 0$$

and so ξ_a are Killing vector fields. By virtue of Theorem 3.1, $(\phi_a = \eta_b \otimes \xi_c - \eta_c \otimes \xi_b$, $\xi_a, \eta_a, g = \sum_a \eta_a \otimes \eta_a)$ is a Sasakian 3-structure on M.

THEOREM 3.3. *A 3-dimensional manifold M admits a Sasakian 3-structure iff it possesses three independent 1-forms η_a which satisfy*

$$\eta_a \wedge d\eta_b = 2(\eta_1 \wedge \eta_2 \wedge \eta_3)\delta_{ab}, \qquad a, b = 1, 2, 3.$$

Proof. Let us suppose that $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is a Sasakian 3-structure on M. Then we have

$$d\eta_a = \eta_b \otimes \eta_c - \eta_c \otimes \eta_b = 2\eta_b \wedge \eta_c,$$

for any cyclic permutation $\{a, b, c\}$ of $\{1, 2, 3\}$, and hence

$$\eta_a \wedge d\eta_b = 2(\eta_1 \wedge \eta_2 \wedge \eta_3)\delta_{ab}.$$

Conversely, from $\eta_a \wedge d\eta_b = 0$, $a \neq b$, it follows $d\eta_a = f\eta_b \wedge \eta_c$ and from $\eta_a \wedge d\eta_a = 2(\eta_1 \wedge \eta_2 \wedge \eta_3)$ we get $f=2$. Let ξ_a be the dual vector fields of the 1-forms η_a. We have

$$d\eta_a(\xi_a, X) = 0, \quad d\eta_a(\xi_b, X) = -d\eta_b(\xi_a, X) = \eta_c(X), \quad \forall X \in \mathfrak{X}(M).$$

We define on M the metric

$$g = \sum_a \eta_a \otimes \eta_a, \qquad g^{-1} = \sum_a \xi_a \otimes \xi_a$$

and

$$\phi_a = g^{-1}(d\eta_a) = \xi_c \otimes \eta_b - \xi_b \otimes \eta_c.$$

Evidently, $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is an amost coquaternion metric structure on M.

From

$$d\eta_a(X, Y) = \frac{1}{2}\{X(\eta_a(Y)) - Y(\eta_a(X)) - \eta_a([X, Y])\}$$
$$= \eta_b(X)\eta_c(Y) - \eta_c(X)\eta_b(Y)$$

we obtain

$$\eta_c([\xi_a, \xi_b]) = -2 \quad \text{or} \quad [\xi_a, \xi_b] = -2\xi_c.$$

Hence ξ_a, $a=1, 2, 3$, generate a unitary semi-simple Lie group of transformations, that is, $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is a Sasakian 3-structure.

4. Examples.

(a) Let

$$S^3 = \{x \mid x \in R^4, \ \|x\| = 1\}$$

be the unit sphere in the Euclidean space R^4 and (J_a, h), $a=1, 2, 3$, be the canonical quaternion Hermitian structure on R^4. If we denote the induced metric on S^3 from the Euclidean metric h on R^4 by g and if we define

$$\xi_a=J_a x, \quad x\in S^3, \quad \eta_a(X)=g(\xi_a, X), \quad \phi_a X=J_a X+\eta_a(X)x,$$

then $(\phi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$, is a Sasakian 3-structure on S^3. In other words, the independent 1-forms η_a satisfy

$$\eta_a\wedge d\eta_b=2(\eta_1\wedge\eta_2\wedge\eta_3)\delta_{ab}, \qquad a, b=1, 2, 3.$$

(b) A 3-dimensional manifold M which admits a Sasakian 3-structure has positive constant curvature. Therefore, if we suppose that M is a complete manifold, then $M\equiv S^3/\Gamma$ (spherical space form), where Γ is a finite subgroup of $O(4)$ which acts freely on S^3. More precisely [6], Γ is any one of subgroups of Clifford translations given by:

(i) $\Gamma=\{\imath d\}$,

(ii) $\Gamma=\{\pm\imath d\}$,

(iii) Γ is the cyclic group of order $q>2$ generated by

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \text{where } A=\begin{pmatrix} \cos\dfrac{2\pi}{2} & -\sin\dfrac{2\pi}{2} \\ \sin\dfrac{2\pi}{2} & \cos\dfrac{2\pi}{2} \end{pmatrix},$$

(iv) Γ is the group of Clifford translations which corresponds to a binary dihedral group, a binary tetrahedral group, a binary octahedral group or a binary icosahedral group.

(c) THEOREM 4.1. *If M is an orientable hypersurface in the Euclidean space R^4 such that its spherical map is regular, then M admits a Sasakian 3-structure.*

Proof. We choose the unit normal vector ζ to M in R^4 such that the positive orientation of M is coherent with the positive orientation of R^4. Then ζ is a differentiable vector field over M and by means of ζ we construct the spherical map of Gauss $s: M\to S^3$.

If M is covered by a system of coordinate neighborhoods $\{U; (u^1, u^2, u^3)\}$ and S^3 is covered by a system of coordinate neighborhoods $\{V; (v^1, v^2, v^3)\}$, then s can be represented locally by

$$v^\alpha=v^\alpha(u^1, u^2, u^3), \qquad \alpha, \beta=1, 2, 3,$$

and by hypothesis

$$\left|\frac{\partial v^\alpha}{\partial u^\beta}\right|\neq 0.$$

On the other hand S^3 possesses a Sasakian 3-structure, that is three independent 1-forms η_a, $a=1, 2, 3$, which satisfy

$$\eta_a \wedge d\eta_b = 2(\eta_1 \wedge \eta_2 \wedge \eta_3)\delta_{ab}, \qquad a, b = 1, 2, 3,$$

or locally

$$\eta_a \wedge d\eta_b = 2\lambda\, dv^1 \wedge dv^2 \wedge dv^3 \delta_{ab}.$$

We denote by s^* the dual map of forms on S^3 into forms on M induced by the map s. Then $s^*\eta_a$ are three 1-forms on M and

$$s^*(\eta_a \wedge d\eta_a) = s^*\eta_a \wedge d(s^*\eta_a), \quad s^*(\eta_1 \wedge \eta_2 \wedge \eta_3) = s^*\eta_1 \wedge s^*\eta_2 \wedge s^*\eta_3.$$

As locally we have

$$s^*\eta_1 \wedge s^*\eta_2 \wedge s^*\eta_3 = \lambda(v(u)) \left| \frac{\partial v^\alpha}{\partial u^\beta} \right| du^1 \wedge du^2 \wedge du^3,$$

the three 1-forms $s^*\eta_a$ are independent.

We deduce

$$s^*\eta_a \wedge d(s^*\eta_b) = 2\lambda(v(u)) \left| \frac{\partial v^\alpha}{\partial u^\beta} \right| du^1 \wedge du^2 \wedge du^3$$

or

$$s^*\eta_a \wedge d(s^*\eta_b) = 2(s^*\eta_1 \wedge s^*\eta_2 \wedge s^*\eta_3)\delta_{ab}$$

Therefore the 1-forms $s^*\eta_a$, $a=1, 2, 3$, give rise to a Sasakian 3-structure on M (Theorem 3.3.).

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