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ON CERTAIN CRITERIA FOR THE LEFT-PRIMENESS OF ENTIRE FUNCTIONS

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1. Introduction. A meromorphic function $F(z)=f(g(z))$ is said to have f and g as left and right factors respectively, provided that f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be prime (pseudo-prime, left-prime, right-prime) if every factorization of the above form into factors implies either f is linear or g is linear (either f is rational or g is a polynomial, f is linear whenever g is transcendental, g is linear whenever f is transcendental). When factors are restricted to entire functions, it is called to be a factorization in entire sense.

Recently several methods on the factorization in the above sense were established. Among them Goldstein's [3] is very powerful and elegant. He proved a general theorem guaranteeing the right-primeness of entire functions, which depends upon the radial growth of the given function. As a corollary he proved the right-primeness of $H_1e^z+H_2$, when H_1, H_2 are entire functions of order less than one and $H_2\not\equiv$const, $H_1\not\equiv 0$. He then proved another general theorem, which gives the left-primeness of $H_1e^z+H_2$. This part of his proof seems to us to be too hard to prove only the result. We shall give another proof of this part, which seems to be very simple. Under the same idea we shall prove two general theorems, which guarantee the left-primeness of entire functions in entire sense. Several applications are discussed then. In order to explain our idea we shall firstly give and discuss the easiest example e^z+z. So the method of proof should be elementary and simple in principle. Compare with the methods in [1], [4], [5], [6], [11].

We freely use the symbols of the Nevanlinna theory such as $N(r, a, f)$, $m(r, f)$, $T(r, f)$.

2. Discussion on e^z+z. *The first step.* Suppose that $e^z+z=f(g(z))$ with transcendental entire f and g. Then by Pólya's result [10] the order $\rho(f)$ of f is equal to zero. It is known that $\rho(f')=\rho(f)$. Hence $\rho(f')=1$. Then $f'(w)$ has infinitely many zeros and hence there is a zero of $f'(w)$ for which $g(z)=w$ has infinitely many roots. At these roots $\{z^*\}$

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$$\begin{cases} e^{z^*}+z^*=f(w) \\ e^{z^*}+1=0\,. \end{cases}$$

Therefore $-1+z^*=f(w)$. We have at least two z^* for one zero w of $f'(w)$. This is impossible. Hence e^z+z is pseudo-prime in entire sense.

The second step. e^z+z has infinitely many zeros whose real parts tend to $+\infty$ and does not have any zero in the left half plane $\Re z<0$. Suppose that $e^z+z=f(g(z))$ with a polynomial $g(z)$. If $g(z)$ is of degree $\geqq 2$, then e^z+z should have zeros in the left half plane. This is impossible. Hence e^z+z is right-prime in entire sense.

The third step. Suppose that $e^z+z=f(g(z))$ with a polynomial f. Consider the derived equation

$$e^z+1=Ag'(z)(g(z)-\alpha_1)\cdots(g(z)-\alpha_n)\,.$$

If $g(z)-\alpha_1$ has two zeros z_1 and z_2, then

$$e^{z_j}+z_j=f(\alpha_1)\,,$$

$$e^{z_j}+1=0\,.$$

Hence $z_j-1=f(\alpha_1)$, $j=1, 2$. This is impossible. If $g(z)-\alpha_1$ has at most one zero, then $g(z)-\alpha_1=Be^{\beta z}$ and hence $g'(z)=(B'+\beta B)e^{\beta z}$ have at most one zero. Since e^z+1 has infinitely many zeros, there is another index, say 2, for which $g(z)-\alpha_2$ has infinitely many zeros. This is again a contradiction.

We shall not discuss the primeness of e^z+z in meromorphic sense. Our idea is to make use of the simultaneous equations $F(z)=f(\alpha)$, $F'(z)=0$.

3. Another proof of the left-primeness of $H_1e^z+H_2$. Here H_1, H_2 are entire functions of order less than one and $H_2\not\equiv$const, $H_1\not\equiv 0$.

Suppose that $H_1e^z+H_2=f(g(z))$ with a polynomial f of degree $n+1$. Let ϕ be $e^z/(e^z+H_2/H_1)$. Then

$$\begin{aligned}(1-\varepsilon)T(r, \phi)&\leqq N(r, 0, \phi)+N(r, 1, \phi)+N(r, \infty, \phi)\\ &\leqq 2N(r, 0, H_1)+N(r, 0, H_2)+N(r, 0, H_1e^z+H_2)\\ &=N(r, 0, H_1e^z+H_2)+o(m(r, e^z))\end{aligned}$$

and $m(r, e^z)\leqq(1+\varepsilon)T(r, \phi)$. Hence

$$m(r, e^z)\leqq(1+\varepsilon)N(r, 0, H_1e^z+H_2)\leqq(1+\varepsilon)m(r, e^z)\,.$$

Let us put

$$f(g(z))=A(g(z)-w_1)\cdots(g(z)-w_{n+1})$$

and

$$g'(z)f'(g(z))=Bg'(z)(g(z)-\alpha_1)\cdots(g(z)-\alpha_n).$$

Then

$$m(r, e^z)\geqq(n+1)(1-\varepsilon)m(r, g)$$

and by the non-constancy of H_2

$$m(r, e^z)\leqq(1+\varepsilon)N(r, 0, (H_1'+H_1)e^z+H_2')$$

$$\leqq(1+\varepsilon)(\sum_{j=1}^{n}N(r, \alpha_j, g)+N(r, 0, g')).$$

Since $N(r, 0, g')\leqq m(r, g')\leqq(1+\varepsilon)m(r, g)$,

$$\sum_{j=1}^{n}N(r, \alpha_j, g)\geqq(1-\varepsilon)nm(r, g).$$

Hence there is an index, say 1, such that

$$N(r, \alpha_1, g)\geqq(1-\varepsilon)m(r, g).$$

Evidently

$$(n+1)m(r, g)\geqq(1-\varepsilon)m(r, e^z).$$

Let $X(z)$ be

$$-\frac{H_1H_2'}{H_1+H_1'}+H_2.$$

$H_1+H_1'\equiv0$ implies $H_1=Ce^{-z}$, $C\neq0$, which is impossible. We need to verify the non-constancy of $X(z)$. Firstly we consider the case $H_1\not\equiv aH_2$. Then there is a point z_0 at which $H_1=0$, $H_2\neq0$ or $H_1\neq0$, $H_2=0$ or $H_1=H_2=0$ with different multiplicities. In these cases it is easy to prove the non-constancy of X. If $H_1\equiv aH_2$, then $X(z)\equiv$const implies $H_1'=AH_1^2-H_1$. Hence

$$1/AH_1=1-e^{A(z+C)}.$$

This is untenable. Thus $X(z)$ is not a constant. $g(z)=\alpha_1$ and $f(\alpha_1)=0$ imply

$$H_1e^z+H_2=f(\alpha_1),$$

$$(H_1'+H_1)e^z+H_2'=0.$$

Hence every root of $g(z)=\alpha_1$ satisfies

$$X(z)=f(\alpha_1).$$

Thus

$$N(r, \alpha_1, g)\leqq N(r, f(\alpha_1), X)$$

$$\leqq T(r, X)\leqq r^\tau, \qquad \tau<1.$$

Hence we have

$$\frac{1-\varepsilon}{n+1}m(r,\, e^z)\leqq N(r,\, \alpha_1,\, g)$$
$$\leqq r^\tau = o(m(r,\, e^z))\,.$$

This is impossible. Thus we have the left-primeness of $H_1e^z+H_2$.

4. Two general theorems. The above observation suggests a certain general idea to make use of the simultaneous equations $F=c$, $F'=0$. Indeed we have the following

THEOREM 1. *Let $F(z)$ be an entire function of finite order whose derivative $F'(z)$ has infinitely many zeros. Assume that the number of common roots of $F(z)=c$ and $F'(z)=0$ is finite for any constant c. Then $F(z)$ is left-prime in entire sense.*

Proof. Suppose that $F(z)=f(g(z))$ with transcendental entire f and g. Then $\rho(F)<\infty$ implies $\rho(f)=\rho(f')=0$ by Pólya's theorem [10]. So there are infinitely many roots of $f'(w)=0$, among which there is a root w_0 of $f'(w)=0$ such that $g(z)=w_0$ has infinitely many roots. At these roots of $g(z)=w_0$ we have

$$\begin{cases} F(z)=f(w_0)\,, \\ F'(z)=0\,. \end{cases}$$

However this has only finitely many common roots by our assumption. This is impossible. Thus F is pseudo-prime in entire sense.

Suppose that $F(z)=P(g(z))$ with a polynomial P and entire g. Assume that P is of degree at least two. Then $P'(w)$ has at least one zero α. If $g(z)=\alpha$ has infinitely many roots, we have a contradiction as in the above. If $g(z)=\alpha$ has only a finite number of roots, then $g(z)=\alpha+Q(z)e^{H(z)}$, $g'(z)=(Q'+QH')e^H$ have only finitely many zeros, where Q and H are polynomials. Since $F'(z)=0$ has infinitely many roots, there must be another zero β of $P'(w)$ for which $g(z)=\beta$ has infinitely many roots. This gives again a contradiction. Thus we have the left-primeness of F in entire sense. q. e. d.

We cannot omit our main assumption. This is shown by

$$z^2\sin z+z^2=\left(z\frac{e^{iz/2}+ie^{-iz/2}}{\sqrt{2i}}\right)^2.$$

In fact, $z^2\sin z+z^2=0$ and $2z(\sin z+1)+z^2\cos z=0$ have infinitely many common roots.

We cannot omit the side condition on the number of zeros of F'. This is shown by $P(z)^p\exp(pH(z))$ with polynomials P and H and a positive integer p. However if $\rho(F)<\infty$ and if F' has 0 as a Picard exceptional value then F is pseudo-prime in entire sense.

THEOREM 2. *Let $F(z)$ be an entire function satisfying $N(r, 0, F')>km(r, F)$ for some $k>0$. Assume that the simultaneous equations*

$$\begin{cases} F(z)=c \\ F'(z)=0 \end{cases}$$

have only finitely many common roots for any constant c. Then F is left-prime in entire sense.

Proof. Suppose that $F=f(g)$ with transcendental f and g. Assume firstly that $f'(w)=0$ has not root. Then

$$\begin{aligned} N(r, 0, F') &= N(r, 0, g') \\ &\leqq m(r, g') \leqq (1+\varepsilon)m(r, g) \end{aligned}$$

for $r \notin E$, where E is a set of r of finite measure. On the other hand

$$\begin{aligned} m(r, F) \geqq N(r, A, F) &\geqq \sum_{\substack{j=1 \\ f(\alpha_j)=A}}^{p} N(r, \alpha_j, g) \\ &\geqq (p-1)m(r, g) - O(\log rm(r, g)) \end{aligned}$$

for $r \notin E$. Here p is an arbitrary positive integer. Hence by $N(r, 0, F')>km(r, F)$, $k>0$

$$(kp-k-1)m(r, g) \leqq O(\log rm(r, g))$$

for $r \notin E$. We have

$$kp-k-1 \leqq 0 .$$

This is impossible, since p is arbitrary.

Assume that $f'(w)=0$ has only one root w_1 and $g(z)=w_1$ has only finitely many roots. Then

$$\begin{aligned} N(r, 0, F') &= N(r, 0, g') + O(\log r) \\ &\leqq m(r, g') + O(\log r) \\ &\leqq m(r, g) + O(\log rm(r, g)) + O(\log r) . \end{aligned}$$

But $N(r, 0, F') \geqq km(r, F)$ implies

$$N(r, 0, F') \geqq k(p-1)m(r, g)(1-\varepsilon)$$

for $r \notin E$. This gives a contradiction.

Assume that $f'(w)$ has only one zero w_1 and $g(z)=w_1$ has infinitely many roots. At these roots

$$\begin{cases} F(z)=f(w_1) \\ F'(z)=0 . \end{cases}$$

But these equations have only finitely many common roots. This is impossible.

Assume that $f'(w)$ has at least two zeros w_1 and w_2. Then we can choose w_j so that $g(z)=w_j$ has infinitely many roots. Hence by considering $F=f(w_j)$, $F'=0$ we have a contradiction.

Suppose that $F=P(g)$ with a polynomial P and entire g. If $P'(w)=0$ has only one root w_1 and $g(z)=w_1$ has a finite number of roots, then $g(z)=w_1+Qe^H$ and $g'=(Q'+H'Q)e^H$ with a polynomial Q and entire H. Hence

$$\begin{aligned} N(r, 0, g') &\leqq N(r, 0, Q'+H'Q)+O(\log r) \\ &\leqq m(r, H')+O(\log r) \\ &\leqq m(r, H)(1+\varepsilon)+O(\log r) \\ &\leqq o(m(r, g)) \end{aligned}$$

for $r \notin E$. On the other hand with $p=\deg P$

$$\begin{aligned} N(r, 0, g') &= N(r, 0, F')+O(\log r) \\ &\geqq km(r, F)(1+\varepsilon) \\ &\geqq k(p-1)m(r, g)(1+\varepsilon') . \end{aligned}$$

This gives a contradiction. If $P'(w)=0$ has one root w_1 but $g(z)=w_1$ has infinitely many roots or if $P'(w)$ has at least two roots and hence $g(z)=w_1$ has infinitely many roots, then we consider $F=f(w_1)$, $F'=0$ at these roots. This gives again a contradiction. q. e. d.

5. Applications of Theorem 1.

COROLLARY 1. *$P_1 \sin z+P_2$ is left-prime if P_1, P_2 are polynomials, $P_1 \not\equiv 0$, $P_2 \not\equiv$ const and further $\deg P_1 \neq \deg P_2$ or $\deg P_1 = \deg P_2$ but the leading coefficients of P_1^2 and P_2^2 are different.*

Proof. Consider $P_1 \sin z+P_2=c$ and $P_1' \sin z+P_1 \cos z+P_2'=0$. By cancelling out $\sin z$ and $\cos z$ we have

$$(*) \quad P_1^4=P_1^2(c-P_2)^2+(-P_2'P_1+P_2P_1'-P_1'c)^2 .$$

When (*) reduces to an identity for a c, we cannot conclude anything. However we have assumed that $\deg P_1 \neq \deg P_2$ or $\deg P_1 = \deg P_2$ but the leading coefficients of P_1^2, P_2^2 are different. Hence (*) is not an identity. Evidently (*) has only finitely many solutions. Hence $F=c$ and $F'=0$ have finitely many common roots for any c. Next consider the number of roots of $P_1' \sin z+P_1 \cos z+P_2'=0$. If this has only finitely many roots, then

$$P_1' \sin z+P_2 \cos z+P_2'=Qe^{\alpha z}$$

with a constant α and a polynomial Q. Then we can make use of the impossibility of Borel's identity [2], [7]. We have the existence of infinitely many roots of $P_1' \sin z+P_2 \cos z+P_2'=0$. Hence $P_1 \sin z+P_2$ with the conditions on P_1 and P_2 is left-prime in entire sense.

In order to prove the left-primeness in meromorphic sense we need another method. Suppose that $F(z)=f(g(z))$ with meromorphic (not entire) f and entire g. Suppose further that f, g are transcendental. Then by a result in [9]

$$f(w)=\frac{f^*(w)}{(w-w_1)^n}, \qquad f^*(w_1)\neq 0,$$

$$g(z)=w_1+Be^{\alpha z}.$$

Hence

$$P_1 \sin z+P_2=B^{-n}e^{-n\alpha z}f^*(w_1+Be^{\alpha z}).$$

Here f^* is transcendental and $\rho(f^*)=\rho(f)=0$ and B, α are constants, n is a positive integer. Hence

$$\begin{aligned}(1+\varepsilon)m(r, \sin z) &\geqq m(r, P_1 \sin z+P_2)\\ &\geqq N(r, 0, P_1 \sin z+P_2)=N(r, 0, f^*(w_1+Be^{\alpha z}))\\ &\leqq \sum_{j=1}^{K} N(r, \alpha_j, w_1+Be^{\alpha z})\\ &\geqq (k-2)m(r, e^{\alpha z})=(K-2)\frac{|\alpha|r}{\pi}.\end{aligned}$$

But $m(r, \sin z)=2r/\pi+O(1)$. Hence

$$2\geqq (K-2)|\alpha|.$$

Here K is arbitrary. This is impossible.

Suppose that f is rational and g is meromorphic. Let a be a pole of f. Then $g(z)-a$ should have no zero. Hence $g_1(z)=1/(g(z)-a)$ is entire. Thus $F(z)=R(g_1(z))$ with rational R and entire g_1. Then R has only one pole b. Therefore $g_1(z)=b+Ae^{\alpha z}$. Thus $F(z)$ is representable as $P(b+Ae^{\alpha z})e^{-m\alpha z}$ with a polynomial P and a positive integer m. This is clearly periodic, but $P_1 \sin z+P_2$ is not. This is untenable. Evidently the case that f is a polynomial and g is meromorphic (not entire) does not occur. q. e. d.

When $P_1^2=P_2^2$, (*) may reduce to an identity, for example, for $c=0$. Then right-primeness of $P_1 \sin z+P_2$ is not true in general. We can decide when it is right-prime. We shall not touch this problem.

COROLLARY 2.

$$F(z)=\int_0^z e^{-t^2}dt+z$$

is prime.

Proof. Consider the simultaneous equations

$$\begin{cases} F=c \\ F'=0\,. \end{cases}$$

$F'=0$ gives $\exp(-z^2)+1=0$. Hence with $z=re^{i\theta}$ $\cos 2\theta=0$, that is, $\theta=\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$. Then on the ray $\theta=\pi/4$

$$\int_0^r e^{-(t^2-\pi/4)i}dt=c-z\,.$$

It is very easy to prove the existence of

$$\int_0^\infty \cos(t^2-\pi/4)dt\,,\qquad \int_0^\infty \sin(t^2-\pi/4)dt$$

by the Leipnitz law for the alternating series. Hence

$$\int_0^r \exp\{-(t^2-\pi/4)i\}dt$$

is bounded for $r\to\infty$. But $c-z$ is not. Hence there are only finitely many roots of $F=c$ on the ray $\theta=\pi/4$. On the other rays $\theta=3\pi/4$, $5\pi/4$, $7\pi/4$ the same holds. Thus $F=c$ and $F'=0$ have only finitely many common roots. Consider $N(r, 0, F')$. Evidently

$$N(r, 0, F')\geqq m(r, F')(1-\varepsilon)\geqq(1-\varepsilon)r^2/\pi\,.$$

Hence by Theorem 1 $F(z)$ is left-prime in entire sense.

Let us consider the right-primeness of $F(z)$. Suppose that $F(z)=f(g(z))$ with a polynomial $g(z)$. Evidently $g(z)$ is of degree four, two or one. If $g(z)$ is of degree four, $F(z)=f(g(z))$ has almost equal values when $|z|$ is sufficiently large and $\theta=0$, $\pi/2$, π and $3\pi/2$. However two of these four give bounded values to F but the remaining two of these give unbounded values to F. This is impossible. If g is quadratic and is $\alpha(z-a)^2+b$, then all the zeros of $F'(z)$ except for only one should be symmetric with respect to the point a. Thus a should be the origin. Hence $g(z)=\alpha z^2+b$. Then $F(-z)=F(z)$. On the other hand $F(z)$ has the power series expansion

$$z+\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)n\,!}\,.$$

Hence $F(z)=-F(-z)$. This is untenable. Thus $F(z)$ is right-prime in entire sense. Therefore $F(z)$ is prime in entire sense. Then we make use of Gross' theorem, which asserts that every non-periodic entire prime function in entire sense is prime [6]. Thus $F(z)$ is prime. q. e. d.

For the function

$$\int_0^z e^{-t^p}dt+Z$$

we can apply the above method, although we need a more delicate consideration. Then we have the primeness of this function. For

$$\int_0^z e^{-t^p}dt, \qquad p \geqq 2$$

the primeness was proved by an entirely different method [8].

6. Applications of Theorem 2.

COROLLARY 3. *$e_2(z)+P(z)$, $e_3(z)+z$, $e_4(z)+z$ are prime, where $e_n(z)$ is defined by* $\exp(e_{n-1}(z))$, $e_1(z)=\exp z$ *and P is a non-constant polynomial.*

Proof. The case $e_2(z)+P(z)$. Consider the equations

$$\begin{cases} F=c \\ F'=0 . \end{cases}$$

By $F(z)=c$ we have

$$e^z=\log(c-P(z))+2p\pi i$$

and with $z=x+iy$

$$e^x \cos y=\log|c-P(z)| .$$

For $x \leqq x_0$, $e^x \cos y$ is bounded, but $\log|c-P|$ is not bounded for $|z| \geqq |z_1|$. Hence for $x \leqq x_0$ there are only finitely many roots of $F=c$. By $F'=0$ we have

$$e_2(z)=-P'e^{-z}$$

and hence

$$e^z=\frac{P'}{P-c} .$$

This gives

$$z=\log\frac{P'}{P-c}+2p\pi i ,$$

$$x=\log\left|\frac{P'}{P-c}\right| .$$

If $z\to\infty$, then the right hand side tends to $-\infty$. Hence for $x \geqq x_0$ there are only finitely many solutions. Hence the equations $F=c$, $F'=0$ have only a finite number of common roots.

Next consider $N(r, 0, F')$. Let

$$\phi=\frac{e_2(z)}{e_2(z)+P'(z)e^{-z}} .$$

Then

$$T(r, \phi) \leqq N(r, \infty, \phi)+N(r, 0, \phi)+N(r, 1, \phi)+O(\log rT(r, \phi))$$

for $r \notin E_\phi$, which has at most a finite measure. Then

$$N(r, 0, F') \geqq T(r, \phi)(1+\varepsilon)$$
$$\geqq m(r, e_2(z))(1+\varepsilon)$$

for $r \notin E_\phi$.

Hence by Theorem 2 (with a slight modification by the existence of E_ϕ) we have the left-primeness in entire sense. We shall not discuss the right-primeness of the function, since this is quite similar as in the following example.

The case $e_4(z)+z$. Consider the equation $e_4(z)+z=c$. Evidently $e_4(z)$ is bounded for $x \leqq x_0$, $z=x+iy$. But $c-z$ is not bounded for $|z| \geqq |z_1|$. Hence for $x \leqq x_0$ there are only finitely many roots of $e_4(z)+z=c$. The equation implies $e_3(z)=\log(c-z)+2p\pi i$. Taking its real part we have

$$\cos(\sin(e^x \sin y)\exp(e^x \cos y))\exp(\cos(e^x \sin y)\exp(e^x \cos y))$$
$$=\log|c-z| .$$

If $\cos y \leqq 0$ and $x \geqq x_0$, then the modulus of the left hand side is not greater than

$$\exp(\cos(e^x \sin y)\exp(e^x \cos y)) \leqq e_2(e^x \cos y) ,$$

which is bounded for $\cos y \leqq 0$ and $x \to +\infty$. If $\cos y > 0$, $\cos(e^x \sin y) \leqq 0$ and $x \to +\infty$, then the left hand side is again bounded. Hence by the unboundedness of $|\log|c-z||$ for $x \to +\infty$ there are only finitely many roots of $e_4(z)+z=c$, if $x \geqq x_0$, $\cos y \leqq 0$ or $x \geqq x_0$, $\cos y > 0$, $\cos(e^x \sin y) \geqq 0$. By $F'=0$

$$e_4(z)=-1/e_3(z)e_2(z)e_1(z)$$

and hence

$$e_3(z)e_2(z)e_1(z)=1/(z-c) .$$

This implies

$$e_2(z)+e_1(z)+z=\log\frac{1}{(z-c)}+2p\pi i$$

and its real part

$$\cos(e^x \sin y)\exp(e^x \cos y)+e^x \cos y+x=-\log|z-c| .$$

If $\cos y>0$, $\cos(e^x \cos y)>0$, the left hand side tends to $+\infty$ as $x \to +\infty$, but the right hand side tends to $-\infty$ as $x \to +\infty$. Hence for $x \geqq x_0$, $\cos y>0$, $\cos(e^x \sin y)>0$ the equations $F=c$ and $F'=0$ have only finitely many common roots. Therefore the equations $F=c$, $F'=0$ have at most finitely many common roots for every c.

Consider $N(r, 0, F')$. By the second fundamental theorem

$$N(r, 0, F') \geqq (1-\varepsilon)m(r, e_4(z)+z)$$

for $r \notin E_\phi$, which is a set of r of finite measure. Here

$$\phi(z)=\frac{e_4(z)}{e_4(z)+1/e_3(z)e_2(z)e_1(z)}.$$

Thus $e_4(z)+z$ is left-prime in entire sense.

Next we shall consider the right-primeness of $e_4(z)+z$. By the above proof there is no zero of $e_4(z)+z$ in $x\leqq x_0$, $|y|\geqq y_0$. Here x_0 is arbitrary and y_0 depends on x_0. Let $\{z_l\}$ be the set of zeros of $e_4(z)+z$ and $z_l=x_l+iy_l$. Then by the above fact $\{x_l\}$ does not have any finite cluster point. Of course there are only finitely many z_l having the same real part. Thus $\{x_l\}$ has at most one cluster point $+\infty$. On the other hand

$$N(r, 0, e_4(z)+z)\sim m(r, e_4(z))$$

for $r\notin E_\phi$, where ϕ is $e_4(z)/(e_4(z)+z)$. Hence $\{x_l\}$ tends to $+\infty$. Let us consider $e_4(z)+z=f(P(z))$ with a polynomial $P(z)$. If $P(z)$ has its degree at least two, then there must be infinitely many zeros of $e_4(z)+z$ in the left half plane. However this is not the case which we have just proved. This is a contradiction. We thus have the right-primeness of $e_4(z)+z$ in entire sense. Hence $e_4(z)+z$ is prime in entire sense. $e_4(z)+z$ is not periodic. Hence $e_4(z)+z$ is prime by Gross' theorem [6].

The case $e_3(z)+z$ is easier than the case $e_4(z)+z$. q. e. d.

Now the primeness of $e_n(z)+z$ is almost evident. The above proof for $n=4$ suggests the proof for n and the proof is an almost routine work. Compare with the method in [5], in which the primeness of $e_2(z)+z$ was proved.

Without any proof we state the following.

COROLLARY 4. *$e_2(z)e(z)+z$, $e_2(z^2)+z$ are prime.*

7. An extension of Theorem 2. Sometimes the side condition on $N(r, 0, F')$ in Theorem 2 makes an obstruction for applications. However we can really weaken it as in the following.

THEOREM 2'. *Let F be entire with $N(r, 0, F')\geqq K\log\log M(4r, F)$ for any positive K. Assume that the equations $F=c$ and $F'=0$ have finitely many common roots for every c. Then F is left-prime in entire sense.*

We shall not give any explicit proof of this Theorem. We only make use of Pólya's estimations $(F=f(g))$

$$M(r, F)\geqq M(cM(dr, g), f), \qquad 0<d<1,\ 0<c<1,$$

$$M(r, F)\leqq M(M(r, g), f)$$

and the known estimations

$$M(r, F)-|F(0)|\leqq rM(r, F')\leqq\frac{M(\alpha r, F)}{(\alpha-1)^2}$$

with $\alpha>1$. The process of the proof is quite similar as in Theorem 2. This Theorem contains Theorem 1.

COROLLARY 5.

$$F(z)=\int_0^z e_2(t)dt+z$$

is prime.

Proof. $F'(z)=e_2(z)+1$. Then by the second fundamental theorem for $e_2(z)$ we have

$$\begin{aligned}N(r, 0, F')&\geqq m(r, F')(1-\varepsilon)\\&\geqq A\frac{e^r}{\sqrt{r}}\end{aligned}$$

for $r\notin E_{F'}$, which is of finite measure. Here A is a constant. Further $M(r, F')\leqq e_2(r)$. Hence

$$\begin{aligned}\log\log M(4r, F)&\leqq\log\log(4rM(4r, F')+O(1))\\&\leqq\log\log(4re_2(4r)+O(1))\\&\leqq B4r\end{aligned}$$

for some constant B and for $r\geqq r_0$. Hence

$$N(r, 0, F')\geqq K\log\log M(4r, F)$$

for any $K>0$ and $r\notin E_{F'}$, $r\geqq r_0$.

$F'(z)=0$ has solutions $z=\log(2p+1)\pi+i(2n\pi+\pi/2)$ for $p\geqq0$ an integer and n an integer and $z=\log\{-(2p+1)\pi\}+i(2n\pi-\pi/2)$ for $p<0$ an integer and n an integer. It is enough to discuss the case $y=2n\pi+\pi/2$, since the case $y=2n\pi-\pi/2$ can be discussed in a quite similar manner. At the above zeros $z=x+iy$ of F'

$$\begin{aligned}\int_0^z e_2(t)dt=&-\int_0^y e^{\cos t}\sin\sin t\,dt+\int_0^x\cos e^t dt\\&+i\int_0^y e^{\cos t}\cos\sin t\,dt+i\int_0^x\sin e^t dt\,.\end{aligned}$$

Taking its imaginary part we have

$$\int_0^y e^{\cos t}\cos\sin t\,dt+\int_0^x\sin e^t dt=\beta-y$$

with $c=\alpha+i\beta$. Evidently

$$\int_0^\infty \sin e^s ds=\int_1^\infty \frac{\sin t}{t} dt$$

exists. Hence

$$\left|\int_0^x \sin e^s ds\right| \leqq M$$

for $x \geqq 0$. If $y>0$, then

$$\int_0^y e^{\cos t} \cos \sin t\, dt \geqq \frac{\cos 1}{e} y .$$

Thus

$$\frac{\cos 1}{e} y-M \leqq \beta-y .$$

This is impossible for $y \geqq y_0$. If $y<0$, then

$$\int_0^y e^{\cos t} \cos \sin t\, dt \leqq \frac{\cos 1}{e} y .$$

Thus

$$\frac{\cos 1}{e} y+M \geqq \beta-y ,$$

which is untenable for $y \leqq -y_0$. Next taking the real part of $F=c$ we have

$$\alpha-x=\int_0^x \cos e^t dt-\int_0^y e^{\cos t} \sin \sin t\, dt .$$

Evidently

$$\int_0^\infty \cos e^t dt=\int_1^\infty \frac{\cos s}{s} \mathrm{ds}$$

exists and hence the first integral in the right hand side is bounded. The second integral is also bounded for $|y| \leqq y_0$. Thus for $x \geqq x_0$ we have a contradiction. Therefore the equations $F=c$ and $F'=0$ have only finitely many common roots for every c. Thus by Theorem 2$'$ F is left-prime in entire sense.

The right-primeness of F in entire sense is almost trivial. Since $F'=0$ has solutions $z=\log(2p\pi+\pi)+i(2n\pi+\pi/2)$, $p \geqq 0$ and $z=\log(-2p\pi-\pi)+i(2n\pi-\pi/2)$, $p<0$. Hence all the solutions lie in the right half plane. Consider $F'=f'(g)g'$ with a polynomial g. Assume that g is of degree$\geqq 2$. Then there must be infinitely many zero of F' in the left half plane. This is impossible.

Therefore F is prime in entire sense and hence by the non-periodicity of F F is prime. q. e. d.

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