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ON THE GROWTH RATE OF COMPOSITIONS OF ENTIRE AND MEROMORPHIC FUNCTIONS

Dedicated to Professor Yûsaku Komatu on his 60th birthday

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1. Let $f(z)$ be a meromorphic function and $T(r, f)$ its Nevanlinna characteristic function. Gross-Yang [4] proposed the following open question:

Suppose that $f_1(z)$ and $f_2(z)$ are meromorphic functions, $g_1(z)$ and $g_2(z)$ are entire functions and that

$$\lim_{r\to\infty}\frac{T(r, f_1)}{T(r, f_2)}=0 \qquad \text{and} \qquad \lim_{r\to\infty}\frac{T(r, g_1)}{T(r, g_2)}=0 .$$

Then is it true that

$$\lim_{r\to\infty}\frac{T(r, f_1\circ g_1)}{T(r, f_2\circ g_2)}=0 \ ?$$

In this paper, firstly, we shall give a negative answer to this question, that is,

THEOREM 1. *There are two meromorphic functions $f_1(z)$, $f_2(z)$ and two entire functions $g_1(z)$, $g_2(z)$ such that*

$$\lim_{r\to\infty}\frac{T(r, f_1)}{T(r, f_2)}=0,\quad \lim_{r\to\infty}\frac{T(r, g_1)}{T(r, g_2)}=0 \quad \textit{and} \quad \overline{\lim_{r\to\infty}}\frac{T(r, f_1\circ g_1)}{T(r, f_2\circ g_2)}=\infty .$$

2. Let $f(z)$ be an entire function and $M(r, f)$ its maximum modulus on $|z|=r$. In our previous paper [5] we discussed the asymptotic behavior of the ratio $\log M(r, h\circ g)/\log M(r, h\circ f)$, where $h(z)$, $g(z)$ and $f(z)$ are entire functions.

Now we investigate the asymptotic behavior of the ratio $\log M(r, g\circ h)/\log M(r, f\circ h)$. We shall prove

THEOREM 2. *Let $g(z)$ and $f(z)$ be entire functions such that*

$$\lim_{r\to\infty}\frac{\log M(\alpha r, g)}{\log M(r, f)}=0 \tag{2.1}$$

for some constant $\alpha>1$. Then for any non-constant entire function $h(z)$

$$\underline{\lim_{r\to\infty}}\frac{\log M(r, g\circ h)}{\log M(r, f\circ h)}=0 .$$

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Clunie [2] found an entire function $h(z)$ such that

$$\overline{\lim_{r\to\infty}}\frac{\log M(r, h)}{\log M(r, \exp\circ h)}=\infty . \tag{2.2}$$

Hence in Theorem 2 we can not replace inferior limit by superior limit. Theorem 2, also, is not valid for $\alpha=1$. In fact we put $g(z)=\exp\exp z$, $f(z)=\exp(-z\exp(-z))$ and $h(z)=z^3+z^2$. Then we have $\log M(r, g)=\exp r$, $\log M(r, f)=r\exp r$, $\log M(r, g\circ h)=\exp(r^3+r^2)$ and $\log M(r, f\circ h)\leqq(r^3+r^2)\exp(r^3+(1/2)r^2+O(r))$ $(r\to\infty)$. Hence we obtain

$$\lim_{r\to\infty}\frac{\log M(r, g)}{\log M(r, f)}=0 \quad \text{and} \quad \lim_{r\to\infty}\frac{\log M(r, g\circ h)}{\log M(r, f\circ h)}=\infty .$$

Next we consider the asmptotic behavior of the ratio $\log M(r, f_1\circ g_1)/\log M(r, f_2\circ g_2)$, where f_j and g_j are entire functions. By the same method used in [5] and in the proof of Theorem 2 we can obtain the following:

THEOREM 3. *Let $f_j(z)$ and $g_j(z)$ $(j=1, 2)$ be non-constant entire functions.*

(I) $$\overline{\lim_{r\to\infty}}\frac{\log M(r, f_1)}{\log M(r, f_2)}<\infty \quad \textit{and} \quad \lim_{r\to\infty}\frac{\log M(\alpha r, g_1)}{\log M(r, g_2)}=0 \quad (\alpha>1)$$

imply

$$\lim_{r\to\infty}\frac{\log M(r, f_1\circ g_1)}{\log M(r, f_2\circ g_2)}=0 .$$

(II) $$\overline{\lim_{r\to\infty}}\frac{\log M(r, f_1)}{\log M(r, f_2)}<\infty \quad \textit{and} \quad \lim_{r\to\infty}\frac{\log M(r, g_1)}{\log M(r, g_2)}=0$$

or

$$\lim_{r\to\infty}\frac{\log M(\alpha r, f_1)}{\log M(r, f_2)}=0 \quad \textit{and} \quad \overline{\lim_{r\to\infty}}\frac{M(r, g_1)}{M(r, g_2)}\leqq\beta \quad (\alpha>\beta\geqq1)$$

imply

$$\underline{\lim_{r\to\infty}}\frac{\log M(r, f_1\circ g_1)}{\log M(r, f_2\circ g_2)}=0 .$$

It is clear from Theorem 4 in [5] and (2.2) that in (II) of Theorem 3 we can not replace inferior limit by superior limit. Moreover we shall show

THEOREM 4. *There are four entire functions $f_j(z)$ and $g_j(z)$ $(j=1, 2)$ such that*

$$\lim_{r\to\infty}\frac{\log M(r, f_1)}{\log M(r, f_2)}=0, \quad \lim_{r\to\infty}\frac{\log M(r, g_1)}{\log M(r, g_2)}=0 \quad \textit{and} \quad \overline{\lim_{r\to\infty}}\frac{\log M(r, f_1\circ g_1)}{\log M(r, f_2\circ g_2)}=\infty .$$

3. Lemmas. In order to prove our theorems we need the following lemmas:

LEMMA 1 ([1, 2]). *Let $f(z)$ and $h(z)$ be entire functions. Then*

$$M(r, f\circ h)\geqq M((1+o(1))M(r, h), f) \qquad \textit{as} \quad r\to\infty$$

outside a set of r of finite logarithmic measure which depends, as does $o(1)$, on $h(z)$.

Combing (2.7) and (2.8) in [5] with Theorem 1 in [3] we obtain the following lemmas:

LEMMA 2. *For any transcendental meromorphic function $f(z)$, there is an entire function $g(z)$ such that*

$$\lim_{r\to\infty}\frac{N(r, 1/g)}{T(r, g)}=1, \quad \lim_{r\to\infty}\frac{N(r, 1/g)}{N(r, f)}=0 \quad \textit{and} \quad \lim_{r\to\infty}\frac{N(r, 1/g)^2}{N(r, f)}=\infty .$$

LEMMA 3. *For any transcendental entire function $f(z)$, there is an entire function $g(z)$ such that*

$$M(r, g)=g(r), \quad \lim_{r\to\infty}\frac{\log M(r, g)}{\log M(r, f)}=0 \quad \textit{and} \quad \lim_{r\to\infty}\frac{(\log M(r, g))^2}{\log M(r, f)}=\infty .$$

4. Proof of Theorem 1. It follows from a slight modification of the proof of Theorem 5 in [2] that we have a transcendental meromorphic function $f_2(z)$, a transcendental entire function $g_2(z)$ and two sequences $\{R_n\}$, $\{M_n\}$ such that

$$\lim_{n\to\infty} R_n=\lim_{n\to\infty} M_n=\infty , \qquad T(R_n, f_2\circ g_2)=M_n(1+o(1)) \qquad (n\to\infty)$$

and

$$N(R_n, f_2)\geqq M_n^4 \log 2 .$$

Let $f_1(z)$ be an entire function obtained by Lemma 2 for the given meromorphic function $f_2(z)$. Then we have

$$\lim_{r\to\infty}\frac{N(r, 1/f_1)}{T(r, f_1)}=1, \quad \lim_{r\to\infty}\frac{N(r, 1/f_1)}{N(r, f_2)}=0 \quad \text{and} \quad \lim_{r\to\infty}\frac{N(r, 1/f_1)}{N(r, f_2)^{1/2}}=\infty .$$

and consequently

$$\overline{\lim_{r\to\infty}}\frac{T(r, f_1)}{T(r, f_2)}\leqq\overline{\lim_{r\to\infty}}\frac{N(r, 1/f_1)}{N(r, f_2)}=0 ,$$

$$(1+o(1))T(R_n, f_1)=N(R_n, 1/f_1)\geqq N(R_n, f_2)^{1/2}\geqq M_n^2(\log 2)^{1/2} \qquad (n\to\infty) .$$

Hence

$$\frac{T(R_n, f_1)}{T(R_n, f_2\circ g_2)}\geqq(1+o(1))(\log 2)^{1/2}M_n \qquad (n\to\infty)$$

and so

$$\overline{\lim_{r\to\infty}}\frac{T(r, f_1)}{T(r, f_2\circ g_2)}=\infty .$$

Therefore, putting $g_1(z)=z$, we obtain Theorem 1.

5. Proof of Theorem 2. It follows from Lemma 1 and (2.1) that there is a set E of r of finite logarithmic measure and

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{\log M(r, g\circ h)}{\log M(r, f\circ h)} \leqq \varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{\log M(M(r, h), g)}{\log M((1+o(1))M(r, h), f)}$$

$$\leqq \varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{\log M(M(r, h), g)}{\log M((1/\alpha)M(r, h), f)} = 0 .$$

Hence we have

$$\varliminf_{r\to\infty} \frac{\log M(r, g\circ h)}{\log M(r, f\circ h)} = 0 ,$$

which gives Theorem 2.

6. Proof of Theorem 4. Let $f_2(z)$ be $\exp z$, $g_2(z)$ an entire function satisfying (2.2) and $g_1(z)$ the entire function obtained by Lemma 3 for the given entire function $g_2(z)$. Then we have

$$\varlimsup_{r\to\infty} \frac{\log M(r, g_2)}{\log M(r, f_2\circ g_2)} = \infty \qquad \text{and} \qquad \log M(r, f_2) = r \tag{6.1}$$

and

$$M(r, g_1) = g_1(r), \quad \lim_{r\to\infty} \frac{\log M(r, g_1)}{\log M(r, g_2)} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{(\log M(r, g_1))^2}{\log M(r, g_2)} = \infty . \tag{6.2}$$

We denote by $f_1(z)$ an entire function such that

$$M(r, f_1) = f_1(r) \quad \text{and} \quad \log M(r, f_1) \sim (\log r)^2 \quad (r\to\infty) . \tag{6.3}$$

The existence of $f_1(z)$ is ensured by Theorem 1 in [3]. (6.1), (6.2) and (6.3) yield

$$\lim_{r\to\infty} \frac{\log M(r, f_1)}{\log M(r, f_2)} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{\log M(r, g_1)}{\log M(r, g_2)} = 0 .$$

It also follows from (6.2) and (6.3) that

$$M(r, f_1\circ g_1) = f_1(g_1(r)) = M(M(r, g_1), f_1)$$

and so

$$\log M(r, f_1\circ g_1) \sim (\log M(r, g_1))^2 \qquad (r\to\infty) .$$

Hence by (6.2) we have

$$\lim_{r\to\infty} \frac{\log M(r, f_1\circ g_1)}{\log M(r, g_2)} = \infty$$

and consequently together with (6.1)

$$\varliminf_{r\to\infty} \frac{\log M(r, f_1\circ g_1)}{\log M(r, f_2\circ g_2)} = \infty .$$

Thus the proof of Theorem 4 is complete.

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