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DOMAIN WITH MANY VANISHING COHOMOLOGY SETS

Dedicated to Professor Yûsaku Komatu on his sixtieth birthday

By Joji Kajiwara

Introduction.

The aim of this paper is to generalize a previous result [10] of the author. Oka [12] proved that a domain of holomorphy in C^n is a Cousin-I domain. Oka [13] also proved that a Cousin-II distribution in a domain of holomorphy in C^n has an analytic solution if and only if it has a topological solution. Grauert [5] proved that the canonical mapping of $H^1(X, \mathcal{A}_L)$ in $H^1(X, \mathcal{C}_L)$ is bijective for a Stein space X and a complex Lie group L where $\mathcal{A}_L$ and $\mathcal{C}_L$ are, respectively, the sheaves over X of all germs of holomorphic and continuous mappings in L. Roughly talking, many cohomology sets have the possibility of vanishing in a Stein space.

Conversely, by Cartain [2] and Behnke-Stein [1], a Cousin-I domain in C^2 is always a domain of holomorphy. By Cartan [3] $C^3-\{(0, 0, 0)\}$ is a Cousin-I domain which is not a domain of holomorphy. By Thullen [15] $D=\{(z_1, z_2)\in C^2;$ $|z_1|<1, |z_2|<1\}-\{(0, 0)\}$ is a Cousin-II domain which is not a domain of holomorphy. By a previous remark [6] of the author, the Thullen's domain D is an example of a Cousin-II domain which satisfies $H^1(D, \mathcal{O}^*)\neq 0$ for the sheaf $\mathcal{O}^*$ of multiplicative groups of all germs of never vanishing holomorphic functions. By the previous result [11] of the author and Kazama, however, a subdomain X of a two-dimensional Stein manifold is a Stein manifold if X satisfies $H^1(X, \mathcal{A}_L)=0$ for a complex Lie group L. In the case of higher dimension, the author [10] proved that a subdomain X of a Stein manifold S with real one-codimensional smooth boundary is a Stein manifold, if, for an abelian complex Lie group L, X satisfies $H^1(X\cap P, \mathcal{A}_L)=0$ for all analytic polydisc P in S.

The aim of this paper is to prove that a subdomain X of a Stein manifold S with real one-codimensional smooth boundary is a Stein manifold if, for a complex Lie group L, X satisfies $H^1(X\cap P, \mathcal{A}_L)=0$ for all analytic polydisc P in S. The above boundary condition for X can not be omitted as the above Cartan's example $C^3-\{(0, 0, 0)\}$ shows. Roughly talking, a subdomain of a Stein manifold with many vanishing cohomology sets is also a Stein manifold. This is the principle which the author wants to maintain. In the proof, we use Lemmata and methods used in [11] and [9]. In this occasion the author ex-

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presses his hearty gratitude to Professor Y. Komatu who gives his continuous and kind encouragements since 1957.

§ 1. Monotonically increasing sequence of domains.

Let M be a complex manifold. If φ is a locally biholomorphic mapping of a complex manifold D in M, (D, φ) is called *an open set over* M. Let (D_1, φ_1) and (D_2, φ_2) be open sets over M. If there is a holomorphic mapping τ of (D_1, φ_1) in (D_2, φ_2) such that $\varphi_1=\varphi_2\circ\tau$, we write $(D_1, \varphi_1)<(D_2, \varphi_2)$. By this relation the set of all open sets over M forms a partially ordered set $\mathfrak{D}$. A sequence $\{(D_p, \varphi_p); p=1, 2, 3, \cdots\}$ of open sets over M is called a *monotonically increasing sequence of open sets over* M if $(D_p, \varphi_p)<(D_{p+1}, \varphi_{p+1})$ for any p. Let $\{(D_p, \varphi_p); p=1, 2, 3, \cdots\}$ be a monotonically increasing sequence of domains over M. Then it is a monotonically increasing sequence in the partially ordered set $\mathfrak{D}$. In the previous paper [8] the author proved the unique existence of its supremum in $\mathfrak{D}$ and called it the *limit of the sequence* $\{(D_p, \varphi_p); p=1, 2, 3, \cdots\}$. Let D be a complex manifold and L be a complex Lie group. The sheaf over D of all germs of holomorphic mappings in L is denoted by $\mathcal{A}_L$. Let $\mathfrak{U}=\{U_i; i\in I\}$ be an open covering of D. We define an element $\{g_{ij}\}$ of $Z^1(\mathfrak{U}, \mathcal{A}_L)$ by putting $g_{ij}=1$ in $U_i\cap U_j$ for any $i, j\in I$. Then $\{g_{ij}\}$ defines an element of $H^1(D, \mathcal{A}_L)$, which is called a *trivial element of* $H^1(D, \mathcal{A}_L)$. If $H^1(D, \mathcal{A}_L)$ consists of only a trivial element, we write $H^1(D, \mathcal{A}_L)=0$ for the sake of brevity.

A complex manifold D is said to be *analytically contractible* if there is a continuous mapping $f(x, t)$ of $D\times[0, 1]$ in D such that $f(x, t)$ is a holomorphic mapping of D in D for any fixed $t\in[0, 1]$, that $f(x, 0)$ is the identity mapping of D and that $f(x, 1)$ is a constant mapping of D in D.

LEMMA 1. *Let* $\{(D_p, \varphi_p); p=1, 2, 3, \cdots\}$ *be a monotonically increasing sequence of open sets over a Stein manifold* S *such that each* D_p *is analytically contractible and connected,* (D, φ) *be its limit and* τ_p *be the canonical mapping of* D_p *in* D *for each* p. *Let* L *be a complex Lie group and* α *be an element of* $H^1(D, \mathcal{A}_L)$. *If the image* $\tau_p^*(\alpha)$ *of* α *by the canonical mapping* τ_p^* *of* $H^1(D, \mathcal{A}_L)$ *in* $H^1(D_p, \mathcal{A}_L)$ *induced by* τ_p *is a trivial element for any* p, *then* α *is a trivial element of* $H^1(D, \mathcal{A}_L)$.

Proof. We denote by τ_q^p the canonical mapping of D_p in D_q for any p and q with $p\leqq q$. Let $\{Q_p; p=1, 2, 3, \cdots\}$ be a sequence such that each Q_p is a relatively compact subdomain of D_p, that $\tau_{p+1}^p(Q_p)\subset Q_{p+1}$ for any p and that $D=\bigcup_{p=1}^{\infty}\tau_p(Q_p)$. Then $(Q_p, \varphi_p|Q_p)$ is a monotonically increasing sequence of domains over S and (D, φ) is its limit. Let $(\tilde{Q}_p, \tilde{\varphi}_p)$ be the envelope of holomorphy of $(Q_p, \varphi_p|Q_p)$ over S and λ_p be the canonical mapping of Q_p in $\tilde{Q}_p$. For any p and q with $p\leqq q$, there is a holomorphic mapping $\tilde{\tau}_q^p$ of $\tilde{Q}_p$ in $\tilde{Q}_q$ such that $\tilde{\varphi}_p=\tilde{\varphi}_q\circ\tilde{\tau}_q^p$ and $\lambda_q\circ(\tau_q^p|Q_p)=\tilde{\tau}_q^p\circ\lambda_p$. Hence $\{(\tilde{Q}_p, \tilde{\varphi}_p)\}$ is a monotonically increasing sequence of domains over S. Let $(\tilde{D}, \tilde{\varphi})$ be its limit. Then $(\tilde{D}, \tilde{\varphi})$ is the envelope of holomorphy of (D, φ). Let λ be the canonical mapping of D in $\tilde{D}$. For any p there is a holomorphic mapping $\tilde{\tau}_p$ of $\tilde{Q}_p$ in $\tilde{D}$ such that $\tilde{\varphi}_p=\tilde{\varphi}\circ\tilde{\tau}_p$ and $\tilde{\varphi}_p=\tilde{\tau}_q\circ\tau_q^p$ for any p and q with $p\leqq q$. Since $\tilde{D}$ is a Stein manifold by Docquier-Grauert [4], there is a sequence $\{P_p\,;\, p=1, 2, 3, \cdots\}$ of relatively compact analytic polycylinders P_p defined by holomorphic functions in $\tilde{D}$ such that P_p is a relatively compact subdomain of P_{p+1} for any p and that $\tilde{D}=\bigcup_{p=1}^{\infty} P_p$. Since $(\tilde{D}, \tilde{\varphi})$ is the limit of $(\tilde{Q}_p, \tilde{\varphi}_p)$, there is a sequence $\{\nu_p\,;\, p=1, 2, 3, \cdots\}$ of positive integers such that $\tilde{\tau}_{\nu_p}$ maps a relatively compact subdomain P'_p of Q_{ν_p} biholomorphically onto P_p for any p. Without loss of generality, we may assume that $\nu_p=p$.

Now we go to prove Lemma 1. Let $\{f_{ij}\}$ be an element of $Z^1(\mathfrak{U}, \mathcal{A}_L)$ for an open covering $\mathfrak{U}=\{U_i\ ; i\in I\}$ of D such that $\{f_{ij}\}$ is an element corresponding to α. We put $\tau_p^{-1}(\mathfrak{U})=\{\tau_p^{-1}(U_i)\,;\, i\in I\}$ for each p. Then $\tau_p^{-1}(\mathfrak{U})$ is an open covering of D_p and $\{f_{ij}\circ\tau_p\}$ is an element of $Z^1(\tau_p^{-1}(\mathfrak{U}), \mathcal{A}_L)$. Since $\tau_p^*(\alpha)$ is trivial, there is an element $\{f_i^p\}$ of $C^0(\tau_p^{-1}(\mathfrak{U}), \mathcal{A}_L)$ for any p such that

$$f_{ij}\circ\tau_p=f_j^p(f_i^p)^{-1}$$

in $\tau_p^{-1}(U_i\cap U_j)$ for any i, $j\in I$. If we put

$$f^p=(f_i^p)^{-1}(f_i^{p+1}\circ\tau_{p+1}^p)$$

in $\tau_p^{-1}(U_i)$, then f^p is a well-defined element of $H^0(D_p, \mathcal{A}_L)$. Since D_p is analytically contractible, $H^0(D_p, \mathcal{A}_L)$ forms a connected topological group. Therefore, any neighborhood of the neutral element of $H^0(D_p, \mathcal{A}_L)$ generates $H^0(D_p, \mathcal{A}_L)$. Let exp be the exponential mapping of the Lie algebra $\boldsymbol{C}^m$ of L in L. There is a polydisc neighborhood W of the origin of $\boldsymbol{C}^m$ such that exp maps an open neighborhood of the closure of W biholomorphically onto a neighborhood of 1 in L. We put $W'=\exp(W)$. There is a finite set $\{f^{p,\nu}\}$ of holomorphic mappings $f^{p,\nu}$ of D_p in L for any p such that

$$f^p=\prod f^{p,\nu}$$

and the $f^{p,\nu}(Q_p)\subset W'$ for any p and ν. Then each $(\exp W')^{-1}\circ(f^{p,\nu}|Q_p)$ is a holomorphic mapping of Q_p in $\boldsymbol{C}^m$. Since $(\tilde{Q}_p, \tilde{\varphi}_p)$ is the envelope of holomorphy of $(Q_p, \varphi_p|Q_p)$, there is a holomorphic mapping $F^{p,\nu}$ of $\tilde{Q}_p$ in $\boldsymbol{C}^m$ for any p and ν such that

$$(\exp W')^{-1}\circ(f^{p,\nu}|Q_p)=F^{p,\nu}\circ\lambda_p\,.$$

Then $F^{p,\nu}\circ(\tilde{\tau}_p|P'_p)^{-1}$ is holomorphic mapping of P_p in $\boldsymbol{C}^m$ for any p and ν. Let $\{\varepsilon_p\,;\, p=1, 2, 3, \cdots\}$ be a sequence of positive numbers. Since P_p is holomorphically convex with respect to $\tilde{D}$, there is a holomorphic mapping $G^{p,\nu}$ of D in $\boldsymbol{C}^m$ for any p and ν such that

$$|F^p\circ(\tilde{\tau}_p|P'_p)^{-1}-G^{p,\nu}|<\varepsilon_p$$

in P_{p-1} for any p and ν. We put

$$g^p=\prod \exp(G^{p,\nu}\circ\lambda)$$

in D for any p. Then g^p is a holomorphic mapping of D in L which approximates f^p in some sense. We put

$$g_\iota^p=f_\iota^p(g^{p-1}\circ\tau_p)(g^{p-2}\circ\tau_p)\cdots(g^1\circ\tau_p)$$

in $\tau_p^{-1}(U_\iota)$. Then $\{g_\iota^p;\ p=1, 2, 3, \cdots\}$ converges to a holomorphic mapping g_ι of U_ι in L uniformly in any compact subset of U_ι if $\{\varepsilon_p\}$ is sufficiently small and decreasing. Then $\{g_\iota\}\in C^0(\mathfrak{U}, \mathcal{A}_L)$ satisfies

$$f_{\iota j}=g_j g_\iota^{-1}$$

in $U_\iota\cap U_j$ for any ι, $j\in I$. Q. E. D.

§ 2. Domains exhausted by L-regular domains.

In the following Lemmata 2 and 3, we put

$$U_1=\{(z_1, z_2)\in \boldsymbol{C}^2;\ 0<|z_1|<1,\ |z_2|<1\}$$

and

$$U_2=\{(z_1, z_2)\in \boldsymbol{C}^2;\ |z_1|<1,\ 0<|z_2|<1\}\,.$$

The following Lemmata 2 and 3 are, respectively, Lemmata 6 and 10 in the previous paper [11], so the proofs are omitted.

LEMMA 2. *Let B be an (m, m)-matrix with a non-zero eigen-value. There are not $g_i\in H^0(U_\iota, \mathcal{A}_{GL(m,\boldsymbol{C})})$ $(\iota=1, 2)$ such that*

$$\exp\Big(\frac{B}{z_1z_2}\Big)=g_2g_1^{-1}$$

in $U_1\cap U_2$.

LEMMA 3. *Let B be a non-zero (m, m)-matrix, whose eigen-values are all zero. There are not $g_i\in H^0(U_\iota, \mathcal{A}_{GL(m,\boldsymbol{C})})$ such that*

$$\exp\Big(\exp\Big(\frac{1}{z_1}+\frac{1}{z_2}\Big)B\Big)=g_2g_1^{-1}$$

in $U_1\cap U_2$.

A complex manifold P is called an *analytic polydisc* if there is a biholomorphic mapping of a polydisc $\{w=(w_1, w_2, \cdots, w_n);\ |w_1|<r_1,\ |w_2|<r_2, \cdots, |w_s|<r_s\}$ $(0\leqq s\leqq n)$ onto P. An analytic polydisc P is analytically contractible and $H^1(P, \mathcal{A}_L)=0$ for any complex Lie group L. Let D be an open subset of a complex manifold M and L be a complex Lie group. If $H^1(D\cap P, \mathcal{A}_L)=0$ for any analytic polydisc P in M, D is called an *L-regular open set in M*. A domain D in a complex manifold M is said to be *exhausted by L-regular open sets* if there is a

sequence $\{D_p\}$ of L-regular open sets D_p in M such that D_p is relatively compact open subset of D_{p+1} for any p and $D=\bigcup_{p=1}^{\infty} D_p$.

LEMMA 4. *Let L be a complex Lie group whose dimension m is positive. Let Ω be a domain in C^n exhausted by L-regular domains Ω_p in C^n. Then Ω is a Stein manifold.*

Proof. In case that $n=1$, there is nothing to prove. In case that $n=2$, each Ω_p and, therefore, Ω is a Stein manifold by the previous paper [11]. So we may assume that $n\geqq 3$. It suffices to prove that Ω is p_7-convex in the sense of Docquier-Grauert [4] by Oka [14]. Assume that Ω were not p_7-convex. For any positive numbers ε and ε' with $1>\varepsilon>\varepsilon'$ we put

$$D(\varepsilon)=\{w=(w_1, w_2, \cdots, w_n)\in C^n ;\ |w_1|<1+\varepsilon,\ |w_\imath|<1(\imath=2, 3, \cdots, n)\}$$

$$\cup\{w=(w_1, w_2, \cdots, w_n)\in C^n ;\ 1-\varepsilon<|w_1|<1+\varepsilon,\ |w_\imath|<1+\varepsilon\ (\imath=2, 3, \cdots, n)\},$$

$$D(\varepsilon, \varepsilon')=\{w=(w_1, w_2, \cdots, w_n)\in C^n ;\ |w_1|<1+\varepsilon-\varepsilon',\ |w_\imath|<1-\varepsilon'\ (\imath=2, 3, \cdots, n)$$

$$\cup\{w=(w_1,w_2,\cdots,w_n)\in C^n ;\ 1-\varepsilon+\varepsilon'<|w_1|<1+\varepsilon-\varepsilon', |w_\imath|<1+\varepsilon-\varepsilon'\ (\imath=2,3,\cdots,n)\}$$

and

$$E(\varepsilon)=\{w=(w_1, w_2, \cdots, w_n)\in C^n ;\ |w_\imath|<1+\varepsilon\ (\imath=2, 3, \cdots, n)\}.$$

Then there are a positive number ε and a biholomorphic mapping τ of $E(\varepsilon)$ in C^n such that $\tau(D(\varepsilon))$ is a subdomain of Ω, that there is a point $a=(a_1, a_2, \cdots, a_n)$ of C^n such that its mage $\tau(a)$ is a boundary point of $\tau(D(\varepsilon))$ and Ω at the same time and that it satisfies $|a_1|\leqq 1-\varepsilon$, $|a_2|=1$, $|a_\imath|<1+\varepsilon$ $(\imath=3, 4, \cdots, n)$. Since the L-regularity and the p_7-convexity which is a local property are invariant under the analytic isomorphism τ, we may assume that τ is the identity mapping of $E(\varepsilon)$. We put

$$H=\{(w_1, w_2, 0, \cdots, 0)\in C^n ;\ (w_1, w_2)\neq(a_1, a_2)\}.$$

There are strictly monotonically decreasing sequences $\{\varepsilon_p\}$ and $\{\delta_p\}$ of positive numbers such that $D'(\varepsilon_p)=D(\varepsilon, \varepsilon_p)\subset\Omega_p$ and the set

$$G_p=\{w=(w_1, w_2, \cdots, w_n)\in\Omega_p ;\ |w_\imath|<\delta_p\ (i=3, 4, \cdots, n)\}$$

satisfies $(w_1, w_2)\neq(a_1, a_2)$ for any point $(w_1, w_2, \cdots, w_n)$ of G_p and that $\varepsilon_p\to 0$ and $\delta_p\to 0$ as $p\to\infty$.

Let $\mathfrak{U}=\{U_\imath ;\ i\in I\}$ be any open covering of H and $\{f_{\imath j}(w_1, w_2)\}$ be any element of $Z^1(\mathfrak{U}, \mathcal{A}_L)$. We define an open covering $\mathfrak{V}_p=\{V_\imath^p\}$ of G_p for any p by putting

$$V_\imath^p=\{w=(w_1, w_2, \cdots, w_n)\in G_p ;\ (w_1, w_2, 0, \cdots, 0)\in U_\imath\}.$$

for any $\imath\in I$. Then $\{f_{\imath j}(w_1, w_2)\}$ defines an element of $Z^1(\mathfrak{V}_p, \mathcal{A}_L)$. Since Ω_p is L-regular in C^n, we have $H^1(G_p, \mathcal{A}_L)=0$. There is an element $\{g_\imath^p\}$ of $C^0(\mathfrak{V}^p, \mathcal{A}_L)$

for any p such that

$$f_{ij}(w_1, w_2)=g_j^p(w_1, w_2, \cdots, w_n)(g_i^p(w_1, w_2, \cdots, w_n))^{-1}$$

in $V_i^p \cap V_j^p$ for any $i, j \in I$. We put

$$\mathfrak{U}^p=\{V_i^p \cap D'(\varepsilon_p) \cap H;\ i \in I\}$$

and

$$\mathfrak{U} \cap D(\varepsilon) \cap H=\{U_i \cap D(\varepsilon) \cap H;\ i \in I\}$$

for any p. Then $\mathfrak{U}^p$ is an open covering of $D'(\varepsilon_p) \cap H$ and $\{g_i^p(w_1, w_2, 0, \cdots, 0)\}$ is an element of $C^0(\mathfrak{U}^p, \mathcal{A}_L)$ such that

$$f_{ij}(w_1, w_2)=g_j^p(w_1, w_2, 0, \cdots, 0)\ (g_i^p(w_1, w_2, 0, \cdots, 0))^{-1}$$

in $V_i^p \cap V_j^p \cap D'(\varepsilon_p) \cap H$. Since $\{D'(\varepsilon_p) \cap H;\ p=1, 2, 3, \cdots\}$ is a monotonically increasing sequence of analytically contractible open sets in $\boldsymbol{C}^2$ and since $D(\varepsilon) \cap H$ is its limit, by Lemma 1 there is an element $\{f_i\}$ of $C^0(\mathfrak{U} \cap D(\varepsilon) \cap H, \mathcal{A}_L)$ such that

$$f_{ij}=f_j f_i^{-1}$$

in $U_i \cap U_j \cap D(\varepsilon) \cap H$ for any $i, j \in I$.

Now we continue to prove Lemma 4. If L is abelian, by the previous paper [11] of the author and Kazama, the limit Ω of L-regular domains Ω_p is a Stein manifold. So we may assume that L is a non-abelian connected m-dimensional complex Lie group. Let $\mathcal{GL}(m, \boldsymbol{C})$ and $\mathcal{L}$ be, respectively, the Lie algebras of $GL(m, C)$ and L. Let $\exp : \mathcal{GL}(m, \boldsymbol{C}) \to GL(m, \boldsymbol{C})$ and $\exp : \mathcal{L} \to L$ be the exponential mappings. Let $ad : \mathcal{L} \to \mathcal{GL}(m, \boldsymbol{C})$ and $Ad : L \to GL(m, \boldsymbol{C})$ be the adjoint representations. We have

$$Ad \exp(tX)=\exp(t\, ad\, X)$$

for any $t \in \boldsymbol{C}$ and $X \in \mathcal{L}$. Since L is not abelian, there is an element X of $\mathcal{L}$ such that

$$B= ad\, X$$

is a non-zero (m, m)-matrix. We consider an open covering $\mathfrak{U}=\{H_1, H_2\}$ of the $H=\{w=(w_1, w_2, 0, \cdots, 0) \in \boldsymbol{C}^n;\ (w_1, w_2) \in \boldsymbol{C}^2-\{(a_1, a_2)\}\}$ defined by

$$H_i=\{w=(w_1, w_2, 0, \cdots, 0) \in \boldsymbol{C}^n;\ w_i \neq a_i\} \qquad (i=1, 2).$$

Two cases may occur. In case that B has a non-zero eigen-value, we put

$$k(w_1, w_2)=\frac{1}{(w_1-a_1)(w_2-a_2)}$$

in $H_1 \cap H_2$. And, in case that all eigen-values of B are zero, we put

$$k(w_1, w_2)=\exp\left(\frac{1}{w_1-a_1}+\frac{1}{w_2-a_2}\right)$$

in $H_1 \cap H_2$. Then, in each case, k is a holomorphic function in $H_1 \cap H_2$. Therefore $\exp(k(w_1, w_2)X) \in H^0(H_1 \cap H_2, \mathcal{A}_L)$ defines an element of $Z^1(\mathfrak{U}, \mathcal{A}_L)$. By the above argument, there are $f_1 \in H^0(H_1 \cap D(\varepsilon), \mathcal{A}_L)$ and $f_2 \in H^0(H_2 \cap D(\varepsilon), \mathcal{A}_L)$ such that

$$\exp(k(w_1, w_2)X) = f_2 f_1^{-1}$$

ih $H_1 \cap H_2 \cap D(\varepsilon)$. We put

$$g_i = \mathrm{Ad}\, f_i$$

in $H_i \cap D(\varepsilon)$ $(i=1, 2)$. Then $g_1 \in H^0(H_1 \cap D(\varepsilon), \mathcal{A}_{GL(m,C)})$ and $g_2 \in H^0(H_2 \cap D(\varepsilon), \mathcal{A}_{GL(m,C)})$ satisfy

$$\exp(k(w_1, w_2)B) = g_2 g_1^{-1}$$

in $H_1 \cap H_2 \cap D(\varepsilon)$. Hence each element of the matrix g_2, $\det g_2$ and $1/\det g_2$ are holomorphic functions in $\{(w_1, w_2, 0, \cdots, 0) \in \boldsymbol{C}^n ; |w_1| < 1+\varepsilon, |w_2| < 1\} \cup \{(w_1, w_2, 0, \cdots, 0) \ni \boldsymbol{C}^n ; 1 < |w_1| < 1+\varepsilon, w_2 \neq a_2\}$. g_2 is continued to an element of $H^0(H_2 \cap E(\varepsilon), \mathcal{A}_{GL(m,C)})$. Hence

$$g_1 = \exp\Big(-\frac{B}{(w_1-a_1)(w_2-a_2)}\Big) g_2$$

is holomorphic in $\{(w_1, w_2, 0, \cdots, 0) \in \boldsymbol{C}^n ; |w_2| < 1+\varepsilon, 1 < |w_1| < 1+\varepsilon\} \cup \{(w_1, w_2, 0, \cdots, 0) \in \boldsymbol{C}^n ; |w_2| < 1+\varepsilon, w_2 \neq a_2, |w_1| < 1+\varepsilon, w_1 \neq a_2\}$. Hence g_1 is continued to an element of $H^0(H_1 \cap E(\varepsilon), \mathcal{A}_{GL(m,C)})$. Each g_i is continued analytically to an element of $H^0(H_i \cap E(\varepsilon), \mathcal{A}_{GL(m,C)})$. Since $\boldsymbol{C}^2 \times \{(0, 0, \cdots, 0)\} \cap E(\varepsilon)$ is an open neighborhood of $(a_1, a_2, 0, \cdots, 0)$ in $\boldsymbol{C}^2 \times \{(0, 0, \cdots, 0)\}$ and since $(H_1 \cap E(\varepsilon)) \cup (H_2 \cap E(\varepsilon)) = \boldsymbol{C}^2 \times \{(0, 0, \cdots, 0)\} \cap E(\varepsilon) - \{(a_1, a_2, 0, \cdots, 0)\}$, this contradicts to Lemma 2 or Lemma 3. Q. E. D.

§ 3. *L*-regular domain with smooth boundary.

An open subset G of a complex manifold M is said to have *smooth boundary* if for any point x^0 of the boundary ∂G of G in M there are a neighborhood V of x^0 in M and a real-valued differentiable function g in V such that

$$\partial G \cap V = \{x \in V ; g(x) = 0\}$$

and $\mathrm{grad}\, g \neq 0$ in V.

THEOREM. *Let L be a complex Lie group with positive dimension. Let D be an L-regular domain with smooth boundary in a Stein manifold S. Then D is a Stein manifold.*

Proof. Let s^0 be any boundary point of D in S. There are n holomorphic functions $z_1(s), z_2(s), \cdots, z_n(s)$ in S such that they form a local coordinate system in a neighborhood V of x^0 and that $z_i(s^0)=0$ $(i=1, 2, \cdots, n)$ where n is the dimension of S. For a sufficiently small ε, we put

$$U = \{s \in V ; |z_i(s)| < \varepsilon\}$$

and we may assume that there is a real-valued differentiable function g in variables $z_1, z_2, \cdots, z_{n-1}, y_n$ such that

$$\partial D \cap U = \{s \in V\,;\ x_n = g(z_1, z_2, \cdots, z_{n-1}, y_n)\}$$

where x_n and y_n are, respectively, the real and imaginary parts of z_n. It suffices to consider the case that

$$D \cap U = \{s \in V\,;\ x_n < g(z_1, z_2, \cdots, z_{n-1}, y_n)\}\,.$$

For $0 \leqq t < 1$ we put

$$E_t = \{s \in V\,;\ x_n < g(z_1, z_2, \cdots, z_{n-1}, y_n) - t\varepsilon/2\,,$$

$$|z_i| < (1-t)\varepsilon/2\ (i=1, 2, \cdots, n)\}\,.$$

Then E_t is an L-regular open set for $0 \leqq t < 1$ and E_0 is exhausted by them. Hence E_0 is a Stein manifold by Lemma 4. Therefore D is pseudoconvex in the sense of Cartan, that is, p_4-convex in the sense of Docquier-Grauert [4]. Therefore D is a Stein manifold by Docquier-Grauert [4].

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