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ON WSN FUNCTIONS AND A THEOREM OF R. NEVANLINNA CONCERNING BOUNDED ANALYTIC FUNCTIONS

Dedicated to Professor Yûsaku Komatu
On the occasion of his sixtieth birthday

BY MAURICE HEINS

§ **1. Introduction.** This paper gives (§§ 2, 3) a generalization of a well-known theorem of R. Nevanlinna [7, Theorem 7, page 48] to a simple class of functions of two complex variables which includes functions central in such important chapters of classical analysis as (a) the coefficient problem for bounded analytic functions treated by I. Schur [10], who introduced the algorithm named after him for this purpose, as (b) the interpolation problem for bounded analytic functions along with cognate boundary questions treated by R. Nevanlinna with the aid of an algorithm generalizing that of Schur in fundamental papers [5, 6, 7] culminating in the paper to which reference was made at the outset, as well as (c)—what is historically first—Weyl's theory of singular boundary problems [15], (d) the theory of Jacobi matrices, cf. [1, 12], and (e) the moment problem, cf. [1, 12]. To be exact, in the three last cited situations and [6] the subject functions enter modified by appropriate Möbius transformations.

We are concerned with functions f defined on $\Delta\times\hat{C}$, where Δ denotes the open unit disk of the complex plane and $\hat{C}$ the extended complex plane, which satisfy the following conditions: (i) $w\to f(z, w)$ is a constant of modulus at most one or else a non-degenerate Möbius transformation mapping $\bar{\Delta}$ into itself, $z\in\Delta$, but is not a constant for all $z\in\Delta$; (ii) $z\to f(z, w)$ is analytic on Δ, $w\in\Delta$. We term such functions *Weyl-Schur-Nevanlinna* ("WSN" for short) functions since in the work of these authors to which reference has been made an important role is played by instances of such functions (possibly modulo appropriate Möbius transformations).

The conditions (i) and (ii) taken together imply that the set $\mathscr{B}$ of $z\in\Delta$ for which $w\to f(z, w)$ is constant satisfies the Blaschke condition

$$\sum_{z\in\mathscr{B}}(1-|z|)<+\infty, \tag{1.1}$$

and that $|f(z, w)|<1$ for z, $w\in\Delta$.

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Actually, the conditions (i) and (ii) imply that f admits exactly one representation of the form

$$f(z,\ w)=\frac{a(z)w+b(z)}{c(z)w+1}\ , \tag{1.2}$$

where a, b, and c are analytic on $\varDelta$ and satisfy: $|a(z)|\leqq 1$, $|b(z)|<1$, $|c(z)|<1$ for all $z\in\varDelta$. [The introduction of a normalized representation of this kind in the Pick-Nevanlinna theory is due to Walsh [14, 298-304]. This assertion may be established as follows.

The uniqueness of $a(z)$, $b(z)$, $c(z)$ is clear for $z\in\varDelta\backslash\mathscr{B}$ since for such z we are concerned with non-degenerate Möbius transformations and so the coefficients are determined up to a nonzero common factor. The hypothesis of analyticity on a, b and c then implies that there is at most one such (a, b, c) satisfying (1.2).

Existence. Given w_1, $w_2(\neq w_1)\in\varDelta$, then $f(z, w_2)-f(z, w_1)=0$ if and only if $z\in\mathscr{B}$. Using this fact and the invariance of crossratio under nondegenerate Möbius transformations, we conclude on introducing three distinct points $w_k\in\varDelta$ and the associated $z\rightarrow f(z, w_k)$ that (1.2) holds for $z\in\varDelta\backslash\mathscr{B}$ with a, b, c analytic on $\varDelta\backslash\mathscr{B}$ (but as yet subject to no further condition beyond the nondegeneracy condition: $a(z)-b(z)c(z)\neq 0$, $z\in\varDelta\backslash\mathscr{B}$). It is to be observed that $w\rightarrow f(z, w)$ does not have a pole in $\varDelta$ when $z\in\varDelta\backslash\mathscr{B}$ by virtue of (i). Hence the normalization of the denominator is allowable.

We remark that if

$$T: w \longrightarrow \frac{\alpha w+\beta}{\gamma w+1}$$

is a nondegenerate Möbius transformation satisfying $T(\bar{\varDelta})\subset\bar{\varDelta}$, then $|\alpha|\leqq 1$, $|\beta|<1$, $|\gamma|<1$. Indeed, $|\beta|=|T(0)|<1$ by the conditions imposed on T thanks to the maximum principle. Let us consider γ. If $|\gamma|>1$, T would have a pole in $\varDelta$, while if $|\gamma|=1$, $T(-\bar{\gamma})=\infty$. Since neither conclusion is admissible, we see that $|\gamma|<1$. Now let S denote the Möbius transformation

$$w \longrightarrow \frac{\gamma w+1}{w+\bar{\gamma}}\ ,$$

which, we observe, maps the unit circumference onto itself. From the analyticity of the product TS at each point of $\{|w|\geqq 1\}$, we infer on applying the maximum principle that α, which is the value of the product TS at ∞, has modulus at most one.

We now conclude on applying the observations of the preceding paragraph] and the fact that $\mathscr{B}$ clusters at no point of $\varDelta$ that the functions a, b, c introduced two paragraphs back are all restrictions of functions analytic on $\varDelta$. For convenience we shall understand that a, b, c will denote the corresponding latter functions with domain $\varDelta$. It is now clear that $|a(z)|\leqq 1$, $|b(z)|<1$, $|c(z)|<1$ for $z\in\varDelta$ and that (1.2) is valid on $\varDelta\times\bar{\varDelta}$. Also $z\rightarrow f(z, w)$ is analytic on $\varDelta$ for each $w\in\bar{\varDelta}$.

In the opposite direction, we see that if f is given by (1.2) with a, b, c subject to the restrictions stated in the preceding paragraph, then at least f satisfies (ii).

We shall not need the fact in the present paper, but we note that a necessary and sufficient condition for a, b, c analytic on Δ to be associated with a WSN function f in the sense of (1.2) is that $|c(z)|<1$, that

$$|a(z)|^2+|b(z)|^2-|c(z)|^2-1+2|a(z)\overline{b(z)}-c(z)|\leqq 0\,, \tag{1.3}$$

for $z\in\Delta$, and that $a-bc$ not be the constant 0.

We also remark without proof that, if f is a WSN function with the representation (1.2), then at a point $z_0\in\mathscr{B}$, the Taylor expansion of

$$z \longrightarrow f[z, g(z)]\,, \tag{1.4}$$

g being analytic and taking values of modulus at most 1 on Δ, has coefficients independent of g for indices less than the multiplicity of $a-bc$ at z_0. An even stronger statement holds: *an inner function component (in the sense of Beurling) of the difference of two functions* (1.4) *formed with distinct g is up to a constant factor of modulus one the product of inner function components of $a-bc$ and the difference of the g.*

§ 2. **Nevanlinna's theorem concerning extremal functions.** Theorem 7 of Nevanlinna's 1929 paper [7, p. 48] asserts that the WSN function there introduced in connection with the indeterminate case of the Pick-Nevanlinna interpolation problem, which we shall denote f, has the property that $z\to f(z, w)$ has Fatou boundary values of modulus one p. p. for each w, $|w|=1$. The functions $z\to f(z, w)$ are called by Nevanlinna the *extremal functions* associated with the interpolation problem. The asserted boundary property of the extremal functions is the property termed (A) by Seidel [11] (referred to as "(U)" in the Japanese literature, cf. [13]) and the property of being *inner* by Beurling [2]. We note without proof that every analytic function on Δ taking values of modulus at most one which is of class (A) is an extremal function in the sense of Nevanlinna.

In the present paper we extend Nevanlinna's Theorem 7 to the setting of the theory of WSN functions. Before we formulate the extended theorem, we shall develop some simple properties of WSN functions. Given $z\in\Delta$ and a WSN function f we denote by $D(z, f)$ the image of $\bar{\Delta}$ with respect to $w\to f(z, w)$. For the f encountered in the Pick-Nevanlinna interpolation problem it is known that if two such f, say f_1 and f_2, satisfy $D(z, f_1)=D(z, f_2)$, $z\in\Delta$, then $f_2(z, w)=f_1[z, \alpha(w)]$ where α is a Möbius transformation independent of z mapping $\bar{\Delta}$ onto itself. cf. [7, §§ 29, 39]. The asserted property persists for unrestricted WSN functions as we shall see.

To that end, let f_1 and f_2 be WSN functions satisfying the stated hypothesis and for each z such that $D(z, f_1)$ does not reduce to a point let α_z be the unique

Möbius transformation satisfying

$$f_2(z, w)=f_1[z, \alpha_z(w)]. \tag{2.1}$$

The transformation α_z maps $\bar{\Delta}$ onto itself. We conclude using this observation and (2.1) that

$$\alpha_z(w)=\frac{A(z)w+B(z)}{C(z)w+1}, \tag{2.2}$$

where A, B, C are analytic on Δ, and $|A(z)|=1$ and $C(z)=A(z)\overline{B(z)}$ for $z\in\Delta$. It follows that A, B, C are constant. Indeed by the maximum principle for analytic functions A is constant. It is now clear that C is analytic and conjugate analytic. It follows that B and C are constant. The Möbius transformation α_z is independent of z and the asserted property of WSN functions follows. Of course, we are merely observing that the argument of Nevanlinna persists in the unrestricted WSN theory.

To continue, we introduce a notion of majorization in the class of WSN functions. Given WSN functions f and g, we say that g is *majorized* by f provided that

$$D(z, g)\subset D(z, f), \qquad z\in\Delta. \tag{2.3}$$

We see that g is majorized by f if and only if there exists a WSN function h satisfying

$$g(z, w)=f[z, h(z, w)], \qquad (z, w)\in\Delta\times\bar{\Delta}. \tag{2.4}$$

such h is necessarily unique.

In the Nevanlinna theory one is concerned with a sequence (f_n), $n=1, 2, \cdots$, of WSN functions, where f_{n+1} is majorized by f_n for all n. Suppose (Grenzkreis case) that $r_n(z)$, the radius of $D(z, f_n)$, does not tend to 0 as $n\to\infty$ for some $z\in\Delta$. In this case one is assured by suitable normalization of the f_n (Denjoy normalization cf. [3]) that (f_n) tends to a WSN function. The question was treated by Nevanlinna for the situation with which he was concerned via the theory of normal families. This approach was not in accord with the canons set by Weyl for himself in [16]. (v. pp. 242-3 ibid.) We take this occasion to show that one can treat the convergence of (f_n) in the presence of Denjoy normalization most simply and without the intervention of the theory of normal families.

As a first step, we note that if $|\zeta|<1$ and F is an analytic function on Δ taking values of modulus at most one, then with $L(z)=(\zeta-z)/(1-\bar{\zeta}z)$, we have

$$|F(z)-1|\leqq|F(\zeta)-1|\frac{1+|L(z)|}{1-|L(z)|}, \qquad z\in\Delta. \tag{2.5}$$

The inequality (2.5) is trivial when F is a constant of modulus 1.

Otherwise we have

$$\frac{F(\zeta)-F}{1-\overline{F(\zeta)}F}=\varphi L \tag{2.6}$$

where φ is an analytic function on $\varDelta$ taking values of modulus at most one (Lemma of Schwarz-Pick), and so

$$F(z)-1=\frac{[F(\zeta)-1]+[\overline{F(\zeta)}-1]\varphi(z)L(z)}{1-\overline{F(\zeta)}\varphi(z)L(z)}, \qquad z\in\varDelta, \tag{2.7}$$

whence we conclude (2.5).

We let $|w|=1$ and take as F the function $z\to f(z, w)/w$, where f is a WSN function. Applying (2.5) we obtain with $\mu=\max_{|w|=1}|f(\zeta, w)-w|$ the inequalities

$$|a(z)-1|, \quad |b(z)|, |c(z)|\leqq 2\mu\frac{1+|L(z)|}{1-|L(z)|}, \quad z\in\varDelta, \tag{2.8}$$

where a, b, c refer to the representation (1.2) of f. Indeed, using (2.5) as indicated, (1.2), and the inequality $|c(z)|<1$, $z\in\varDelta$, we obtain

$$|-c(z)w^2+[a(z)-1]w+b(z)|\leqq 2\mu\frac{1+|L(z)|}{1-|L(z)|}, \quad z\in\varDelta, \ |w|=1, \tag{2.9}$$

and (2.8) by the Cauchy inequalities.

A WSN function f is said to be *Denjoy normalized* at $\zeta\in\varDelta\backslash E$ provided that $c(\zeta)=0$, $a(\zeta)>0$. Given a WSN function f and $\zeta\in\varDelta\backslash E$, there exists a unique Möbius transformation τ mapping $\bar{\varDelta}$ onto itself such that $f[z, \tau(w)]$ is Denjoy normalized at ζ.

We return to the situation considered four paragraphs back and suppose that $\lim r_n(\zeta)>0$ and that each f_n is Denjoy normalized at ζ. We affect the coefficients a, b, c entering the representation (1.2) of f_n with the subscript "n". Given $m<n$, we have

$$f_n(z, w)=f_m[z, \varphi_{mn}(z, w)], \tag{2.10}$$

for exactly one WSN function φ_{mn}. cf. (2.4). Further φ_{mn} is Denjoy normalized at ζ. We denote the functions a, b, c entering the representation (2.1) of φ_{mn} by $\alpha_{mn}, \beta_{mn}, \gamma_{mn}$ respectively, and note that $\lim\limits_{m,n\to\infty}\alpha_{mn}(\zeta)=1$, and that $\gamma_{mn}(\zeta)=0$ for all allowed m, n. Since $w\to\varphi_{mn}(\zeta, w)$ maps $\bar{\varDelta}$ into itself, we have $\alpha_{mn}(\zeta)+|\beta_{mn}(\zeta)|\leqq 1$, whence we see that $\lim\limits_{m,n\to\infty}\beta_{mn}(\zeta)=0$. We conclude that

$$\lim_{m,n\to\infty}\max_{|w|=1}|\varphi_{mn}(\zeta, w)-w|=0. \tag{2.11}$$

On representing a_n, b_n, c_n in terms of a_m, b_m, c_m and $\alpha_{mn}, \beta_{mn}, \gamma_{mn}$ and using the inequalities (2.8) relative to φ_{mn}, we conclude that the sequences (a_n), (b_n), (c_n) are uniformly Cauchy on compact subsets of $\varDelta$. The respective limit functions of the three sequences, say A, B, C, are analytic on $\varDelta$ and take values of modulus at most one. Since $c_n(\zeta)=0$, $n=1, 2, \cdots$, we have $|C(z)|<1$ for $z\in\varDelta$. Since

$a_n(\zeta)+|b_n(\zeta)|\leqq 1$, $n=1, 2, \cdots$, we have $\sup|b_n(\zeta)|<1$ and hence $|B(z)|<1$, $z\in\Delta$. We conclude that f defined by

$$f(z, w)=\frac{A(z)w+B(z)}{C(z)w+1} \tag{2.12}$$

is a WSN function and $f_n\to f$ uniformly on compact subsets of $\Delta\times\bar{\Delta}$. One verifies that $D(z, f)$ is the limit of the monotone sequence $(D(z, f_n))$ and reduces to a point if and only if $D(z, f_n)$ does for some finite index n.

The following result may now be concluded: *Let (f_n) be a sequence of WSN functions satisfying:* (1) *f_n majorizes f_{n+1}, all n,* (2) *(f_n) is in the Grenzkreis case,* (3) *(f_n) converges uniformly to f on compact subsets of $\Delta\times\bar{\Delta}$. Further, let $g_n(z, w)=f_n[z, \beta_n(w)]$ where β_n is the unique Möbius transformation mapping $\bar{\Delta}$ onto itself such that g_n is Denjoy normalized at a point ζ satisfying* $\inf r_n(\zeta)>0$, *$r_n(\zeta)$ being taken relative to f_n. Let $g=\lim_{n\to\infty} g_n$. Then*

$$f(z, w)=g[z, \alpha(w)]\,, \qquad (z, w)\in\Delta\times\bar{\Delta}\,, \tag{2.13}$$

where α is either a constant of modulus one or a Möbius transformation mapping $\bar{\Delta}$ onto itself.

Nevanlinna's Theorem 7, quoted at the beginning of this section treats a situation where f is the limit of a sequence of WSN functions (f_n), f_n having a representation (1.2) with a, b, c rational functions, $w\to f_n(z, w)$ mapping $\bar{\Delta}$ onto itself for each z satisfying $|z|=1$, and f_n majorizing f_{n+1}, $n=1, 2, \cdots$. The stated properties of the f_n arise out of their genesis by a generalized Schur algorithm. The following theorem extends Nevanlinna's theorem to the general theory of WSN functions.

THEOREM. *Let (f_n) be a sequence of WSN functions. We suppose that (f_n) tends to the WSN function f uniformly on compact subsets of $\Delta\times\bar{\Delta}$ (here, $\bar{\Delta}$ may be replaced by a subset of three distinct points) and that each f_n majorizes f,* $n=1, 2, \cdots$. *We suppose further that there exists a Lebesque measurable subset E of the unit circumference,* meas. $E>0$, *such that for each n the a, b, c entering the representation* (1.2) *of f_n, say a_n, b_n, c_n have Fatou radial limits existing at $z\in E$ and*

$$w \longrightarrow \frac{a_n(z)w+b_n(z)}{c_n(z)w+1} \tag{2.14}$$

maps $\bar{\Delta}$ onto itself, for each $z\in E$. [Here $a_n(z)$, $b_n(z)$, $c_n(z)$ will by convention denote the Fatou radial limits of a_n, b_n, c_n respectively at z and the corresponding convention will be understood throughout.] Then for almost all $z\in E$

$$w \longrightarrow \frac{a(z)w+b(z)}{c(z)w+1} \tag{2.15}$$

maps $\bar{\Delta}$ onto itself, a, b, c referring to the representation (1.2) *of f. In particular, there is a subset E_1 of E having zero measure such that $z\to f(z, w)$ has Fatou* *radial limits of modulus one on $E\backslash E_1$ for all w, $|w|=1$.*

The Fatou radial limit function of a_n (resp. b_n, c_n) converges in measure on E to the Fatou radial limit function of a (resp. b, c).

The proof of this theorem will be given in § 3. It has two aspects. One looks to the proof given by Nevanlinna of his cited theorem in its appeal to the skilful use of hyperbolic distance made by him. The other aspect is based on the observation that $\log|a_n-b_nc_n|$ and $-\log(1-|c_n|^2)$ are both *subharmonic*. This observation and the formula

$$r_n(z)=\frac{|a_n(z)-b_n(z)c_n(z)|}{1-|c_n(z)|^2} \tag{2.16}$$

for the radius of $D(z, f_n)$ yield useful information about c_n thanks to the fact that $r_n(z)\leqq 1$ for $z\in\Delta$. [For (2.16) cf. [7], p. 40, (23).] The normalized form of a WSN function which we are using plays a simplifying role.

The convergence in measure properties of the a_n, b_n, c_n will be seen to follow from their convergence properties in Δ taken together with a standard L_2 argument which establishes strong convergence in the presence of weak convergence and convergence of the norm.

The WSN functions merit further study. A question of interest is that of the pointwise convergence of $a_n(z)$, $b_n(z)$, $c_n(z)$, $z\in E$. A very special case of this question is the one corresponding to interpolation to zero on a set satisfying the Blaschke condition. It is essentially the open question concerning the convergence set on the unit circumference of the sections of a convergent Blaschke product.

§ 3. **Proof of the Theorem.** Since (2.14) maps $\bar{\Delta}$ onto itself for $z\in E$ we have

$$|a_n(z)|=1\,, \qquad c_n(z)=a_n(z)\overline{b_n(z)}\,, \qquad z\in E,\ n=1, 2, \cdots, \tag{3.1}$$

and hence

$$|c_n(z)|=|b_n(z)| \tag{3.2}$$

for the same z and n. Further, from formula (2.16) and the fact that $r_n(z)\leqq 1$ for $z\in\Delta$, $n=1, 2, \cdots$, we see that

$$\log\frac{1}{1-|c_n(z)|^2}\leqq\log\frac{1}{|a_n(z)-b_n(z)c_n(z)|}\,, \qquad z\in\Delta,\ n=1, 2, \cdots. \tag{3.3}$$

Now the left side of (3.3) is *subharmonic* and the right side is *superharmonic* on Δ. Since $a(z)-b(z)c(z)\neq 0$ for some $z\in\Delta$, say ζ, and, in addition, f_n majorizes f, all n, and $a_n(\zeta)\to a(\zeta)$, $b_n(\zeta)\to b(\zeta)$, $c_n(\zeta)\to c(\zeta)$, it follows that

$$\{|a_n(\zeta)-b_n(\zeta)c_n(\zeta)|\}$$

has a positive lower bound. Since the value at ζ of the least harmonic majorant of the left side of (3.3) does not exceed the value at ζ of the right side of (3.3), it follows from the classical Harnack inequalities for positive harmonic functions on Δ and the subharmonicity of

$$\log\frac{1}{1-|c_n|^2}$$

that there exists a positive number M such that

$$\int_0^{2\pi}\log\frac{1}{1-|c_n(re^{i\theta})|^2}\,d\theta\leqq M, \qquad 0\leqq r<1,\ n=1, 2, \cdots. \tag{3.4}$$

Using the Fatou radial limit theorem and Fatou's lemma, we conclude that

$$\int_0^{2\pi}\log\frac{1}{1-|c_n(e^{i\theta})|^2}\,d\theta\leqq M, \tag{3.5}$$

and, a fortiori,

$$\int_E\log\frac{1}{1-|c_n(z)|^2}\,d\theta\leqq M, \tag{3.6}$$

whence by (3.2) we have

$$\int_E\log\frac{1}{1-|b_n(z)|^2}\,d\theta\leqq M, \tag{3.7}$$

the inequalities holding for all n. In the last two integrals $d\theta$ is construed in the sense of Lebesgue measure on the unit circumference.

We now let m denote a positive number less than one, and let $e(m, n)$ denote the subset of E on which $|b_n(z)|\leqq m$.

From (3.7) we conclude that

$$\left(\log\frac{1}{1-m^2}\right)[\text{meas. } E-\text{meas. } e(m, n)]\leqq M \tag{3.8}$$

and consequently

$$\text{meas. } e(m, n)\geqq\text{meas. } E-M\left(\log\frac{1}{1-m^2}\right)^{-1}. \tag{3.9}$$

The proof continues along the lines of Nevanlinna's argument using the hyperbolic metric in Δ. Given $u, v\in\Delta$, we recall that $d[u, v]$, the hyperbolic distance from u to v, is given by

$$d[u, v]=\frac{1}{2}\log\frac{1+|(v-u)/(1-\bar{u}v)|}{1-|(v-u)/(1-\bar{u}v)|}, \tag{3.10}$$

and that d satisfies the triangle inequality and is invariant under conformal automorphisms of Δ.

We fix ω, $|\omega|=1$, and A, $0<A<1$. Let $\mathcal{E}$ denote the subset of E on which $|f(z, \omega)|\leqq A$. [With g a WSN function and $|z|=1$, we understand $g(z, \omega)$ as the Fatou radial limit $\lim\limits_{t\to 1} g(tz, \omega)$.] We let φ_n denote the unique WSN function satisfying

$$f_n[z, \varphi_n(z, w)]=f(z, w), \qquad (z, w)\in\Delta\times\bar{\Delta}. \tag{3.11}$$

We have

$$f_n[z, \varphi_n(z, \omega)]=f(z, \omega), \qquad z\in\mathcal{E}, \tag{3.12}$$

as we see with the aid of the hypothesis on E, and also have $|\varphi_n(z, \omega)|<1$ for $z\in\mathcal{E}$. Thanks to (3.12) and the invariance property of d we have

$$d[f(z, \omega), f_n(z, 0)]\leqq d[\varphi_n(z, \omega), 0], \qquad z\in\mathcal{E}. \tag{3.13}$$

Using the triangle inequality for d we obtain

$$\begin{aligned} d[f(z, \omega), f_n(z, 0)] &\leqq d[f(z, \omega), 0]+d[0, f_n(z, 0)] \\ &= d[f(z, \omega), 0]+d[0, b_n(z)] \\ &\leqq d[A, 0]+d[0, b_n(z)], \qquad z\in\mathcal{E}. \end{aligned} \tag{3.14}$$

By (3.13), (3.14) and the definition of $e(m, n)$ we conclude that given m there exists C, $0<C<1$, such that

$$|\varphi_n(z, \omega)|\leqq C, \qquad z\in\mathcal{E}\cap e(m, n), \qquad n=1, 2, \cdots. \tag{3.15}$$

From (3.15) and

$$\begin{aligned} \log|\varphi_n(0, \omega)| &\leqq \frac{1}{2\pi}\int_0^{2\pi}\log|\varphi_n(e^{i\theta}, \omega)|\,d\theta \\ &\leqq \frac{1}{2\pi}\int_{\mathcal{E}\cap e(m,n)}\log|\varphi_n(z, \omega)|\,d\theta, \end{aligned} \tag{3.16}$$

we obtain

$$\log|\varphi_n(0, \omega)|\leqq\frac{1}{2\pi}(\log C)\,\text{meas.}\,[\mathcal{E}\cap e(m, n)]. \tag{3.17}$$

Now for some $\zeta\in\Delta$, $\varphi_n(\zeta, \omega)\to\omega$, thanks to the hypothesis of the theorem. Hence by (2.5) we conclude that $\varphi_n(0, \omega)\to\omega$, and thereupon with the aid of (3.17) that

$$\lim_{n\to\infty}\text{meas.}\,[\mathcal{E}\cap e(m, n)]=0. \tag{3.18}$$

Since

$$e(m, n)\subset(E\backslash\mathcal{E})\cup[\mathcal{E}\cap e(m, n)], \tag{3.19}$$

we obtain

$$\limsup_{n\to\infty}\text{meas.}\,e(m, n)\leqq\text{meas.}\,(E\backslash\mathcal{E}). \tag{3.20}$$

Using (3.20) and (3.9) we obtain

$$\text{meas.}\,(E\backslash\mathcal{E})\geqq\text{meas.}\,E-M\left(\log\frac{1}{1-m^2}\right)^{-1}, \tag{3.21}$$

whence we conclude on letting $m\to1$ that meas. $(E\backslash\mathcal{E})\geqq$ meas. E and thereupon that meas. $\mathcal{E}=0$. It now follows that $|f(z,\omega)|=1$ p.p. on E.

We now wish to show that $w\to f(z,w)$, $|w|\leqq1$, is the restriction of a Möbius transformation mapping $\bar{\Delta}$ onto itself for almost all $z\in E$. To that end, we start by fixing w_1, w_2, w_3, distinct points on the unit circumference, and noting that for given j, $k(\neq j)$ we have

$$f(z,w_j)\neq f(z,w_k) \tag{3.22}$$

for almost all $z\in E$. Otherwise, by the uniqueness theorem of F. and M. Riesz, $f(z,w_j)=f(z,w_k)$ for all $z\in\Delta$ and f would not be a WSN function. Hence $f(z,w_j)$, $j=1,2,3$, are distinct and of modulus one for almost all $z\in E$.

For each non-exceptional $z\in E$ let L_z denote the unique Möbius transformation satisfying $L_z(w_j)=f(z,w_j)$, $j=1,2,3$. Thanks to the facts that the $f(z,w_j)$ are distinct and that $L_z(w_j)=\lim_{r\to1}f(rz,w_j)$, $j=1,2,3$, we shall conclude that a, b, c have Fatou radial limits at z and that $a(z)-b(z)c(z)\neq0$ and $|c(z)|<1$. For this purpose we note that the determinant with rows

$$(w_jf(z,w_j),\ w_j,\ 1)\,,\qquad j=1,2,3\,, \tag{3.23}$$

does not vanish. Otherwise, we should conclude that L_z has a pole at 0, and hence, by the cross ratio formula for a Möbius transformation and a continuity argument, that $w\to f(rz,w)$ would have a pole in Δ for r near 1. This is not possible. Since the determinant in question does not vanish, we conclude by a continuity argument that $\lim_{r\to1}a(rz)$, $\lim_{r\to1}b(rz)$, $\lim_{r\to1}c(rz)$ exist. At least two of the w_j are distinct from $-1/c(z)$. For them we have

$$\frac{a(z)w_j+b(z)}{c(z)w_j+1}=f(z,w_j)\,. \tag{3.24}$$

Since the $f(z,w_j)$ are distinct for j distinct, we see that $a(z)-b(z)c(z)\neq0$. If $|c(z)|=1$, then $\lim_{r\to1}f[rz,-\overline{c(z)}]=\infty$. This conclusion being false, we see that $|c(z)|<1$. Using these facts we infer that

$$w\longrightarrow\frac{a(z)w+b(z)}{c(z)w+1} \tag{3.25}$$

is the Möbius transformation L_z and that

$$\lim_{r\to1}f(rz,w)=L_z(w)\,,\qquad |w|\leqq1\,. \tag{3.26}$$

By its construction L_z maps the unit circumference onto itself and $L_z(\bar{\Delta})\subset\bar{\Delta}$ by (3.26). Hence

$$L_z(\bar{\varDelta})=\bar{\varDelta}\,. \tag{3.7}$$

The first assertion of the Theorem is thereby established. From (3.27) we obtain the relation

$$a(z)\overline{b(z)}=c(z)\,. \tag{3.28}$$

Remark 1. Let ζ be a point of the unit circumference. Let f be an arbitary WSN function. Let M_r denote the map

$$w \longrightarrow f(r\zeta, w)\,, \qquad w\in\varDelta\,, \tag{3.29}$$

r satisfying $0\leqq r<1$. Suppose that $\lim_{r\to 1} M_r$ is a conformal automorphism of $\varDelta$. Using a determinant argument of the kind introduced above we conclude that $\lim_{r\to 1} a(r\zeta)$, $\lim_{r\to 1} b(r\zeta)$, $\lim_{r\to 1} c(r\zeta)$ all exist and are equal to the respective coefficients of the normalized representation $(Aw+B)/(Cw+1)$ of $\lim_{r\to 1} M_r$. If, in addition, g is an analytic function on $\varDelta$ taking values of modulus at most one and g has a Fatou radial limit at ζ, then $\lim_{r\to 1} f[r\zeta, g(r\zeta)]$ exists and is equal to

$$\frac{Ag(\zeta)+B}{Cg(\zeta)+1}\,. \tag{3.30}$$

This observation leads to the conclusion that *if f is a WSN function associated with the Grenzkreis case of the Pick-Nevanlinna interpolation problem and g is a function of class* (A) *in the sense of Seidel* (*equivalently, if g is an inner function*), *then $z\to f[z, g(z)]$, $z\in\varDelta$, also has the property* (A).

Remark 2. (concerning the proof of the first assertion of the Theorem). Since a, b, c take values of modulus at most 1, it is clear that one may show that $a(z)-b(z)c(z)\neq 0$ p.p. on E with the aid of the uniqueness theorem of F. and M. Riesz and thereupon that $|c(z)|<1$ p.p. on E.

Remark 3. The requirements on a_n, b_n, c_n of the third sentence of the Theorem may be replaced by the equivalent one that $w\to f_n(rz, w)$ tends on $\varDelta$ to a conformal automorphism of $\varDelta$.

We now turn to the proof of the second assertion of the Theorem. In any case, by virtue of the convergence hypothesis on (f_n) we see with the aid of the above determinant argument and Stieltjes' convergence theorem for sequences of uniformly bounded analytic functions that $a_n\to a$, $b_n\to b$, $c_n\to c$ uniformly on compact subsets of $\varDelta$. Since the analytic functions a_n, b_n, c_n, a, b, c all take values of modulus at most 1, they are all representable as the Poisson integrals of their Fatou boundary functions. Thus

$$a(z)=\frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}d\theta\,, \qquad z\in\varDelta\,, \tag{3.31}$$

corresponding representations holding for the remaining functions in question.

Since the functions with domain R given by

$$\theta \longrightarrow \frac{1-|z|^2}{|e^{i\theta}-z|^2}, \qquad z\in\Delta, \tag{3.32}$$

span a dense linear subset of the space of complex-valued continuous functions on R having period 2π, their restrictions to $[0, 2\pi]$ span a dense linear subspace of $L_2[0, 2\pi]$ considered over C. Thanks to this observation, the pointwise convergence of (a_n) to a on Δ, and the fact that $|a_n|, |a|\leqq 1$, we conclude that the Fatou boundary function of a_n *converges to that of* a *weakly* in the sense of L_2. Consequently, the corresponding statement holds for the "restrictions" of the boundary functions of a_n and a to E, the weak convergence being understood in the sense of $L_2(E)$. Since $|a_n(z)|=1$, $z\in E$, $n=1, 2, \cdots$, and $|a(z)|=1$ p.p. on E, on using the classical device of considering

$$\int_E |a_n-a|^2 d\theta = 2\int_E [1-\Re e(a_n\bar{a})]d\theta, \tag{3.33}$$

we see that $a_n\to a$ in the mean of order 2 on E. Since the boundary functions of a_n and a are uniformly bounded and meas. $E<+\infty$, we obtain the *equivalent* conclusion that $a_n\to a$ *in measure on* E.

To complete the proof, we note that for w satisfying $|w|=1$ we have $|f_n(z, w)|=1$, $z\in E$. By the first assertion of the Theorem and the convergence of (f_n) to f on $\Delta\times\bar{\Delta}$, we see by the argument of the preceding paragraph that

$$\lim_{n\to\infty}\int_E |f_n(z, w)-f(z, w)|^2 d\theta=0. \tag{3.34}$$

On representing $f_n(z, w)$ and $f(z, w)$ explicitly with the aid of (2.14) and (2.15) respectively and noting that $|c_n(z)w+1|, |c(z)w+1|\leqq 2$ for almost all $z\in E$, we conclude that

$$\lim_{n\to\infty}\int_E |P_{n,w}(z)|^2 d\theta=0, \tag{3.35}$$

where

$$P_{n,w}(z)=[f_n(z, w)-f(z, w)][c_n(z)w+1][c(z)w+1]. \tag{3.36}$$

On examining the expansion of $P_{n,w}(z)$ as a quadratic polynomial in w, we see taking three distinct values of w, each of modulus 1, say w_1, w_2, w_3, that $b_n(z)-b(z)$ is a linear combination of the $P_{n,w_k}(z)$ with coefficients independent of z and n. We conclude that

$$\lim_{n\to\infty}\int_E |b_n(z)-b(z)|^2 d\theta=0. \tag{3.37}$$

Since

$$\begin{aligned} c_n(z)-c(z) &= a_n(z)\overline{b_n(z)}-a(z)\overline{b(z)} \\ &= [a_n(z)-a(z)]\overline{b_n(z)}+a(z)\overline{[b_n(z)-b(z)]} \end{aligned} \tag{3.38}$$

holds p.p. on E and a and b_n are bounded in modulus by 1, we conclude, using the mean convergence of (a_n) to a and (b_n) to b on E, that

$$\lim_{n\to\infty}\int_E |c_n(z)-c(z)|^2 d\theta=0. \tag{3.39}$$

The second assertion of the Theorem follows. The proof of the Theorem is thereby completed.

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