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KŌDAI MATH. SEM. REP
26 (1975), 240—244

REMARKS ON THE SCALAR CURVATURE OF IMMERSED MANIFOLDS

Dedicated to Prof. S. Ishihara on his 50-th birthday

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§1. Introduction.

For surfaces in a $(2+N)$-dimensional Euclidean space E^{2+N}, T. Ōtsuki [11] has introduced some kinds of curvature and then B. Y. Chen [2] has proved the following theorem:

THEOREM A. *Let $x: M^2 \to E^{2+N}$ be an immersion of a closed surface M^2 in a $(2+N)$-dimensional Euclidean space E^{2+N}. Then*

(I) *The last curvature $\lambda_N \geqq 0$ if and only if M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}, and*

(II) *The first curvature $\lambda_1 = \alpha$ (constant) and the last curvature $\lambda_N = 0$ $(N \geqq 2)$ if and only if M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N} with radius $1/\sqrt{\alpha}$.*

On the other hand, B. Y. Chen has considered the notion of α-th scalar curvature: $\lambda_1, \lambda_2, \cdots, \lambda_N$ and find the relationship between the scalar curvature R and them for an n-dimensional Riemannian manifold isometrically immersed in a Euclidean space E^{n+N}. And he has proved the following [3]:

THEOREM B. *Let M^n $(n \geqq 3)$ be an n-dimensional closed manifold in E^{n+N}. Then*

$$\int_{M^n} (\lambda_1)^{n/2} dV = C_n \qquad \text{and} \qquad \lambda_2 = \lambda_3 = \cdots = \lambda_N = 0$$

if and only if M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+N}, where dV means the volume element of M^n and C_n the area of the unit spere.

The purpose of this note is to show the following:

THEOREM. *Let $x: M^n \to E^{n+N}$ be an immersion of a closed manifold with N-th*

Received Sept. 6, 1973.

*) Manifolds, mappings, metrics,⋯, etc. are assumed to be differentiable and of class C^∞.

scalar curvature $\lambda_N \geqq 0$ in an $(n+N)$-dimensional Euclidean space. Then we have

$$\int_{M^n} (R/n(n-1))^{n/2} dV \geqq C_n , \tag{1}$$

where R is the scalar curvature of M^n. The equality sign of (1) holds when and only when M^n is imbedded as a hyperesphere in an $(n+1)$-dimensional linear subspace of E^{n+N}, or, when and only when the dimension of M^n is 2 and M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}.

§ 2. Preliminaries.

Let M^n be an n-dimensional closed manifold with an immersion $x: M^n \to E^{n+N}$. Let $F(M^n)$ and $F(E^{n+N})$ be the bundles of orthonormal frames of M^n and E^{n+N}, respectively. Let B be the set of elements $b=(p, e_1, e_2, \cdots, e_{n+N})$ such that $(p, e_1, e_2, \cdots, e_n) \in F(M^n)$ and $(x(p), e_1, e_2, \cdots, e_{n+N}) \in F(E^{n+N})$ whose orientation is coherent with the one of E^{n+N}, identifying e_i with $dx(e_i)$ $(i, j, k, \cdots = 1, 2, \cdots, n)$. Then $B \to M^n$ may be considered as a principal bundle with fibre $O(n) \times SO(N)$, and $\tilde{x}: B \to F(E^{n+N})$ is naturally defined by $\tilde{x}(b)=(x(p), e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+N})$. Let B_ν be the bundle of unit normal vector of $x(M^n)$ so that a point of B_ν is a pair (p, e) where e is a unit normal vector at $x(p)$.

The structure equations of E^{n+N} are given by

$$\begin{gathered} dx=\sum_A \theta_A e_A , \qquad de_A=\sum_B \theta_{AB} e_B , \qquad \theta_{AB}+\theta_{BA}=0 , \\ d\theta_A=\sum_B \theta_B \wedge \theta_{BA} , \qquad d\theta_{AB}=\sum_C \theta_{AC} \wedge \theta_{CB} , \\ (A, B, C, \cdots = 1, 2, \cdots, n+N) , \end{gathered} \tag{2}$$

where θ_A and θ_{AB} are differential 1-forms on $F(E^{n+N})$. Let ω_A and ω_{AB} be the induced 1-forms on B respectively from θ_A and θ_{AB} by the mapping $\tilde{x}$. Then we have

$$\begin{gathered} \omega_r=0 , \qquad \omega_{ri}=\sum_j A_{rij}\omega_j , \qquad A_{rij}=A_{rji} , \\ (i, j, \cdots = 1, 2, \cdots, n : r, s, t, \cdots = n+1, \cdots, n+N) . \end{gathered} \tag{3}$$

The symmetric matrix (A_{rij}) is called the second fundamental form at (p, e_r). We define the k-th mean curvature $K_k(p, e_r)$ at $(p, e_r) \in B_\nu$ by

$$\det(\delta_{ij}+tA_{rij})=1+\sum \binom{n}{k} K_k(p, e_r) t^k .$$

Using (2), we get

$$\begin{gathered} d\omega_j=\sum_k \omega_k \wedge \omega_{kj} , \\ d\omega_{ij}=\sum_k \omega_{ik} \wedge \omega_{kj} + (1/2) \sum_{k,h} R_{ijkh} \omega_k \wedge \omega_h , \end{gathered} \tag{4}$$

$$R_{ijkh}=\sum_r A_{rih}A_{rjk}-\sum_r A_{rik}A_{rjh}\,.$$

The volume element of M^n can be written as $dV=\omega_1\wedge\cdots\wedge\omega_n$. The $(N-1)$-form $d\sigma_{N-1}=\omega_{n+N\,n+1}\wedge\cdots\wedge\omega_{n+N\,n+N-1}$ can be regard as an $(N-1)$-form on B_ν. The $(n+N-1)$-form $d\sigma_{N-1}\wedge dV$ can be regarded as the volume element of B_ν. The integral $K_i^*(p)=\int|K_i(p, e)|^{n/i}d\sigma_{N-1}$ over the sphere of unit normal vectors at $x(p)$ is called the i-th total absolute curvature of the immersion x at p, and the integral $\int_{M^n}K_i^*(p)dV$ is called the i-th total absolute curvature of M^n.

The following theorem is well known [6, 7]:

THEOREM C. *Let $x: M^n\to E^{n+N}$ be an immersion of an n-dimensional closed manifold M^n into E^{n+N}. Then we have*

$$\int_{M^n}K_i^*(p)dV\geqq 2C_{n+N-1}\,,\qquad (i=1, 2, \cdots, n)\,. \tag{5}$$

The equality sifn of (5) *holds if and only if M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+N} if $i<n$, and as a convex hypersurface in an $(n+1)$-dimensional linear subspace of E^{n+N} if $i=n$.*

For each unit normal vector $e=\sum_{r=n+1}^{n+N}\cos\beta_r e_r$, the 2nd mean curvature $K_2(p, e)$ is given by

$$\binom{n}{2}K_2(p, e)=\sum_{i<j}[(\sum_r\cos\beta_r A_{rii})(\sum_s\cos\beta_s A_{sjj})-(\sum_t\cos\beta_t A_{tij})^2]\,,$$

in which the right hand side is a quadratic form of $\cos\beta_{n+1}, \cdots, \cos\beta_{n+N}$. Hence, by choosing a suitable cross-section, we can write $K_2(p, e)$ as

$$K_2(p, e)=\sum_{r=n+1}^{n+N}\lambda_{r-n}\cos^2\beta_r,\quad \lambda_1\geqq\cdots\geqq\lambda_N\,. \tag{6}$$

This local cross-section of $B\to F(E^{n+N})$ is called a Frenet frame. λ_α $(\alpha=1, 2, \cdots, N)$ is called the α-th scalar curvature of M^n in E^{n+N} [3]. We know that the scalar curvature R of M^n satisfies

$$R/n(n-1)=\sum_{\alpha=1}^{N}\lambda_\alpha \tag{7}$$

with respect to the Frenet frame.

§ 3. Lemma.

To prove the Theorem, we shall prove the following lemma:

LEMMA. *Let $a_1, \cdots, a_N$ be N non-negative constants and S^{N-1} be the unit hypersphere of E^N centered at the origin $0=(0, \cdots, 0)$. Let F be the function on*

S^{N-1} *defined by* $F(x)=\sum_{i=1}^{N} a_i x_i^2$, *where* $x=(x_1, \cdots, x_N)$. *For a positive integer* $2d$ *we have*

$$(8)\qquad (\sum_{i=1}^{N} a_i)^d \geqq (C_{2d}/2C_{N+2d-1})\int_{S^{N-1}} (\sum_{i=1}^{N} a_i x_i^2)^d dS^{N-1},$$

where dS^{N-1} *is the volume element of* S^{N-1}. *The equality sign of* (8) *holds when and only when we have either at least* $N-1$ *of* $a_1, \cdots, a_N$ *are zero or* $d=1$.

Proof. For non-negative integer e, we get

$$(9)\qquad \int_{S^{N-1}} |x_i|^e dS^{N-1} = [2\Gamma((1+e)/2)\Gamma(1/2)^{N-1}]/\Gamma((N+e)/2).$$

Taking account of Minkowski's inequality and (9), we have

$$\Big[\int_{S^{N-1}} (\sum a_i x_i^2)^d dS^{N-1}\Big]^{1/d} \leqq \sum_{i=1}^{N} a_i\Big(\int_{S^{N-1}} |x_i|^{2d} dS^{N-1}\Big)^{1/d}$$
$$= \sum a_i [(2C_{N+2d-1})/C_{2d}]^{1/d},$$

which means inequality (8). Moreover, by the property of Minkowski's inequality, we find that the sign of equality holds in (8) if and only if at least $N-1$ of $a_1, \cdots, a_N$ are zero or $d=1$. This completes the proof of Lemma.

Remark. B. Y. Chen [4, 8] has proved this lemma, $2d$ being positive even integer. When $2d$ is positive odd integer, by virtue of this lemma and Schwarz's inequality, we have

$$[(C_{N+4d-3}C_{N+1})/\pi C_{4d-1}]^{1/2} (\sum a_i)^d \geqq \int_{S^{N-1}} (\sum a_i x_i^2)^d dS^{N-1}.$$

The equality sign of this holds if and only if at least $N-1$ of $a_1, \cdots, a_N$ are zero.

§ 4. The proof of Theorem.

Now, let us prove the Theorem stated in § 1. By assumption $\lambda_N \geqq 0$, we may use the Lemma. Taking account of (7)∼(9), we obtain

$$[R/n(n-1)]^{n/2} \geqq (C_n/2C_{n+N-1})\int |K_2(p, e)|^{n/2} d\sigma_{N-1}$$
$$= (C_n/2C_{n+N-1}) K_2^*(p).$$

Accordingly, from Theorem C, we get the inequality (1). If the equality sign of (1) holds, then we see by Lemma that either at least $N-1$ of $a_1, \cdots, a_N$ are rero or that $n=2$. If $a_1, \cdots, a_N$ are all zero, then $K_2(p, e)=0$. But this case does not occur as a consequence of Theorem C. Therefore, making use of Theorem C, we can find that M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+N}, or that the dimension of M^n is 2 and M^2 is imbedded as a convex surface in a 3-dimensional linear subspace of E^{2+N}. The converse of this is trivial by virtue of Theorem A and B.

The author wishes to express his sincere thanks to Prof. S. Tachibana and Prof. Y. Ogawa who gave criticisms and advices.

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