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MORSE FUNCTIONS ON SOME ALGEBRAIC VARIETIES

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1. Introduction.

For any n-tuple of integers $a=(a_1, \cdots, a_n)$ $(a_i \geqq 2)$ Brieskorn variety $\Sigma(a)$ is, by definition, a real algebraic variety given by the following equations in $z=(z_1, \cdots, z_n) \in \mathbf{C}^n$:

$$z_1^{a_1}+\cdots+z_n^{a_n}=0\,,$$

$$z_1\bar{z}_1+\cdots+z_n\bar{z}_n=1\,,$$

$\Sigma(a)$ is known to be the boundary of some parallelizable $(2n-2)$-manifold while, if $n \geqq 4$, any homotopy $(2n-3)$-sphere being the boundary of the parallelizable manifold becomes diffeomorphic to some Brieskorn variety. Moreover, in case where $\Sigma(a)$ is the homotopy sphere, E. Brieskorn [1] and H. Hirzebruch and K. H. Mayer [3] have shown that the diffeomorphism type of $\Sigma(a)$ can be completely classified in terms of $a=(a_1, \cdots)$ using the famous theory due to M. Kervaire and J. Milnor [2]. In the present paper we shall show that two Brieskorn varieties

$$\Sigma(a_2, a_3, \cdots, a_n) \quad \text{and} \quad \Sigma(a_1, a_3, \cdots, a_n) \qquad (n \geqq 3)$$

are cobordant and this cobordism is realized by a real algebraic variety W defined by the following equations in $(z, t) \in \mathbf{C}^n \times [0, 1]$:

$$f(z)=z_1^{a_1}+\cdots+z_n^{a_n}=0\,, \tag{1}$$

$$g(z, t)=tz_1+(1-t)z_2=0\,, \tag{2}$$

$$h(z)=|z|^2-1=z_1\bar{z}_1+\cdots+z_n\bar{z}_n-1=0\,. \tag{3}$$

Besides, in many cases the real valued function t on W becomes a Morse function, hence the study of the function t gives us the information on the homotopy type of W. More precisely, we shall prove the following theorem.

THEOREM. *In case $n \geqq 3$ and $a_2 > a_1$, W is a smooth $(2n-4)$-manifold which gives a cobordism between $\Sigma_1=\Sigma(a_2, a_3, \cdots, a_n)$ and $\Sigma_2=\Sigma(a_1, a_3, \cdots, a_n)$. If $10 \geqq a_2 > a_1 = 2$ or $a_2 > a_1 > 2$, then t is a Morse function on W. The Morse index at the critical point (z, t) is given by*

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$$(n-2)-\{\text{the number of } i\text{'s such as } i>2,\ a_i>2 \text{ and } z_i=0\}$$

and the number of critical points of index k is given by

$$(a_2-a_1)\sum a_{i_1}a_{i_2}\cdots a_{i_l}$$

where $l=k-\{$the number of i's such as $i>2$ and $a_i=2\}$ and the summation is taken over the subsets $\{a_{i_1},\cdots,a_{i_l}\}$ of $\{a_3,\cdots,a_n\}$ such as $a_{i_j}>2$ $(j=1, 2,\cdots, l)$.

COROLLARY. *If $n\geqq 5$, $a_2>a_1$ and Σ_1 and Σ_2 are homotopy spheres, then we have*

$$H_0(W)=H_{2n-5}(W)=\mathbf{Z},$$

$$H_{n-2}(W)=\text{a free abelian group of rank } (a_2-a_1)\prod_{i\geqq 3}(a_i-1),$$

$$H_i(W)=0 \quad \text{for} \quad i\neq 0,\ n-2,\ 2n-5.$$

For the cases omitted in the statements of Theorem above, we shall obtain the following facts.

In case $a_1=2$ and $a_2>10$, if $a_i=a_j=2$ for some $i\neq j>2$, then t has degenerate critical points which form spheres, but we can modify t so as to be a Morse function having two critical points near the sphere. In case $a_1=a_2$, if a_1 and a_2 are even, then W is diffeomorphic to $\Sigma_1\times[0, 1]$ since t has no critical point, while if a_1 and a_2 are odd, W does not become even a topological manifold.

2. Proof of the theorem.

We shall first prove that W becomes a smooth manifold. It is well known that W is a smooth manifold if the following matrix has the maximal rank everywhere on W:

$$E=\frac{\partial}{\partial(x, y, t)}\begin{pmatrix} 2\,\mathcal{R}e\, f \\ 2\,\mathcal{I}m\, f \\ 2\,\mathcal{R}e\, g \\ 2\,\mathcal{I}m\, g \\ h \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -i & i & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -i & i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\frac{\partial}{\partial(z, \bar{z}, t)}\begin{pmatrix} f \\ \bar{f} \\ g \\ \bar{g} \\ h \end{pmatrix}\frac{\partial(z, \bar{z}, t)}{\partial(x, y, t)},$$

where $z=x+iy$ denotes a complex n-vector. Note that E does not attain the maximal rank at (z, t) if there exists a non zero real vector $v=(a, b, c, d, e)$ with

$$\begin{aligned} 0&=v\times E \\ &=(a-ib,\ a+ib,\ c-id,\ c+id,\ e)\frac{\partial}{\partial(z, \bar{z}, t)}{}^t(f, \bar{f}, g, \bar{g}, h)\frac{\partial(z, \bar{z}, t)}{\partial(x, y, t)}, \\ &=\frac{\partial}{\partial(z, \bar{z}, t)}(-\alpha f-\overline{\alpha f}+\beta g+\overline{\beta g}+eh)\frac{\partial(z, \bar{z}, t)}{\partial(x, y, t)}, \end{aligned}$$

where we put $\alpha=-a+ib$ and $\beta=c-id$. Assume that such a non zero vector v exists. Then substituting (1), (2) and (3) into this, we have the following equations:

$$\alpha a_1 z_1^{a_1-1}=e\bar{z}_1+\beta t, \tag{4}$$

$$\alpha a_2 z_2^{a_2-1}=e\bar{z}_2+\beta(1-t), \tag{5}$$

$$\alpha a_i z_i^{a_i-1}=e\bar{z}_i, \qquad i=3, 4, \cdots, n, \tag{6}$$

$$\mathcal{R}e\, \beta(z_1-z_2)=0. \tag{7}$$

Note that $v\neq 0$ means $|\alpha|+|\beta|+|e|\neq 0$. It is easy to show that $t\neq 0, 1$. Then it follows readily that $e\neq 0$. (In fact, if $e=0$ then clearly $\alpha=0$ does not occur. Now by (6) $z_i=0$ for $i>2$ so that by (1), (4), (5) and (2) we have

$$0=\alpha(z_1^{a_1}+z_2^{a_2})=\beta tz_1/a_1+\beta(1-t)z_2/a_2=\beta tz_1(1/a_1-1/a_2).$$

As $t\neq 1$ it follows from (2) and (3) that $z_1\neq 0$. So we have $\beta=0$ and $z_1=z_2=0$ from (4) and (5), which contradicts (3).) Hence we may assume without loss of generality that $e=1$. Then we have

$$0=\alpha\sum z_i^{a_i}=\sum z_i\bar{z}_i/a_i+\beta tz_1/a_1+\beta(1-t)z_2/a_2$$
$$=\sum z_i\bar{z}_i/a_i+\beta tz_1(1/a_1-1/a_2).$$

Note that $\sum z_i\bar{z}_i/a_i>0$ and $1/a_1-1/a_2>0$. Thus we have

$$\beta z_1<0 \qquad \text{and} \qquad \beta z_2>0, \tag{8}$$

which contradicts (7). Hence we have proved that W is a smooth manifold.

On the other hand, in case where $a_1=a_2$ and they are odd, the equations (1), $\cdots$, (7) have solutions:

$$z=(-e^{i\theta}/\sqrt{2}, e^{i\theta}/\sqrt{2}, 0, \cdots, 0),\ t=1/2,\ e=0,\ \beta=ie^{-i\theta},\ \alpha=\beta t/a_1 z_1^{a_1-1}.$$

Hence the matrix E does not attain the maximal rank at such point $(z, t)\in W$. Actually, in this case W does not become a manifold. This is proved as follows.

For sufficiently small $\varepsilon>0$

$$W\cap\{(z, t)\in\mathbf{C}^n\times[0, 1]\,|\,z_3\bar{z}_3+\cdots+z_n\bar{z}_n\leqq\varepsilon\}$$

is homeomorphic to the quotient space of

$$\{(z_3, \cdots, z_n)\in\mathbf{C}^{n-2}\,|\,z_3^{a_3}+\cdots+z_n^{a_n} \text{ is real, } z_3\bar{z}_3+\cdots+z_n\bar{z}_n\leqq\varepsilon\}\times[0, 2\pi]$$

with $(z_3, \cdots, z_n)\times 0$ and $(e^{-i2\pi a_2/a_3}z_3, \cdots, e^{-i2\pi a_2/a_n}z_n)\times 2\pi$ identified. The homeomorphism is given by

$$(z_1, \cdots, z_n, t)\longmapsto(e^{-ia_2\theta_2/a_3}z_3, \cdots, e^{-ia_2\theta_2/a_n}z_n),$$

where $\theta_2=\arg(z_2)$.

Moreover the first component of this space is homeomorphic to a cone C on

$$D=\{(z_3, \cdots, z_n)\in \mathbf{C}^{n-2} \mid z_3^{a_3}+\cdots+z_n^{a_n} \text{ is real, } z_3\bar{z}_3+\cdots+z_n\bar{z}_n=\varepsilon\}.$$

Hence we can consider $\mathbf{C}\times[\theta-\delta, \theta+\delta]$ as a neighbourhood of $(z, t)=(-e^{i\theta}/\sqrt{2}, e^{i\theta}/\sqrt{2}, 0, \cdots, 0, 1/2)\in W$ for some small $\delta>0$, and this is homeomorphic to a cone on the suspension of D. If $n=3$, then D consists of $2a_3(\geqq 4)$ points so that W cannot be locally homeomorphic to the Euclidean space at the point. In case $n>3$, let

$$F_{\pm}=D\cap\{(z_3, \cdots, z_n)\in \mathbf{C}^{n-2} \mid z_3^{a_3}+\cdots+z_n^{a_n} \gtreqless 0\},$$

then F_+ is diffeomorphic to F_- and $H_{n-3}(F_+)\neq 0$ (see [4]) so that $H_{n-3}(D)\neq 0$. Hence W cannot be a manifold in either case.

Now we shall seek the critical point of the function t. Let (z, t) be a critical point of t. Then (z, t) is characterized by the condition that the rank of the matrix

$$\frac{\partial}{\partial(x, y, t)}{}^t(2\,\mathcal{R}e\, f, 2\,\mathcal{I}m\, f, 2\,\mathcal{R}e\, g, 2\,\mathcal{I}m\, g, h, t)$$

is less than 6 (see [4]). By the argument similar to the above we have the system of equations in z, t with parameters α, β, e, e':

(*) $\qquad$ (1), $\cdots$, (6) $\quad$ and

(7)$'$ $\qquad$ $\mathcal{R}e\, \beta(z_1-z_2)=e'$,

where α, β are complex and e, e' real with $|\alpha|+|\beta|+|e|+|e'|\neq 0$. We proceed to solve these equations. As in the above case, we may assume $e=1$ and we can also get the inequalities (8) in this case. We shall express the complex numbers z_i, α and β by the polar coordinates:

$$z_1=-r_1\omega,\ z_2=r_2 w,\ z_i=r_i\omega_i\ (i=3, 4, \cdots, n),\ \alpha=a\varepsilon,\ \beta=b\bar{\omega}.$$

Since by (5) and (8)

$$\alpha a_2 z_2^{a_2}=a\varepsilon a_2 r_2^{a_2}\omega^{a_2}=z_2\bar{z}_2+\beta(1-t)z_2>0,$$

we have

(9) $$\varepsilon\omega^{a_2}=1.$$

The equality (6) is rewritten as

$$a\varepsilon a_i r_i^{a_i-1}\omega_i^{a_i-1}=r_i\bar{\omega}_i,$$

whence we have

(10) $\qquad$ if $r_i\neq 0$, then $\varepsilon\omega_i^{a_i}=1$ and $r_i^{a_i-2}=(aa_i)^{-1}$ $(i=3, 4, \cdots, n)$.

From (1), (9) and (10) we get

$$(-1)^{a_1}\omega^{a_1-a_2}r_1^{a_1}+r_2^{a_2}+\cdots+r_n^{a_n}=0,$$

whence the following equalities can be obtained:

(11) $$\omega^{a_2-a_1}=(-1)^{a_1+1}\,,$$

(12) $$r_1^{a_1}=r_2^{a_2}+\cdots+r_n^{a_n}\,.$$

By (2) and (3) we have

(13) $$t=r_2/(r_1+r_2)\,,$$

(14) $$r_1^2+r_2^2+\cdots+r_n^2=1\,.$$

On the other hand, from (4) and (5) it follows

$$a\varepsilon a_1(-1)^{a_1-1}r_1^{a_1-1}\omega^{a_1-1}=-r_1\bar{\omega}+b\bar{\omega}t\,,$$

$$a\varepsilon a_2 r_2^{a_2-1}\omega^{a_2-1}=r_2\bar{\omega}+b\bar{\omega}(1-t)\,,$$

so that by (9) and (11) we have

$$aa_1r_1^{a_1-1}=-r_1+bt\,,$$

$$aa_2r_2^{a_2-1}=r_2+b(1-t)\,.$$

Using (13), we can solve these equations with respect to a and b. We have then

(15) $$a=(r_1^2+r_2^2)/(-a_1r_1^{a_1}+a_2r_2^{a_2})\,,$$

(16) $$b=r_1r_2(r_1+r_2)(a_1r_1^{a_1-2}+a_2r_2^{a_2-2})/(-a_1r_1^{a_1}+a_2r_2^{a_2})\,.$$

In conclusion, the system of equations (*) can be reduced to the following one in (r_i, a):

(**) (10), (12), (14) and (15).

In fact, if r_i and a are obtained, the other unknowns t and b are determined by (13) and (16), while ω, ε and ω_i are obtained from (11), (9) and (10), thus finally e' from (7)'. To solve (**) we shall use the following lemma, the proof of which will be given later.

LEMMA 1. *In case $n\geqq 3$, if $10\geqq a_2>a_1=2$ or $a_2>a_1>2$, then we have necessarily $a>1/2$. Hence if $a_i=2$ $(i>2)$, then $r_i=0$ by* (10).

Let $J(r_i, a)=\{i\,|\,r_i>0,\ i=1, 2, \cdots, n\}$. From (8) and Lemma 1 it follows immedicately that

(17) $$J(r_i, a)\ni 1, 2 \text{ and } a_i>2 \text{ for } i\in J(r_i, a)-\{1, 2\}\,.$$

The condition (17), however, suffices to assure the existence of our solution. Actually the following proposition holds.

PROPOSITION. *Given a subset J of $\{1, 2, \cdots, n\}$ satisfying* (17), *then there exists a unique solution (r_i, a) of* (**) *such that $J=J(r_i, a)$.*

Proof. It is sufficient to prove in case where $a_i>2$ for every $i>2$ and $J=\{1, 2, \cdots, n\}$. From (15), (14) and (10) we have

$$-a_1 r_1^{a_1}+a_2 r_2^{a_2}=a^{-1}(1-\sum r_i^2)=a^{-1}-\sum a_i r_i^{a_i},$$

where the summation (here and in the following) is taken over only the indices $i>2$. This equation, combined with (12), yields

$$r_1^{a_1}=(a_2-a_1)^{-1}(a^{-1}+\sum(a_2-a_i)r_i^{a_i}),$$

$$r_2^{a_2}=(a_2-a_1)^{-1}(a^{-1}+\sum(a_1-a_i)r_i^{a_i}).$$

Putting h for a^{-1}, from (10) and the above equalities we can regard r_i's as functions of h. Note that every r_i $(i>2)$ is an increasing function of h. Hence if r_1 and r_2 are proved to be also increasing functions, then the proposition follows from (14). Differentiating $r_2^{a_2}$ and $r_i^{a_i-2}$ by h we have

$$a_2 r_2^{a_2-1}r_2'=(a_2-a_1)^{-1}(1+\sum(a_1-a_i)a_i r_i^{a_i-1}r_i'),$$

$$(a_i-2)r_i^{a_i-3}r_i'=a_i^{-1},$$

so that we have

$$a_2 r_2^{a_2-1}r_2'=(a_2-a_1)^{-1}(1+\sum r_i^2(a_1-a_i)/(a_i-2)).$$

Suppose $r_2'(h)=0$ for some h. Let h_0 be the first zero of $r_2'(h)$. Then at h_0 we have

$$1+\sum_{a_i\leqq a_1} r_i^2(a_1-a_i)/(a_i-2)=\sum_{a_i>a_1} r_i^2(a_i-a_1)/(a_i-2).$$

Since $a_1\geqq 2$, we can get

$$1\leqq \sum_{a_i>a_1} r_i^2(h_0).$$

As $r_i'(h)\geqq 0$ in $[0, h_0]$, it follows that $r_1^2+r_2^2+\cdots+r_n^2$ increases in $[0, h_0]$ and is greater than 1 at h_0 by the above inequality. Hence there exists a unique $h_1\in[0, h_0]$ such that the equality (14) holds there. On the other hand, if $h\geqq h_0$, we have

$$r_1^2+r_2^2+\cdots+r_n^2>\sum_{a_i>a_1} r_i^2\geqq \sum_{a_i>a_1} r_i^2(h_0)\geqq 1,$$

which completes the proof.

From (10) and (11), it follows that the solutions (z, t) of (*) corresponding to (r_i, a) are $(a_2-a_1)\prod a_{i_j}$ in number, where the index of a_{i_j} runs over all $i_j\in J(r_i, a)-\{1, 2\}$. This proves the last part of the theorem.

Next we shall calculate the index of the function t at the critical point (z, t).

LEMMA 2. *Putting $z_i=x_i+iy_i$, we can take $(x_3, \cdots, x_n, y_3, \cdots, y_n)$ as local coordinates near the critical point (z, t).*

Proof. Assume the contrary, then there exists a non zero real $(2n+1)$-vector $(0, 0, x_3, \cdots, x_n, y_3, \cdots, y_n, 0)$ which is a linear combination of the row vectors of the matrix E at (z, t). It follows that there exist complex numbers α, β and γ_i $(i=3, \cdots, n)$ with $\sum|\gamma_i|\neq 0$ and a real number e such that these satisfy (4), (5), (7) and

$$(6)' \qquad \alpha a_i z_i^{a_i-1}=e\bar{z}_i+\gamma_i \qquad i=3, 4, \cdots, n.$$

Note that α, β and e should be understood to be irrelevant to those taken in (*). It follows easily that $e\neq 0$. (In fact, if $e=0$, then clearly $\alpha\neq 0$ and $\beta\neq 0$. Hence from (4) ann (5) we have

$$a_1z_1^{a_1}/a_2z_2^{a_2}=tz_1/(1-t)z_2\,.$$

Taking the absolute values of both sides we have

$$a_1r_1^{a_1}/a_2r_2^{a_2}=1\,,$$

which contradicts (15).) Hence we may assume that $e=1$. As $a_1z_1^{a_1-1}(1-t)-a_2z_2^{a_2-1}t\neq 0$, we can solve (4) and (5) with respect to α and β and obtain $\beta=b\bar{\omega}$ (which is equal to β in (*)). Hence we have $\mathcal{R}e\,\beta(z_1-z_2)=-b(r_1+r_2)\neq 0$, which contradicts (7).

We shall take these coordinates. Next we replace (z, t) by $(-\omega z_1, \omega z_2, \omega_3 z_3, \cdots, \omega_n z_n, t)$, then the equality (1) is transformed into

$$(1)' \qquad -z_1^{a_1}+z_2^{a_2}+\cdots+z_n^{a_n}=0\,,$$

and the critical point (z, t) is transformed into $(r_1, r_2, \cdots, r_n, t)$. Moreover we put

$$z_1=ue^{i\theta}\,, \qquad z_2=ve^{i\theta}\,,$$

and for brevity, set

$$a_1=p\,, \qquad a_2=q\,, \qquad r_1=r\,, \qquad r_2=s\,,$$

$$\frac{\partial u}{\partial x_i}=u_i\,, \quad \frac{\partial u}{\partial y_i}=u_{i'}\,, \quad \frac{\partial^2 u}{\partial x_i\partial y_j}=u_{ij'} \quad \text{and so on } (i, j>2)\,.$$

Then from (1)′ and (14) we have

$$\begin{aligned}
&-u^p\cos p\theta+v^q\cos q\theta=-\mathcal{R}e\sum z_i^{a_i}\,,\\
&-u^p\sin p\theta+v^q\sin q\theta=-\mathcal{I}m\sum z_i^{a_i}\,,\\
&u^2+v^2=1-\sum(x_i^2+y_i^2)\,,
\end{aligned}$$

where the summation is taken over the indices $i>2$. Differentiating these equalities we have

$$\begin{aligned}
&-pu^{p-1}\cos p\theta\, u_i+pu^p\sin p\theta\,\theta_i+qv^{q-1}\cos q\theta\, v_i-qv^q\sin q\theta\,\theta_i\\
&=-\mathcal{R}e\, a_iz_i^{a_i-1}\,,\\
&-pu^{p-1}\cos p\theta\, u_{i'}+pu^p\sin p\theta\,\theta_{i'}+vq^{q-1}\cos q\theta\, v_{i'}-qv^q\sin q\theta\,\theta_{i'}\\
&=-\mathcal{R}e\, a_iz_i^{a_i-1}i\,,
\end{aligned}$$

$$(17) \qquad -pu^{p-1}\sin p\theta\, u_i-pu^p\cos p\theta\,\theta_i+qv^{q-1}\sin q\theta\, v_i+qv^q\cos q\theta\,\theta_i$$

$$=-\mathcal{I}m\, a_i z_i^{a_i-1},$$

$$-pu^{p-1}\sin p\theta\, u_{i'}-pu^p\cos p\theta\,\theta_{i'}+qv^{q-1}\sin q\theta\, v_{i'}+qv^q\cos q\theta\,\theta_{i'}$$

$$=-\mathcal{I}m\, a_i z_i^{a_i-1}i,$$

$$uu_i+vv_i=-x_i,$$

$$uu_{i'}+vv_{i'}=-y_i.$$

At the critical point we have

$$-pu^{p-1}u_i+qv^{q-1}v_i=-a_ir_i^{a_i-1},$$

$$-pu^{p-1}u_{i'}+qv^{q-1}v_{i'}=0,$$

$$-pu^p\theta_i+qv^q\theta_i=0,$$

$$-pu^p\theta_{i'}+qv^q\theta_{i'}=-a_ir_i^{a_i-1},$$

$$ru_i+sv_i=-r_i,$$

$$ru_{i'}+sv_{i'}=0.$$

From these equations we have at the critical point

$$u_i=-r_i(-a_ir_i^{a_i-2}+qs^{q-2})/r(pr^{p-2}+qs^{q-2}),$$

$$v_i=-r_i(a_ir_i^{a_i-2}+pr^{p-2})/s(pr^{p-2}+qs^{q-2}),$$

$$u_{i'}=0,$$

$$v_{i'}=0,$$

$$\theta_i=0,$$

$$\theta_{i'}=-a_ir_i^{a_i-1}/(-pr^p+qs^q).$$

By (10) and (15) we have

$$u_i=-rr_i/(r^2+s^2),$$

$$v_i=-sr_i/(r^2+s^2), \tag{18}$$

$$\theta_{i'}=-r_i/(r^2+s^2).$$

Differentiating equalities (17) at the critical point, we have

$$-p(p-1)r^{p-2}u_iu_j-pr^{p-1}u_{ij}+q(q-1)s^{q-2}v_iv_j+qs^{q-1}v_{ij}$$

$$=-a_i(a_i-1)r_i^{a_i-2}\delta_{ij},$$

$$-pr^{p-1}u_{ij'}+qs^{q-1}v_{ij'}=0,$$

$$-pr^{p-1}u_{i'}u_{j'}+pr^p\theta_{i'}\theta_{j'}+qs^{q-1}v_{i'}v_{j'}-qs^{q-1}\theta_{i'}\theta_{j'}$$

$$=a_i(a_i-1)r_i^{a_i-2}\delta_{ij},$$

$$u_iu_j+ru_{ij}+v_iv_j+sv_{ij}=-\delta_{ij}\,,$$
$$ru_{ij'}+sv_{ij'}=0\,,$$
$$ru_{i'j'}+sv_{i'j'}=-\delta_{ij}\,.$$

Then we have

$$\begin{aligned}
&u_{ij}=(-A_{ij}+B_{ij}qs^{q-2})/r(pr^{p-2}+qs^{q-2})\,,\\
&v_{ij}=(A_{ij}-B_{ij}pr^{p-2})/s(pr^{p-2}+qs^{q-2})\,,\\
&u_{ij'}=0\,,\\
&v_{ij'}=0\,,\\
&u_{i'j'}=(-A'_{ij}+B'_{ij}qs^{q-2})/r(pr^{p-2}+qs^{q-2})\,,\\
&v_{i'j'}=(A'_{ij}-B'_{ij}pr^{p-2})/s(pr^{p-2}+qs^{q-2})\,,
\end{aligned}\tag{19}$$

where

$$\begin{aligned}
A_{ij}&=-a_i(a_i-1)r_i^{a_i-2}\delta_{ij}+p(p-1)r^{p-2}u_iu_j-q(q-1)s^{q-2}v_iv_j\\
&=-a_i(a_i-1)r_i^{a_i-2}\delta_{ij}+(p(p-1)r^p-q(q-1)s^q)r_ir_j/(r^2+s^2)^2\,,\\
B_{ij}&=\delta_{ij}+u_iu_j+v_iv_j=\delta_{ij}+r_ir_j/(r^2+s^2)\,,\\
A'_{ij}&=a_i(a_i-1)r_i^{a_i-2}\delta_{ij}+(-p^2r^p+q^2s^q)\theta_{i'}\theta_{j'}\\
&=a_i(a_i-1)r_i^{a_i-2}\delta_{ij}+(-p^2r^2+q^2s^q)r_ir_j/(r^2+s^2)^2\,,\\
B'_{ij}&=\delta_{ij}\,.
\end{aligned}$$

Here we have used (18).
Now from $t(u+v)=v$ we have at the critical point

$$t_{ij}=(-su_{ij}+rv_{ij})/(r^2+s^2)\,.$$

Substituting (19) into this, we have

$$t_{ij}=(A_{ij}(r^2+s^2)+B_{ij}(-pr^p+qs^q))/rs(r+s)^2(pr^{p-2}+qs^{q-2})\,.$$

Hence for any real numbers $c_3, c_4, \cdots, c_n$ we have

$$\begin{aligned}
&\sum c_it_{ij}c_j\times rs(r+s)^2(pr^{p-2}+qs^{q-2})\\
&=\sum(-a_i(a_i-1)r_i^{a_i-2}(r^2+s^2)+(-pr^p+qs^q))c_i^2\\
&\quad+((p(p-1)r^p-q(q-1)s^q)+(-pr^p+qs^q))(r^2+s^2)^{-1}(\sum r_ic_i)^2\,.
\end{aligned}$$

If $r_i>0$, then from (10) and (15) we have

$$\begin{aligned}
&-a_i(a_i-1)r_i^{a_i-2}(r^2+s^2)+(-pr^p+qs^q)\\
&=-(a_i-1)a^{-1}(r^2+s^2)+(-pr^p+qs^q)\\
&=-(a_i-2)(-pr^p+qs^q)\,.
\end{aligned}$$

Hence the above Hessian form is equal to

$$-(2(r^2+s^2)-(-pr^p+qs^q))\sum_1 c_i^2+(-pr^p+qs^q)\sum_2 c_i^2$$

$$-(-pr^p+qs^q)\sum_3(a_i-2)c_i^2-(-p(p-2)r^p+q(q-2)s^q)(r^2+s^2)^{-1}(\sum_3 r_i c_i)^2\,,$$

where the summation $\sum_1$ takes over the indices i's such as $a_i=2$, $\sum_2$ i's such as $a_i>2$ and $r_i=0$, $\sum_3$ i's such as $a_i>2$ and $r_i>0$. Smilarly we have

$$t_{ij'}=0\,,$$

$$\sum c_i t_{i'j'} c_j \times rs(r+s)^2(pr^{p-2}+qs^{q-2})$$

$$=(2(r^2+s^2)+(-pr^p+qs^q))\sum_1 c_i^2+(-pr^p+qs^q)\sum_2 c_i^2$$

$$+(-pr^p+qs^q)\sum_3 a_i c_i^2+(-p^2r^p+q^2s^q)(r^2+s^2)^{-1}(\sum_3 r_i c_i)^2\,.$$

Now from (15) and Lemma 1, it follows that

$$-pr^p+qs^q>0\,,\qquad 2(r^2+s^2)-(-pr^p+qs^q)>0\,,$$

moreover as $p<q$ we have

$$-p(p-2)r^p+q(q-2)s^q>0\,,\qquad -p^2r^p+q^2s^q>0\,.$$

Hence the function t is a Morse function and the index is as mentioned in Theorem. This completes the proof of Theorem.

3. Proofs of the corollary and the lemma.

We shall first consider the case where $10\geqq a_2>a_1=2$ or $a_2>a_1>2$. Then by Theorem the function t is a Morse function with indices $\leqq n-2$, so that $W/\sum_2$ has the homotopy type of a $(n-2)$-CW complex. Hence $H_{n-2}(W,\sum_2)=H_{n-2}(W/\sum_2)$ is a free abelian group and $H_i(W,\sum_2)=H_i(W/\sum_2)=0$ for $i>n-2$. From the exact sequence of the pair $(W,\sum_2)$ it follows that $H_i(W)=H_i(\sum_2)$ for $i>n-2$ and $H_i(W)=H_i(W,\sum_2)$ for $i\neq 0$, $2n-5$, $2n-4$, which proves the corollary for $i\geqq n-2$ except for the calculation of the rank of $H_{n-2}(W)$. Next we shall use the Morse function $1-t$. As the indices of $1-t\geqq n-2$, W has the homotopy type of what is constructed by adjoining cells of dimension $\geqq n-2$ to $\sum_1$, so that the map $\pi_i(\sum_1)\to\pi_i(W)$ is surjective for $i<n-2$. It follows that W is $(n-3)$-connected, which proves the corollary for $i<n-2$. To calculate the rank of $H_{n-2}(W)$ we shall use the Morse equality which means that

$$\sum(-1)^k R_k=\sum(-1)^k C_k\,,$$

where

$$R_k=\text{the rank of } H_k(W,\textstyle\sum_2) \text{ and}$$

$$C_k = \text{the number of critical points of index } k.$$

In our case

$$R_k = \text{the rank of } H_{n-2}(W) \qquad k=n-2,$$
$$=0 \qquad \text{otherwise},$$
$$C_k = (a_2 - a_1)\sum a_{i_1} \cdots a_{i_l}.$$

It is easy to get the rank of $H_{n-2}(W)$ as mentioned in the corollary. Before considering the case where $a_1=2$ and $a_2>10$, we shall prove Lemma 1.

Proof of Lemma 1. Put

$$r_1^2 = x, \qquad r_2^2 = y, \qquad a_1 = 2l, \qquad a_2 = 2m.$$

Moreover, we set

$$\sum r_i^2 = A, \qquad \sum r_j^2 = B, \qquad \sum r_j^{a_j} = C,$$

where the subscript i runs over such i's with $a_i=2$, while the subscript j such j's with $a_j>2$. Suppose the contrary, so that we assume $a \leqq 1/2$. Then from (12), (14) and (15), we can get

$$x^l = y^m + A + C, \tag{20}$$

$$x + y + A + B = 1, \tag{21}$$

$$-lx^l + my^m \geqq x + y. \tag{22}$$

(20) and (21) yield

$$x^l + x + y + B = y^m + C + 1$$

which, combined with (22), leads to the following inequalities:

$$y \geqq m(1-B+C)/(m-1) - x - x^l(m-l)/(m-1) = f_1(x), \tag{23}$$

$$y^m \geqq (1-B+C)/(m-1) + x^l(l-1)/(m-1) = f_2(x), \tag{24}$$

Also, we need a supplementary inequality

$$y \leqq 1 - B - x = f_3(x), \tag{25}$$

being obtained from (21).

Now it is clear that the graphs $y=f_1(x)$ and $y=f_3(x)$ inrersect at a unique point (x_1, y_1) and the graphs $x=x_1$ and $y=f_2(x)$ at a unique point (x_1, y_2). Acutually we have

$$x_1 = \left(\frac{1-B+mC}{m-l}\right)^{1/l}, \qquad y_1 = 1 - B - x_1, \qquad y_2 = \left(\frac{1-B+C}{m-l}\right)^{1/m}.$$

Since there exists a point (x, y) which satisfies (23), (24) and (25) we can easily conclude that $y_1 \geqq y_2$. Hence we have

$$1-B \geqq \left(\frac{1-B+mC}{m-l}\right)^{1/l}+\left(\frac{1-B+C}{m-l}\right)^{1/m},$$

so that

$$1-B \geqq \left(\frac{1-B}{m-l}\right)^{1/l}+\left(\frac{1-B}{m-l}\right)^{1/m}. \tag{26}$$

Thus we have arrived the inequality

$$1 \geqq (m-l)^{-1/l}+(m-l)^{-1/m}=f(l, m).$$

The condition of the lemma means that l and m are half integers such that

$$5 \geqq m>l=1 \quad \text{or} \quad m>l \geqq 3/2.$$

In both cases, the inequality above, however, is impossible. In fact, $f(l, m)$ is increasing in l when $m>l\geqq 1$ and by estimating the first and second derivatives, we can show that the behaviour of the curves $f(1, m)$ and $f(3/2, m)$ are described as follows:

m	1		5.1 $\cdots$		13. $\cdots$		∞
$f(1, m)$	∞	$\searrow$	1	$\searrow$	0.9093 $\cdots$	$\nearrow$	1

m	3/2		44. $\cdots$		37$\cdot.\cdot\cdot$		∞
$f(3/2, m)$	∞	$\searrow$	1.0004 $\cdots$	$\nearrow$	1.0036 $\cdots$	$\searrow$	1

This completes the proof of the lemma.

Now we return to the proof of the corollary in case where $a_1=2$ and $a_2>10$. In this case, (**) has solutions (r_i, a) such as $a\leqq 1/2$. For such a solution, we have that $r_j=0$ if $a_j>2$. (In fact, if $r_j>0$ for some $a_j>2$, then it is easy to see that

$$B \geqq r_j^2=(aa_j)^{-2/(a_j-2)} \geqq (1/2\cdot 3)^{-2/(3-2)}=4/9,$$

and using this inequality, we have

$$1-B<\left(\frac{1-B}{m-1}\right)^{1/m},$$

which contradicts (26).)

If we put $a=1/2$, then we have a set of solutions of (**):

$$\begin{aligned} r_2^2=y=(m-1)^{-1/m}, \qquad & r_1^2=x=(m/(m-1)-y)/2, \\ r_i^2=A=x-1/(m-1), \qquad & r_j=0, \qquad a=1/2, \end{aligned} \tag{27}$$

where we used the same notations as in the proof of the lemma. Note that

$$r_1^2>A=(1-(1/(m-1)+(m-1)^{-1/m}))/2=(1-f(1, m))/2>0.$$

Hence the solutions above exist if and only if there exists some $i>2$ such as

$a_i=2$. Next we assume $a<1/2$, then from (10) we have that $r_i=0$ if $a_i=2$. So the proposition applies to the case where $J=\{1, 2\}$ and for this solution we have acutually $a<1/2$. Because, from (12) and (14) it follows that

$$f(1, m)=2/(a_2-2)+(2/(a_2-2))^{2/a_2}<1=r_1^2+r_2^2=r_1^2+(r_1^2)^{2/a_2},$$

so that $r_1^2>2/(a_2-2)$. Hence we have

$$a=(r_1^2+r_2^2)/(-2r_1^2+a_2r_2^{a_2})=1/(a_2-2)r_1^2<1/2.$$

Now the critical points corresponding to the solutions (27) form $(N-1)$-spheres which are (a_2-a_1) in number (where N denotes the number of i's such as $\imath>2$ and $a_i=2$). At these points the Hessian form of t is given by

$$\begin{aligned}&\sum c_i t_{ij} c_j \times rs(r+s)^2(pr^{p-2}+qs^{q-2})\\ &=-(p(p-2)r^p+q(q-2)s^q)(r^2+s^2)^{-1}(\textstyle\sum_1 r_i c_\imath)^2+(-pr^p+qs^q)\textstyle\sum_2 c_\imath^2,\\ &t_{\imath j'}=0,\\ &\textstyle\sum c_i t_{i'j'} c_j \times rs(r+s)^2(pr^{p-2}+qs^{q-2})\\ &=(2(r^2+s^2)+(-pr^p+qs^q))\textstyle\sum_1 c_i^2\\ &\quad+(-p^2r^p+q^2s^q)(r^2+s^2)^{-1}(\textstyle\sum_1 r_i c_\imath)^2+(-pr^p+qs^q)\textstyle\sum_2 c_\imath^2.\end{aligned}$$

Hence t is not a Morse function if $N\geqq 2$. We can, however, modify t near each critical sphere so as to have only two nondegenerate critical points of index 1 and N respectively. To see this, note first that we can take local coordinates $(x_3, \cdots, x_n, y_3, \cdots, y_n)$ in some small neighbourhood of the sphere. Next we take new coordinates $(\Theta, R, Y_1, \cdots, Y_M)$, where (Θ, R) are polar coordinates of the N-plane containing the sphere and (Y) are coordinates of the orthogonal $(2n-4-N)$-plane. Then for any fixed Θ, t is a Morse function of (R, Y) and the Hessian from at the critical point $(R_0, 0)$ is given by

$$\sum c_i t_{ij} c_j=-ac_0^2+b\sum c_i^2+c\sum c_\imath^2+d(\sum c_j d_j(\Theta))^2,$$

where we denote $R=Y_0$ and a, b, c and d are positive constants and $d_j(\Theta)$ is a function of Θ. Hence we can take new coordinates (Θ, R', Y') such that $t(\Theta, R', y')=-(R'-R_0)^2+Y_1'^2+\cdots+Y_M'^2$. Let $(X_1', \cdots, X_N')$ be orthogonal coordinates corresponding to the polar coordinates (Θ, R'), then it is easy to see that $t(X', Y')+\varphi(|X'|^2-R_0^2)\varphi(|Y'|^2)X_1'$ has the desired property for a suitable function φ having its support in a small neibourhood of 0. Now the index of the critical point corresponding to the solution of (**) such as $J(r_\imath, a)=\{1, 2\}$ is equal to 0 since $a<1/2$. Note that in the first case of the proof, the index of such critical point is equal to N and that the contribution of three critical points of index 0, 1 and N respectively to the right hand side of the Morse equality is equal to one of a critical point of index N. Hence using the modified Morse function, we obtain the same rank of $H_{n-2}(W)$ as in the first case. This completes the proof of the corollary.

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