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KŌDAI MATH. SEM. REP.
26 (1975), 204—209

HARMONIC L^p-FUNCTIONS ON RIEMANNIAN MANIFOLDS

Dedicated to Professor Y. Komatu on his sixtieth birthday

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The harmonic classification scheme

$$O_{HG}<O_{HP}<O_{HB}<O_{HD}=O_{HC}$$

was recently completed [1, 2] between Riemannian manifolds which do not carry harmonic Green's functions, or nonconstant harmonic functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. These classes and inclusion relations were first investigated on Riemann surfaces and then generalized to Riemannian manifolds of any dimension. An important class which is meaningless for Riemann surfaces is the class O_{HL^p} of Riemannian manifolds which do not admit nonconstant harmonic functions of finite L^p norm. An interesting question is: which classes O_{HX}, with $X=G$, P, B, D or C, does O_{HL^p} contain, and which classes is it contained in? We shall show that O_{HL^p} exhibits in this regard a new behavior, entirely different from that of the other null classes: there are no inclusion relations whatever. Explicitly, if $\tilde{O}^N$ is the complement of a given class O^N of Riemannian N-manifolds, then the classes

$$O^N_{HL^p}\cap O^N_{HX},\quad O^N_{HL^p}\cap\tilde{O}^N_{HX},\quad \tilde{O}^N_{HL^p}\cap O^N_{HX},\quad \tilde{O}^N_{HL^p}\cap\tilde{O}^N_{HX}$$

are all nonvoid for every $1\leqq p\leqq\infty$, $N\geqq 2$ and $X=G$, P, B, D, C (except for the triviality that for $p=\infty$ the classes $O^N_{HL^p}\cap\tilde{O}^N_{HX}$ are void for $X=B$, D, C, and the classes $\tilde{O}^N_{HL^p}\cap O^N_{HX}$ void for $X=G$, P, B).

1. We shall make use of a well-known characterization of L^p. Let $p\geqq 1$, $q\geqq 1$ be real numbers with $p^{-1}+q^{-1}=1$.

If $h\in HL^p$ *for some* $1\leqq p\leqq\infty$, *then*

$$\sup_{\varphi\in L^q}\frac{|(h,\varphi)|}{\|\varphi\|_q}<\infty. \tag{1}$$

Received July 17, 1973.
* The work was sponsored by the U.S. Army Research Office, Durham, Grant DA-ARO-D-31-124-73-G39, University of California, Los Angeles.
MOS Classification: 31B05.

Conversely, if this inequality is true for a harmonic h and some $1<p\leqq\infty$, *then* $h\in HL^p$.

The sufficiency is trivial: If there exists an $h\in HL^p$, then by Hölder's inequality, $|(h, \varphi)|\leqq\|h\|_p\|\varphi\|_q=K\|\varphi\|_q$ for $1\leqq p\leqq\infty$ and all $\varphi\in L^q$. Conversely, if $|(h, \varphi)|\leqq K\|\varphi\|_q$ for all $\varphi\in L^q$, then for $1\leqq q<\infty$ there exists a function $k\in L^p$ such that $(h, \varphi)=(k, \varphi)$ for all $\varphi\in L^q$. From $(h-k, \varphi)=0$ it follows that $\|h\|_p<\infty$.

2. First we exclude both the HL^p and HX functions.

THEOREM 1. $O^N_{HL^p}\cap O^N_{HX}\neq\emptyset$ *for* $1\leqq p\leqq\infty$, $N\geqq 2$, *and* $X=G, P, B, D, C$.

Proof. Consider the 'beam'

$$T: |x|<\infty\,, \qquad |y_i|\leqq\pi\,, \qquad i=1, \cdots, N-1\,,$$

with the pair of opposite faces $y_i=\pi$ and $y_i=-\pi$ identified by a parallel translation perpendicular to the x-axis, for every i. Endow T with the Euclidean metric $ds^2=dx^2+\sum_{i=1}^{N-1}dy_i^2$. For the Laplace-Beltrami operator $\Delta=d\delta+\delta d$ and a function h_0 of x only, the equation $\Delta h_0=-h_0''=0$ gives the harmonic functions $h_0=ax+b$. The harmonic measure ω_c of $\{x=c\}$ on $\{0\leqq x\leqq c\}$ with $c>0$ is x/c, and $\omega_c\to 0$ as $c\to\infty$. The analogue is true for $c<0$, and therefore $T\in O^N_{HG}\subset O^N_{HX}$.

For a trial solution of the general equation $\Delta h=0$ take

$$h=f(x)\prod_{i=1}^{N-1}g_i(y_i)\,.$$

Then

$$\Delta h=-\Big(f''\prod_{i=1}^{N-1}g_i+f\sum_{i=1}^{N-1}g_i''\prod_{k\neq i}g_k\Big)$$

$$=-h\Big(f''f^{-1}+\sum_{i=1}^{N-1}g_i''g_i^{-1}\Big)=0\,.$$

Each term in ( ) depends on one variable only and is therefore constant. The eigenvalues $n_i\geqq 0$ give the equation $g_i''=-n_i^2g_i$ and the eigenfunctions

$$g_{i1}=\cos n_iy_i\,, \qquad g_{i2}=\sin n_iy_i\,.$$

We shall use the notation

$$n=(n_1, \cdots, n_{N-1})\,, \qquad 0=(0, \cdots, 0)\,, \qquad \eta^2=\sum_{i=1}^{N-1}n_i^2\,, \qquad \eta\geqq 0\,.$$

Given a function j from n to $\{1, 2\}$, set

$$G_{nj}=\prod_{i=1}^{N-1}g_{ij(n_i)}\,.$$

Then f_nG_{nj} is harmonic if f_n satisfies $f_n''=\eta^2f_n$, that is, $f_n=e^{\pm\eta x}$. It is readily seen that an arbitrary harmonic function h on T has an expansion

$$h=h_0+\sum_{n\neq 0}\sum_j (a_{nj}e^{\eta x}+b_{nj}e^{-\eta x})G_{nj}$$

on $\{|x|=c\}$, hence on $\{|x|\leqq c\}$ and a fortiori on T.

Suppose $h\in HL^p$. If some $a_{nj}\neq 0$, take a continuous function $\rho_0(x)\geqq 0$ with supp $\rho_0\subset(0, 1)$ and $\int_0^1\rho_0 dx=1$. For a number $t>0$ set $\rho_t(x)=\rho_0(x-t)$ and $\varphi_t=\rho_t G_{nj}$. Then

$$(h, \varphi_t)=c\int_t^{t+1}(a_{nj}e^{\eta x}+b_{nj}e^{-\eta x})\rho_t dx\,.$$

Here and later c is a constant, not always the same. As $t\to\infty$,

$$(h, \varphi_t)\sim ce^{\eta t}\int_t^{t+1}\rho_t dx=ce^{\eta t}\,,$$

whereas, for $1<p\leqq\infty$,

$$\|\varphi_t\|_q=c\Big(\int_t^{t+1}\rho_t^q\, dx\Big)^{\frac{1}{q}}=O(1)\,,$$

and for $p=1$, $\|\varphi_t\|_q=\text{const.}<\infty$. Thus relation (1) gives a contradiction for $\eta t>0$, that is, for all $\eta>0$. We conclude that $a_{nj}=0$ for all $n\neq 0$. An analogous argument using $t<0$ and $t\to-\infty$ gives $b_{nj}=0$ for all $n\neq 0$. Therefore $h=h_0$. For $1\leqq p\leqq\infty$, we have $\|h_0\|_p=\infty$u nless $a=b=0$. We have proved that every $h\in HL^p$ reduces to zero, hence $T\in O_{HL^p}$.

3. Next we exclude the HL^p functions only.

THEOREM 2. *$O^N_{HL^p}\cap\tilde{O}^N_{HX}\neq\emptyset$ for $1\leqq p<\infty$, $N\geqq 2$, and $X=G$, P, B, D, C.*

Note that for $p=\infty$, $O^N_{HL^p}=O^N_{HB}$, and the classes $O^N_{HL^p}\cap\tilde{O}^N_{HX}$ and trivially nonvoid if $X=G$, P, void if $X=B$, D, C.

Proof. Consider the manifold

$$T: |x|<1\,,\qquad |y_i|\leqq\pi\,,\qquad i=1,\cdots,N-1\,,$$

with the metric

$$ds^2=\lambda^2dx^2+\lambda^{-\frac{2}{N-1}}\sum_{i=1}^{N-1}dy_i^2\,,$$

where $\lambda=\lambda(x)$ is in C^2. The identification of the opposite faces $y_i=\pi$ and $y_i=-\pi$ by pairs is henceforth tacitly understood. The equation

$$\Delta h_0(x)=-\lambda^{-2}(\lambda^2\lambda^{-2}h_0')'=0$$

gives again $h_0=ax+b$. Since

$$D(x)=c\int_{-1}^1\lambda^{-2}\lambda^2 dx<\infty\,,$$

we have $x\in HC$, hence $T\in\tilde{O}^N_{HC}\subset\tilde{O}^N_{HX}$ for all X.

Let G_{nj} be as in No. 2. Then $h=f_n(x)G_{nj}$ is harmonic if

$$\Delta h=-\lambda^{-2}\Big(f_n'' G_{nj}-\lambda^{\frac{2(N-2)}{N-1}}\sum_{i=1}^{N-1} n_i^2 f_n G_{nj}\Big)=0,$$

i. e.,

$$f_n''=\eta^2\lambda^{\frac{2(N-1)}{N-1}}f_n.$$

First we consider the case $N>2$ and choose

$$\lambda=(1-x^2)^{-\frac{N-1}{N-2}}.$$

Then

$$(1-x^2)^2 f_n''=\eta^2 f_n.$$

To solve this differential equation, set

$$f_n(x)=(1-x^2)^{\frac{1}{2}}t_n(z), \qquad \frac{dz}{dx}=(1-x^2)^{-1}.$$

A simple computation yields

$$f_n'=(1-x^2)^{-\frac{1}{2}}(-xt_n+t_n'),$$

$$f_n''=(1-x^2)^{-\frac{3}{2}}(t_n''-t_n),$$

and our equation is transformed into

$$t_n''(z)=(1+\eta^2)t_n(z).$$

It is satisfied by $t_n=e^{\pm\sqrt{1+\eta^2}\,z}$, and in view of

$$z=\frac{1}{2}\log\frac{1+x}{1-x},$$

we obtain the solutions

$$\begin{cases} f_{n1}=(1+x)^{\frac{1}{2}(1+\sqrt{1+\eta^2})}(1-x)^{\frac{1}{2}(1-\sqrt{1+\eta^2})}, \\ f_{n2}=(1+x)^{\frac{1}{2}(1-\sqrt{1+\eta^2})}(1-x)^{\frac{1}{2}(1+\sqrt{1+\eta^2})}. \end{cases}$$

For $k=1, 2$, the function $f_{nk}G_{nj}$ is harmonic, and an arbitray harmonic function h can be written

$$h=\sum_n\sum_j\sum_k a_{njk}f_{nk}G_{nj},$$

where the summation with respect to n now includes $n=0=(0, \cdots, 0)$.

Suppose $h\in HL^p$. If some $a_{nj1}\neq 0$, take numbers $0<\alpha<\beta<1$ and a continuous function $\rho_0\geqq 0$ with supp $\rho_0\subset(\alpha, \beta)$. For a number $0<t<1$ set $\rho_t(x)=\rho_0((1-x)/t)$, $\varphi_t=\rho_t G_{nj}$. Now supp $\rho_t\subset(1-\beta t, 1-\alpha t)$, and as $t\to 0$,

$$(h, \varphi_t)\sim c\int_{1-\beta t}^{1-\alpha t}(1-x)^{\frac{1}{2}(1-\sqrt{1+\eta^2}}(1-x^2)^{-\frac{2(N-1)}{N-2}}\rho_t dx,$$

which, by virtue of $\int_{1-\beta t}^{1-\alpha t}\rho_t dx=O(t)$, gives

$$(h, \varphi_t) \sim ct^{\frac{1}{2}\left(1-\sqrt{1+\eta^2}\right)-\frac{2(N-1)}{N-2}+1}.$$

On the other hand, if $p>1$, then

$$\|\varphi_t\|_q \sim c\left(\int_{1-\beta t}^{1-\alpha t} \rho_t^q (1-x^2)^{-\frac{2(N-1)}{N-2}} dx\right)^{\frac{1}{q}} = O\left(t^{\frac{1}{q}\left(-\frac{2(N-1)}{N-2}+1\right)}\right).$$

In view of $1-q^{-1}=p^{-1}$, relation (1) gives a contradiction for

$$\frac{1}{p}\left(-\frac{2(N-1)}{N-2}+1\right) < \frac{1}{2}\left(\sqrt{1+\eta^2}-1\right),$$

i.e., for every $\eta \geqq 0$. We infer that all $a_{nk1}=0$. If $p=1$, then $\|\varphi_t\|_q=\text{const.}<\infty$ and we have a contradiction for

$$\frac{1}{2}\left(1-\sqrt{1+\eta^2}\right)-\frac{2(N-1)}{N-2}+1<0,$$

hence again for every $\eta \geqq 0$.

If some a_{nj2} is $\neq 0$, we choose $\rho_t=\rho_0((x+1)/t)$ and obtain the same estimates as above for $|(h, \varphi_t)|$ and $\|\varphi_t\|_q$ as $t \to 0$. Thus all $a_{nj2}=0$, hence $h\equiv 0$, and $T \in O_{HL^p}$ for $N>2$.

For $N=2$ choose

$$\lambda=(1-x^2)^{-1}.$$

In the conformal metric $ds=\lambda|dx|$ the harmonic functions are the same as in the Euclidean metric, and we obtain as in No. 2,

$$h=h_0+\sum_{n\neq 0}\sum_j (a_{nj}e^{nx}+b_{nj}e^{-nx})G_{nj},$$

where now n stands for n_1. Suppose $h \in HL^p$. If $a_{nj}e^n+b_{nj}e^{-n}\neq 0$ for some $n>0$, then for $\rho_t(x)=\rho_0((1-x)/t)$, $\varphi_t=\rho_t G_{nj}$, $t\to 0$, we have

$$(h, \varphi_t)=c\int_{1-\beta t}^{1-\alpha t}(a_{nj}e^{nx}+b_{nj}e^{-nx})\rho_t(1-x^2)^{-2}dx$$

$$\sim c_1 t^{-2}\int_{1-\beta t}^{1-\alpha t}\rho_t dx=c_2 t^{-1},$$

whereas, for $1<p<\infty$,

$$\|\varphi\|_q=c\left(\int_{1-\beta t}^{1-\alpha t}\rho_t^q(1-x^2)^{-2}dx\right)^{\frac{1}{q}}=O(t^{-\frac{1}{q}}).$$

In view of $-1<-q^{-1}$, relation (1) gives a contradiction, and we conclude that $a_{nj}e^n+b_{nj}e^{-n}=0$ for all $n>0$. An analogous reasoning by means of $\rho_t(x)=\rho_0((x+1)/t)$, $t\to 0$, gives $a_{nj}e^{-n}+b_{nj}e^n=0$. As a consequence, $a_{nj}=b_{nj}=0$ for all $n>0$, that is, $h=h_0$. Since

$$\|h_0\|_p=\left(\int_{-1}^{1}|ax+b|^p(1-x^2)^{-2}dx\right)^{\frac{1}{p}}=\infty$$

unless $a=b=0$, we obtain $h\equiv 0$, hence $T\in O_{HL^p}^2$ for $1<p<\infty$.

The conclusion remains true for $p=1$, since $\|\varphi_i\|_\infty=\text{const.}<\infty$, and the contradiction is for $n>0$ such that $t^{-1}>O(1)$, hence again $a_{nj}=b_{nj}=0$ for all $n>0$. The reasoning for $n=0$ is the same as before.

The proof of Theorem 2 is herewith complete.

4. Next we exclude the HX functions only.

THEOREM 3. *$\tilde{O}_{HL^p}^N \cap O_{HX}^N \neq \emptyset$ for $1\leqq p<\infty$, $N\geqq 2$, and $X=G, P, B, D, C$.*

For $p=\infty$, $\tilde{O}_{HL^p}^N \cap O_{HX}^N$ is trivially void for $X=G, P, B$, nonvoid for $X=D, C$.

Proof. Take the manifold

$$T:\ |x|<\infty\,, \qquad |y_i|\leqq\pi\,, \qquad i=1,\cdots,N-1\,,$$

with the metric

$$ds^2=\lambda^2 dx^2+\lambda^{-\frac{2}{N-1}}\sum_{i=1}^{N-1} dy_i^2\,,$$

where $\lambda=\lambda(x)$ is in C^2. As in No. 3, $h_0(x)=ax+b$, and we conclude as in No. 2 that $T\in O_G^N\subset O_{HX}^N$.

Choose $\lambda=e^{-x^2}$. Then for $1\leqq p<\infty$

$$\|x\|_p=c\Big(\int_{-\infty}^{\infty}|x|^p e^{-2x^2}dx\Big)^{\frac{1}{p}}<\infty\,,$$

and the theorem follows.

5. It is obvious that $\tilde{O}_{HL^p}^N \cap \tilde{O}_{HX}^N \neq \emptyset$ for $1\leqq p\leqq\infty$, $N\geqq 2$, and $X=G, P, B, D, C$. In fact, on the space

$$T:\ |x|<1\,, \qquad |y_i|\leqq\pi\,, \qquad i=1,\cdots,N-1\,,$$

with the Euclidean metric, the function x belongs to $HL^p\cap HX$.

We combine our results:

THEOREM 4. *The classes*

$$O_{HL^p}^N\cap O_{HX}^N,\quad O_{HL^p}^N\cap \tilde{O}_{HX}^N,\quad \tilde{O}_{HL^p}^N\cap O_{HX}^N,\quad \tilde{O}_{HL^p}^N\cap \tilde{O}_{HX}^N$$

are all nonvoid for $1\leqq p<\infty$, $N\geqq 2$, and $X=G, P, B, D, C$. The same is true for $p=\infty$, except that trivially the classes $O_{HL^p}^N\cap \tilde{O}_{HX}^N$ are void for $X=B, D, C$, and the classes $\tilde{O}_{HL^p}^N\cap O_{HX}$ void for $X=G, P, B$.

REFERENCES

[1] D. HADA-L. SARIO-C. WANG, N-manifolds carrying bounded but no Dirichlet finite harmonic functions, Nagoya Math. J. (to appear).

[2] Y.K. KWON, Bounded harmonic but no Dirichlet-finite harmonic, Bull. Amer. Math. Soc., **79** (1973), 491–492.