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GEODESIC CONFORMAL TRANSFORMATIONS AND SYMMETRIC SPACES

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Introduction. In [3], S. Tachibana introduced the notion of a (local) geodesic conformal transformation around a point in a Riemannian manifold M and showed that when M has constant scalar curvature and possesses around each point a non-homothetic geodesic conformal transformation, then M is harmonic (see [2] for the theory of harmonic spaces in this sense). In this note, we show that these conditions also imply that M is locally symmetric (and so the universal covering space of M is symmetric). This is of interest for two reasons: (1) There is a conjecture, still unresolved to the author's knowledge, that a harmonic Riemannian space (Riemannian always means positive-definite metric) is locally symmetric (see [2], p. 231). (2) The harmonic Riemannian spaces which are decomposable are locally flat and the indecomposable harmonic symmetric Riemannian spaces are precisely the rank one symmetric spaces which are completely classified (see [2], pp. 235, 230; for the theory of Riemannian symmetric and locally symmetric spaces, see [1]). In particular, it should now be easy to determine which of these spaces actually possess local geodesic conformal transformations but we shall not pursue this.

Derivation of results. Let M be an $n(>2)$ dimensional connected C^∞ Riemannian manifold with a normal coordinate $(x^1, \cdots, x^n)$ with origin at the point 0 and orthonormal at 0. Let g_{ij}, Γ^k_{ij} be the components of the metric tensor and the Christoffel symbols in this coordinate system. We have

(1) $$g_{ij}x^j = g_{ij}(0)x^j = x^i$$

from which we get

(2) $$\sum_i x^i g^{ik} = x^k .$$

Differentiating (1) with respect to x^k gives

(3) $$\frac{\partial g_{ij}}{\partial x^k} x^j + g_{ik} = \delta_{ik} .$$

Of course

(4) $$\Gamma^k_{ij} = \frac{1}{2} g^{kh}\left(\frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^h} + \frac{\partial g_{jh}}{\partial x^i}\right).$$

Combining these, we get

(5) $$\sum_k \Gamma_{ij}^k x^k = \delta_{ij} - g_{ij} - \frac{1}{2} x^h \frac{\partial g_{ij}}{\partial x^h}.$$

Now introduce the function

(6) $$s = (\sum_i (x^i)^2)^{1/2}$$

and note

(7) $$\frac{\partial s}{\partial x^i} = \frac{x^i}{s} \quad \text{for} \quad s > 0.$$

Now Tachibana considers a function ϕ defined in a punctured neighborhood of 0 and of the form $\phi : x^i \to \rho(s) x^i$. The given Riemannian metric on M is pulled back via ϕ to give a new Riemannian metric g_{ij}^* on this punctured neighborhood. It is assumed that g_{ij}^* is conformally related to g_{ij} by

(8) $$g_{ij}^* = e^\sigma g_{ij}$$

where σ is shown to be a function of s alone. This is what is meant by saying that ϕ is a geodesic conformal transformation and the further condition that ϕ is non-homothetic means that σ' is nowhere zero in some interval $(0, \varepsilon)$.

Assume from now on that M has constant scalar curvature $n(n-1)k$ and possesses a non-homothetic geodesic conformal transformation around each point. Then Tachibana derives the formulas

(9) $$\tau_{ij} = \frac{1}{n} \tau_h^h g_{ij}$$

(10) $$\frac{1}{n} \tau_h^h = \sigma'' - \frac{1}{2} \sigma'^2$$

where essentially τ_{ij} and τ_h^h are defined by

(11) $$\tau_{ij} = \frac{\partial \sigma}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial \sigma}{\partial x^k} - \frac{\partial \sigma}{\partial x^i} \frac{\partial \sigma}{\partial x^j} + \frac{1}{2} \sigma'^2 g_{ij}$$

(12) $$\tau_h^h = g^{ij} \tau_{ij}.$$

We also have, using (7),

(13) $$\frac{\partial \sigma}{\partial x^j} = \sigma' \frac{x^j}{s}$$

(14) $$\frac{\partial \sigma}{\partial x^i \partial x^j} = \sigma'' \frac{x^i}{s} \frac{x^j}{s} + \frac{\sigma'}{s} \delta_{ij} - \frac{\sigma'}{s} \frac{x^i}{s} \frac{x^j}{s}.$$

Using (9), (10), (11), (13) and (14) gives

(15) $$\left(\sigma'' - \sigma'^2 - \frac{\sigma'}{s}\right) \frac{x^i}{s} \frac{x^j}{s} - \frac{\sigma'}{s} \sum_k \Gamma_{ij}^k x^k + \frac{\sigma'}{s} \delta_{ij} = (\sigma'' - \sigma'^2) g_{ij}.$$

Using (4) in (15) gives

$$(16)\qquad \left(\sigma''-\sigma'^2-\frac{\sigma'}{s}\right)\frac{x^i}{s}\frac{x^j}{s}+\frac{1}{2}\sigma'\frac{x^h}{s}\frac{\partial g_{ij}}{\partial x^h}$$
$$=\left(\sigma''-\sigma'^2-\frac{\sigma'}{s}\right)g_{ij}.$$

Now Tachibana also derives the formula

$$(17)\qquad \frac{1}{2}(\log g)'=(n-1)(\sigma''/\sigma'-\sigma'-1/s)$$

where $g=\det(g_{ij})$ is a function of s alone, since M is harmonic. Let $X=a^i\frac{\partial}{\partial x^i}\Big|_0$ be a unit tangent vector at 0 and let γ be the geodesic emanating from 0 with velocity vector X. Then s can be taken as the arclength parameter along γ and γ has the equations $x^i=a^i s$. If we restrict equation (16) to the geodesic γ (treating $g_{ij}(s)\equiv g_{ij}(\gamma(s))$ as a function of s along γ) and use (17), we get

$$(18)\qquad a^i a^j(\log g)'+(n+1)\frac{d}{ds}g_{ij}=(\log g)'g_{ij}$$

for $s>0$ and by continuity also for $s=0$. Then, given the function $g(s)$ and the constants a^i, the function $g_{ij}(s)$ is completely determined by the first order differential equation (18) and the initial condition

$$(19)\qquad g_{ij}(0)=\delta_{ij}.$$

If we do the same for the geodesic corresponding to $-X$, we must replace each a^i by $-a^i$ but we get the same differential equation (18) and the same initial conditions (19). This shows that $g_{ij}(x)=g_{ij}(-x)$ for x sufficiently close to 0 and hence the geodesic symmetry map $x\to -x$ is an isometry at each point 0. This is of course equivalent to saying M is locally symmetric.

REFERENCES

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