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ON MANIFOLDS WITH SASAKIAN 3-STRUCTURE OVER QUATERNION KAEHLER MANIFOLDS

Dedicated to Professor Yūsaku Komatu
on his sixtieth birthday

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§1. Introduction. Any complete Riemannian manifold admitting a regular K-contact 3-structure is a $S^3(1)$ or RP^3-principal bundle over an almost quaternion manifold, where $S^3(1)$ denotes a sphere of curvature 1 and $RP^3=S^3(1)/\{\pm I\}$ (See Tanno [7]). If the contact 3-structure is Sasakian, then the manifold is Einstein space and the base space becomes a quaternion Kaehler manifold with positive scalar curvature.

On the other hand, every quaternion Kaehler manifold M admits a principal bundle P over it, whose structure group is $SO(3)$ (Sakamoto [6]). In this note, we construct in P, 3-structure which is canonically associated with a given quaternion Kaehler structure. That is, we shall prove Theorem 2 in §4, which is corresponding to the theorem for a compact Hodge manifold, i.e.,

THEOREM 1. *Let M be a compact Hodge manifold. Then there exists a circle bundle over M, which admits a normal contact metric structure* (Hatakeyama [1]).

§2. Sasakian 3-structure.

Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold and ξ be a unit Killing vector. Define a tensor field of type (1.1) by

$$\phi=\tilde{\nabla}\xi,$$

where $\tilde{\nabla}$ denotes the Riemannian connection. Then we call ξ a *K-contact structure* if ϕ satisfies

$$\phi^2=-I+\alpha\otimes\xi, \tag{2.1}$$

α being a 1-form defined by $\alpha(\tilde{X})=\tilde{g}(\xi, \tilde{X})$. Next we denote by N the Nijenhuis tensor of ϕ and by Φ the 2-form defined by $\Phi(\tilde{X}, \tilde{Y})=\tilde{g}(\phi\tilde{X}, \tilde{Y})$. If the tensor

$$S=N+2\Phi\otimes\xi$$

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vanishes, we call ξ a *Sasakian structure.*

Next we consider a set of mutually othogonal unit Killing vectors $\{\xi, \eta, \zeta\}$ satisfying

$$[\xi, \eta]=2\zeta, \quad [\eta, \zeta]=2\xi, \quad [\zeta, \xi]=2\eta, \tag{2.2}$$

which is called a *triple of Killing vectors.* We put

$$\phi=\tilde{\nabla}\xi, \quad \psi=\tilde{\nabla}\eta, \quad \theta=\tilde{\nabla}\zeta$$

and

$$\alpha(\tilde{X})=\tilde{g}(\xi, \tilde{X}), \quad \beta(\tilde{X})=\tilde{g}(\eta, \tilde{X}), \quad \gamma(\tilde{X})=g(\zeta, \tilde{X}).$$

If each of ξ, η and ζ is a K-contact structure and satisfies

$$\begin{aligned} &\psi\phi=\theta+\alpha\otimes\eta, \quad \theta\psi=\phi+\beta\otimes\zeta, \quad \phi\theta=\psi+\gamma\otimes\xi, \\ &\phi\psi=-\theta+\beta\otimes\xi, \quad \psi\theta=-\phi+\gamma\otimes\eta, \quad \theta\phi=-\psi+\alpha\otimes\zeta, \end{aligned} \tag{2.3}$$

then $\{\xi, \eta, \zeta\}$ is called a *K-contact 3-structure.* For a K-contact 3-structure, if each of ξ, η and ζ is a Sasakian structure, then $\{\xi, \eta, \zeta\}$ is called a *Sasakian 3-structure.*

§ 3. **Quaternion Kaehler manifold** (See Ishihara [3]).

Let M be a differentiable manifold of dimension $n(=4m)$. Assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over M satisfying the following condition.

a) In any coordinate neighborhood U of M, there is a local base $\{F, G, H\}$ of V such that

$$\begin{aligned} &F^2=-I, \quad G^2=-I, \quad H^2=-I, \\ &HG=-GH=F, \quad FH=-HF=G, \quad GF=-FG=H, \end{aligned} \tag{3.1}$$

I denoting the identity tensor field of type (1.1) in M. Then the bundle V is called an *almost quaternion structure* in M and (M, V) an *almost quaternion manifold.*

In an almost quaternion manifold (M, V), we take two intersecting coordinate neighborhoods U, U', and local bases $\{F_U, G_U, H_U\}$, $\{F_{U'}, G_{U'}, H_{U'}\}$ satisfying (3.1) in U, U', respectively. Then they have relations in $U\cap U'$ as

$$\begin{aligned} F_{U'}&=s_{11}F_U+s_{12}G_U+s_{13}H_U \\ G_{U'}&=s_{21}F_U+s_{22}G_U+s_{23}H_U \\ H_{U'}&=s_{31}F_U+s_{32}G_U+s_{33}H_U \end{aligned} \tag{3.2}$$

where s_{ji} $(j, i=1, 2, 3)$ form an element $s_{UU'}=(s_{ji})$ of the special orthogonal group $SO(3)$ of dimension 3.

Let P be the associated principal bundle of V. That is, P is the bundle whose transition functions and structure group are the same as V, but whose fibre is $SO(3)$ (=the real projective space RP^3 of dimension 3). Then the Lie algebra $\mathfrak{so}(3)$ of the structure group of P admits a base $\{e_1, e_2, e_3\}$ such that

$$e_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 2\\ 0 & -2 & 0\end{pmatrix},\quad e_2=\begin{pmatrix}0 & 0 & -2\\ 0 & 0 & 0\\ 2 & 0 & 0\end{pmatrix},\quad e_3=\begin{pmatrix}0 & -2 & 0\\ 2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and hence they satisfy

$$[e_1, e_2]=2e_3, \quad [e_2, e_3]=2e_1, \quad [e_3, e_1]=2e_2. \tag{3.3}$$

In any almost quaternion manifold (M, V), there is a Riemannian metric such that

$$g(F_UX, Y)+g(X, F_UY)=0, \quad g(G_UX, Y)+g(X, G_UY)=0,$$
$$g(H_UX, Y)+g(X, H_UY)=0$$

hold for any local base $\{F_U, G_U, H_U\}$ and any vector fields X, Y. Assume that the Riemannian connection ∇ of (M, g) satisfies for any local base $\{F_U, G_U, H_U\}$

$$\begin{aligned}\nabla_X F_U&= \qquad\qquad -2r_U(X)G_U-2q_U(X)H_U\\ \nabla_X G_U&=2r_U(X)F_U \qquad\qquad +2p_U(X)H_U\\ \nabla_X H_U&=2q_U(X)F_U-2p_U(X)G_U\end{aligned} \tag{3.4}$$

where p_U, q_U and r_U are certain 1-forms defined in U. Then (M, g, V) is called a *quaternion Kaehler manifold* and (g, V) a *quaternion Kaehler structure.*

For each neighborhood U in a quaternion Kaehler manifold (M, V), we define a $\mathfrak{so}(3)$-valued 1-form on U by

$$\omega_U=p_Ue_1+q_Ue_2+r_Ue_3.$$

Then, by virtue of (3.2) and (3.3), in the intersection of neighborhoods U and U', we find

$$\omega_{U'}(X)=ad(s_{UU'}{}^{-1})\cdot\omega_U(X)+(s_{UU'})_*(X)\cdot s_{UU'}{}^{-1}$$

for every vector field X on P, where ad denotes the adjoint representation of $SO(3)$ in $\mathfrak{so}(3)$, and $(s_{UU'})_*$ denotes the differential of the mapping $s_{UU'}: U\cap U' \to SO(3)$. Hence there exists a connection form ω on P such that

$$\tau^*\omega=\omega_U \tag{3.5}$$

where τ is a certain local cross-section of P over U (for detail, see p. 66 in Kobayashi-Nomizu [4]).

We denote by Ω the curvature form defined by the connection ω. Then Ω is the $\mathfrak{so}(3)$-valued 2-form expressed by

$$\Omega(\tilde{X}, \tilde{Y})=d\omega(\tilde{X}, \tilde{Y})+\frac{1}{2}[\omega(\tilde{X}), \omega(\tilde{Y})]$$

where $\hat{X}$, $\tilde{Y}$ are vector fields in P and $[\ ,\ ]$ denotes the bracket operation in $\mathfrak{so}(3)$. Then we have

$$\tau^*\Omega=(dp_U+q_U\wedge r_U)e_1+(dq_U+r_U\wedge p_U)e_2 + (dr_U+p_U\wedge q_U)e_3 \tag{3.6}$$

for each cross section $\tau: U\to P$ and 1-forms p_U, q_U, r_U on U.

On the other hand, since any quaternion Kaehler manifold is an Einstein space (See Theorem 3.3 in Ishihara [3]), we have following relations

$$dp_U+q_U\wedge r_U=cA_U\,,\qquad dq_U+r_U\wedge p_U=cB_U\,,\qquad dr+p_U\wedge q_U=cC_U\,, \tag{3.7}$$

where $4m(m+2)c$ is a constant equal to the scalar curvature of (M, g), and $A_U(X, Y)=g(F_UX, Y)$, $B_U(X, Y)=g(G_UX, Y)$, $C_U(X, Y)=g(H_UX, Y)$.

§ 4. Construction of Sasakian 3-structure.

Let (M, g) be a quaternion Kaehler manifold of dimension $n=4m$, and P be the associated RP^3-principal bundle over M. We denote by $\omega=\sum_{i=1}^{3}\omega_ie_i$ the infinitesimal connection in P defined in the previous section. We define a pseudo-Riemannian metric g in P by

$$\tilde{g}=c\pi_*g+\sum_{i=1}^{3}\omega_i\otimes\omega_i \tag{4.1}$$

where c is the constant appearing in (3.6). If the scalar curvature of M is positive, then g is a Riemannian metric and if negative, g is a pseudo-Riemannian metric of signature $(3, n)$. In both cases, (M, g) is necessarily irreducible (Ishihara [3]).

We put

$$\omega_1=\alpha\,,\qquad \omega_2=\beta\,,\qquad \omega_3=\gamma\,,$$

then α, β and γ are 1-forms in P. Let ξ, η, ζ be fundamental vector fields corresponding to e_1, e_2, e_3, respectively. Then we have from (3.3)

$$[\xi, \eta]=2\zeta\,,\qquad [\eta, \zeta]=2\xi\,,\qquad [\zeta, \xi]=2\eta\,,$$

and

$$\alpha(\xi)=1\,,\qquad \alpha(\eta)=0\,,\qquad \alpha(\zeta)=0\,,$$
$$\beta(\xi)=0\,,\qquad \beta(\eta)=1\,,\qquad \beta(\zeta)=0\,,$$
$$\gamma(\xi)=0\,,\qquad \gamma(\eta)=0\,,\qquad \gamma(\zeta)=1\,.$$

Hence we have

PROPOSITION 1. *In the associated principal bundle P over a quaternion Kaehler manifold, there exists a triple of Killing vectors $\{\xi, \eta, \zeta\}$ with respect to* *the metric defined by* (4.1), *i.e.* ξ, η *and* ζ *are mutually orthogonal unit Killing vectors satisfying*

$$[\xi, \eta]=2\zeta, \qquad [\eta, \zeta]=2\xi, \qquad [\zeta, \xi]=2\eta, \tag{4.2}$$

Proof. It remains to prove that ξ, η, ζ are all Killing vectors with respect to $\tilde{g}$ in (4.1). This is clear from the fact that $\sum_{i=1}^{3}\omega_i\otimes\omega_i$ is invariant under the action of $SO(3)$.

PROPOSITION 2. *The triple of Killing vectors* $\{\xi, \eta, \zeta\}$ *defined in proposition* 1 *is a K-contact* 3*-structure, if* $c>0$.

Proof. We define

$$\phi=\tilde{\nabla}\xi, \qquad \psi=\tilde{\nabla}\eta, \qquad \theta=\tilde{\nabla}\zeta, \tag{4.3}$$

$\tilde{\nabla}$ being Riemannian connection formed with $\tilde{g}$. Then we have

$$\begin{gathered}\phi\xi=0, \qquad \psi\eta=0, \qquad \theta\zeta=0, \\ \theta\eta=-\psi\zeta=\xi, \qquad \phi\zeta=-\theta\xi=\eta, \qquad \psi\xi=-\phi\eta=\zeta,\end{gathered} \tag{4.3}$$

since ξ, η and ζ are mutually orthogonal unit vectors. Denoting by $T_{\tilde{p}}{}^V(P)$ the the tangent space of a fibre at $\tilde{p}\in P$ and by $T_{\tilde{p}}{}^H(P)$ its orthogonal complemented space in $T_{\tilde{p}}(P)$, we see from (4.4) that $T_{\tilde{p}}{}^V(P)$ and $T_{\tilde{p}}{}^H(P)$ are invariant under the actions of the linear endomorphisms ϕ, ψ acd θ of $T_{\tilde{p}}(P)$. Hence we can put

$$\begin{gathered}\phi=\phi^H+\gamma\otimes\eta-\beta\otimes\zeta, \qquad \psi=\psi^H+\alpha\otimes\zeta-\gamma\otimes\xi, \\ \theta=\theta^H+\beta\otimes\xi-\alpha\otimes\eta,\end{gathered}$$

where ϕ^H, ψ^H and θ^H denote the restricted actions of ϕ, ψ and θ on $T_{\tilde{p}}{}^H(P)$ for each $\tilde{p}$.

On the other hand, taking account of (3.5)$\sim$(3.7), for each neighborhood U in M and a local cross section $\tau: U\to P$, we have

$$\begin{aligned}(d\alpha-\beta\wedge\gamma)(\tau_*X, \tau_*Y)&=cA_U(X, Y), \\ (d\beta-\gamma\wedge\alpha)(\tau_*X, \tau_*Y)&=cB_U(X, Y), \\ (d\gamma-\alpha\wedge\beta)(\tau_*X, \tau_*Y)&=cC_U(X, Y),\end{aligned}$$

τ_* denoting the differential of τ. Since the curvature form is horizontal, we have

$$\begin{aligned}(\phi-\gamma\otimes\eta+\beta\otimes\zeta)(\tau_*X)&=(\tau_*F_UX)^H, \\ (\psi-\alpha\otimes\zeta+\gamma\otimes\xi)(\tau_*X)&=(\tau_*G_UX)^H, \\ (\theta-\beta\otimes\xi+\alpha\otimes\eta)(\tau_*X)&=(\tau_*H_UX)^H,\end{aligned}$$

i. e.

$$\phi(\tau_* X)=(\tau_* F_U X)^H+\gamma(\tau_* X)\eta-\beta(\tau_* X)\zeta\,,$$

$$\psi(\tau_* X)=(\tau_* G_U X)^H+\alpha(\tau_* X)\zeta-\gamma(\tau_* X)\xi\,,$$

$$\theta(\tau_* X)=(\tau_* H_U X)^H+\beta(\tau_* X)\xi-\alpha(\tau_* X)\eta\,,$$

where $(\tau_* F_U X)^H$ denotes the projection of $\tau_* F_U X$ to $T_{\tau(p)}{}^H(P)$. Next we show that ϕ, ψ and θ satisfy (2.1) and (2.3). From (4.4) and (4.5) we have

$$\begin{aligned}\phi^2(\tau_* X)&=\phi((\tau_* F_U X)^H+\gamma(\tau_* X)\eta-\beta(\tau_* X)\zeta)\\&=(\tau_* F_U{}^2 X)^H-\gamma(\tau_* X)\zeta-\beta(\tau_* X)\eta\\&=-(\tau_* X)^H-\beta(\tau_* X)\eta-\gamma(\tau_* X)\zeta\\&=-\tau_* X+\alpha(\tau_* X)\xi\,.\end{aligned}$$

and

$$\begin{aligned}\phi(\psi(\tau_* X))&=\phi((\tau_* G_U X)^H+\alpha(\tau_* X)\zeta-\gamma(\tau_* X)\xi)\\&=(\tau_* F_U G_U X)^H+\alpha(\tau_* X)\eta\\&=-(\tau_* H_U X)^H+\alpha(\tau_* X)\eta\\&=-\theta(\tau_* X)+\beta(\tau_* X)\xi\,,\end{aligned}$$

$$\begin{aligned}\psi(\phi(\tau_* X))&=\psi((\tau_* F_U X)^H+\gamma(\tau_* X)\eta-\beta(\tau_* X)\zeta)\\&=(\tau_* G_U F_U X)^H+\beta(\tau_* X)\xi\\&=(\tau_* H_U X)^H+\beta(\tau_* X)\xi\\&=\theta(\tau_* X)+\alpha(\tau_* X)\eta\end{aligned}$$

because of (3.1). Similarly we obtain the other relations in (2.3). That is, $\{\xi, \eta, \zeta\}$ defines a K-contact 3-structure.

Going through the process of having induced quaternion (Kaehler) structure from regular K-contact (Sasakian) 3-structure (cf. Ishihara [2] and Konishi [5]), our construction of K-contact 3-structure $\{\xi, \eta, \zeta\}$ is quite natural. That is to say, we have obtained a fibred Riemannian space $(P, \tilde{g})$ with K-contact 3-structure $\{\xi, \eta, \zeta\}$ which induces the given quaternion Kaehler structure in the base space. As shown in [5], such a K-contact 3-structure is necessarily a Sasakian 3-structure. Thus we have

THEOREM 2. *Let M be a quaternion Kaehler manifold of dimension $n=4m$. Then there exists a canonically associated RP^3-principal bundle P over M. If the scalar curvature of M is positive, P admits a Sasakian* 3*-structure and if negative, the induced metric by* (4.1) *has signature* (3, *n*).

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