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ON THE LOCAL VERSION OF PU'S PROBLEM

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1. Introduction. Let $\mathfrak{M}=\{(P^n, g)\}$ be the space of riemannian structures on the n-dimensional real projective space P^n. Let $\mathrm{vol}(P^n, g)$ denote the volume of P^n with respect to the canonical measure ν_g derived from g, and $L_g(c)$ denote the length of a closed curve c relative to g. Now we define

$$\mathrm{quot}_1(P^n, g)\equiv \mathrm{vol}(P^n, g)/[\mathrm{Inf}\,\{L_g(c)\,|\,c\,;\ \text{homologically non-trivial piecewise smooth closed curve on } P^n\}]^n.$$

Thus $\mathrm{quot}_1(P^n, g)$ may be considered as a function over $\mathfrak{M}$.

Now Pu's problem states that "$\mathrm{quot}_1(P^n, g)\geqq \mathrm{quot}_1(P^n, g_0)$ holds where g_0 denotes the canonical structure of constant curvature, and the equality holds if and only if (P^n, g) is of constant curvature". That is, the function $\mathrm{quot}_1(P^n, g)$ on $\mathfrak{M}$ takes the minimum value exactly at the riemannian structure of constant curvature. For $n=2$ the problem is solved affirmatively (Pu [6]). But for $n>2$ the problem is completely open. Some authors have tried to solve the problem for some classes of riemannian structures on P^n (i.e. for some subsets of $\mathfrak{M}$) (See I. Chavel [2], [3], [4], P.M. Pu [6], T. Sakai [7]).

$\mathrm{Quot}_1(P^n, g)$ is not a differentiable function on $\mathfrak{M}$. That is, even if $g(t)$ is a differentiable one parameter family of riemannian structures on P^n, generally $\mathrm{quot}_1(P^n, g)$ doesn't depend differentiably on t. In the present note we shall consider the following function f on $\mathfrak{M}$ instead of quot_1.

Let $G=SO(n+1)/SO(n-1)\times SO(2)$ be the Grassmann manifold of all real projective lines (i.e. closed geodesics of length π with respect to the metric g_0 of constant curvature 1) of $P^n=SO(n+1)/SO(n)\times(\pm I_{n+1})$. G is assumed to carry the bi-invariant riemannian structure which is derived from the Killing form of the Lie algebra of $O(n+1)$.

Now we define for (P^n, g)

$$f(P^n, g)\equiv \mathrm{vol}(P^n, g)/\{c(P^n, g)\}^n, \qquad \text{where}$$

$$c(P^n, g)\equiv (1/(\mathrm{vol}\,G))\int_G\Big(\int_0^\pi \sqrt{g(\dot c(s), \dot c(s))}\,ds\Big)\nu_G$$

In the above definition, $\mathrm{vol}\,G$ denote the volume of G with respect to the bi-invariant metric defined above, and $c(s)$ is a closed geodesic in (P^n, g_0) of length

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π which is parametrized by arc length. This function f has been considered in M. Berger ([1]) in a more general setting.

In the first part of the present note we shall consider this function. It is known that f is a differentiable function on $\mathfrak{M}$ and that g_0 of constant curvature is a critical point of f; i.e. $d/dt(f(g_t))_{t=0}=0$ holds for any differentiable one parameter family $g(t)$ with $g(0)=g_0$ in $\mathfrak{M}$ (See Berger [1]). We shall show that conversely any riemannian structure on P^n which is a critical point of f must be a metric of constant curvature.

Thus we have a following characterization of the riemannian structure of constant curvature on P^n.

THEOREM A. *(P^n, g) is a critical point of f on $\mathfrak{M}$ if and only if (P^n, g) is of constant curvature.*

Remark. We have $f(g_0)=\text{quot}_1(P^n, g_0)$ and $f(g)\leqq\text{quot}_1(P^n, g)$ for any $(P^n, g)\in\mathfrak{M}$.

In the second part of the present note we shall treat the generalized Pu's problem which has been proposed by Berger ([1]). Let $K_1=\boldsymbol{R}$, $K_2=\boldsymbol{C}$, $K_4=\boldsymbol{H}$, $K_8=\boldsymbol{Ca}$ be the fields of real numbers, complex numbers, quaternions, and Cayley numbers respectively. Let P_i^a be an a-(K_i)dimensional projective space ($i=1$, 2, 4, 8) and $\mathfrak{M}=\mathfrak{M}_i^a=\{(P_i^a, g)\}$ be the space of riemannain structures on P_i^a. We set for $1\leqq b\leqq a-1$

$$\text{carc}_b(P_i^a, g)=\text{Inf}\,\{\text{vol}\,(Y, g|_Y)|\iota: Y\subset P_i^a \text{ is a } (bi)\text{-dimensional compact orientable submanifold of } P_i^a \text{ and the image of a generator of } H_{bi}(Y) \text{ by } \iota_* \text{ is a non-zero element of } H_{bi}(P_i^a)\}$$

$$\text{carc}'_b(P_i^a, g)=\text{Inf}\,\{\text{vol}\,(Y, g|_Y)|\,Y(\iota\subset P_i^a) \text{ is diffeomorphic to } P_b^i \text{ and } \iota_* \text{ maps a generator of } H_{bi}(Y) \text{ onto a generator of } H_{bi}(P_i^a)\}$$

$$\text{quot}_b(P_i^a, g)=(\text{vol}\,(P_i^a, g))^b/(\text{carc}_b(P_i^a, g))^a$$

$$\text{quot}'_b(P_i^a, g)=(\text{vol}\,(P_i^a, g))^b/(\text{carc}'_b(P_i^a, g))^a\,.$$

Now Berger's problems state as follows;

$I(a, b\,; i)$: $\text{quot}_b(P_i^a, g)\geqq\text{quot}_b(P_i^a, g_0)$ for $^\forall g\in\mathfrak{M}$,

$I'(a, b\,; i)$: $\text{quot}'_b(P_i^a, g)\geqq\text{quot}'_b(P_i^a, g_0)$ for $^\forall g\in\mathfrak{M}$,

$IC(a, b\,; i)$: "$I(a, b\,; i)$" and the equality holds if and only if $g=g_0$,

$IC'(a, b\,; i)$: "$I'(a, b\,; i)$" and the equality holds if and only if $g=g_0$,

where g_0 denotes the canonical riemannain structure of symmetric space of rank one on P_i^a.

In this general case, we shall consider the following function $f=f_i^{a,b}$ in stead of $\text{quot}_b(P_i^a, g)$. Let $G=G_i^{a,b}$ be a Grassmann manifold of b-dimensional projective subspaces of P_i^a. G carries a riemannian structure of symmetric space.

Now we define following Berger ([1]),

$$c(P_i^a, g) \equiv (\mathrm{vol}\, G)^{-1} \int_{Y \in G} \mathrm{vol}\,(Y, g|_Y) \nu_G$$

$$f(P_i^a, g) \equiv (\mathrm{vol}\,(P_i^a, g))^b / (c(P_i^a, g))^a .$$

Then f is a "differentiable function" on $\mathfrak{M}$, and we have $f(P_i^a, g_0) = \mathrm{quot}_b(P_i^a, g_0)$ and $f(P_i^a, g) \leqq \mathrm{quot}'_b(P_i^a, g) \leqq \mathrm{quot}_b(P_i^a, g)$. It is known that g_0 (the canonical riemannian structure of symmetric space of rank one) is a critical point of f; i.e. $d/dt(f(g(t))|_{t=0} = 0$ for and differentiable one parameter family $g(t)$ of riemannain structures on P_i^a with $g(0) = g_0$.

In the present note we shall calculate the second variation $d^2/dt^2(f(g(t)))|_{t=0}$ at g_0 and consider the problem; "For what $g(t)$, $d^2/dt^2(f(g(t)))_{t=0} > 0$ holds?". This implies the following result concerned with Berger's problem.

THEOREM B. *Let $(P_i^a, g(t))$ be a differentiable one parameter family of riemannain structures on P_i^a such that $g(0) = g_0$ is the canonical metric on P_i^a and $g'(0) = \lambda g_0$ holds, where λ is a not constant function on P_i^a. Then there exists a positive number ε such that*

$$\mathrm{quot}_b(P_i^a, g(t)) \geqq \mathrm{quot}'_b(P_i^a, g(t)) > \mathrm{quot}_b(P_i^a, g_0)$$

holds for any $0 < |t| < \varepsilon$.

Remark. For $b = i = 1$, Chavel ([3]) has proved that for any $g(t)$ such that $g(0) = g_0$ and $g'(0)$ is not a constant multiple of g_0, there exists a positive number ε such that $\mathrm{quot}_1(P^n, g(t)) > \mathrm{quot}_1(P^n, g_0)$ holds for $0 < |t| < \varepsilon$. But his proof depends on a theorem of Michel ([5]) which is valid only for real projective space. So Chavel's method seems to be not valid for this general case.

Remark. If (P_i^a, g) is conformally related to the canonical structure, then it is known that $\mathrm{quot}_b(P_i^a, g) \geqq \mathrm{quot}_b(P_i^a, g_0)$, where the equality holds if and only if $g = g_0$ up to the homothety. Since the condition "$g'(0) = \lambda g_0$" means that $g(t)$ is conformally related to g_0 up to the first order, the theorem may be considered as the local version of the above result. Of course there are many $g(t)$ which satisfy $g'(0) = \lambda g_0$ but are not conformally related to g_0.

2. *Proof of Theorem* A. First we shall sum up the formulas which are used in the sequel.

LEMMA 1. *Let M be a compact C^∞-manifold and $g(t)$ be a differentiable one parameter family of riemannian structures on M. Then we have,*

$$\{\mathrm{vol}\,(M, g(t))\}' = 1/2 \int_M \mathrm{trace}_{g(t)} g'(t) \nu_{g(t)}, \tag{2.1}$$

where $\mathrm{trace}_{g(t)} g'(t)$ means the trace of the symmetric covariant 2-tensor $g'(t)$ with respect to $g(t)$, i.e. $\mathrm{trace}_{g(t)} g'(t) = g^{ij}(t) g'_{ij}(t)$.

Proof of Lemma 1). This is well known. In fact we have

$$\{\nu_{g(t)}\}'=1/2\,\mathrm{trace}_{g(t)}g'(t) \qquad \text{(See Berger [1])}.$$

LEMMA 2. *Let (M, g) be a compact riemannain manifold of dimension n, and $p: UM\to M$ be the unit tangent bundle. Then for any symmetric covariant 2-tensor h on M we have*

$$\int_M \mathrm{trace}_g h\nu_g=\frac{n}{\omega_{n-1}}\int_{m\in M}\Big\{\int_{p^{-1}(m)} h(x, x)\nu_{p^{-1}(m)}\Big\}\nu_g\,, \tag{2.2}$$

where ω_{n-1} denotes the volume of the unit sphere and $\nu_{p^{-1}(m)}$ denotes the measure of the U_mM derived from g.

Proof of Lemma 2). This is also standard. For every $m\in M$, choose an orthonormal frame with respect to which h takes the form

$$h(x, x)=\lambda_1x_1^2+\lambda_2x_2^2+\cdots+\lambda_nx_n^2.$$

Then

$$\int_{p^{-1}(m)}h(x, x)\nu_{p^{-1}(m)}=\sum_{i=1}^n\lambda_i\int_{x_1^2+\cdots+x_n^2=1}x_i^2dS^{n-1}=\frac{\omega_{n-1}}{n}\mathrm{trace}_g h\,.$$

Let (P^n, g_0) denotes the n-dimensional real projective space with the canonical riemannian structure of constant curvature 1. Then the unit tangent bundle UP^n has the natural induced riemannian structure h_0 derived from g_0 (See the proof of Lemma 3). On the other hand let G be the Grassmann manifold of real projective lines (i. e. closed geodesic of length π in (P^n, g_0)) of P^n. We endow with G the canonical riemannian structure k_0 of symmetric space. To every unit tangent vector $x\in UP^n$, we assign $q(x)\in G$ which is the closed geodesic of (P^n, g_0) with the initial direction x. Then we have the bundle structure $q: UP\to G$. In the above notation we have

LEMMA 3. *$p: (UP^n, h_0)\to(P^n, g_0)$ and $q: (UP^n, h_0)\to(G, k_0)$ are riemannian submersions. The fiber $p^{-1}(m)$, $m\in M$ is an $(n-1)$-dimensional sphere of constant curvature 1 and the fiber $q^{-1}(c)$, $c\in G$ is a closed geodesic of length π in (UP^n, h_0), where $q^{-1}(c)$ may be identified with real projective line c.*

Proof of Lemma 3). This follows from the following representations of (P^n, g_0), (UP^n, h_0), (G, h_0) as the riemannain homogeneous spaces:

$$(P^n, g_0)=SO(n+1)/SO(n)\times\{\pm I_{n+1}\}$$

$$(UP^n, h_0)=SO(n+1)/SO(n-1)\times\{\pm I_{n+1}\}$$

$$(G, k_0)=SO(n+1)/SO(n-1)\times SO(2)\,,$$

where these homogeneous spaces are assumed to carry the bi-invariant normal homogeneous riemannian structures which are derived from the Killing form of the Lie algebra of $O(n+1)$. Then p, q are riemannian submersions and we have $p^{-1}(m)\cong SO(n)/SO(n-1)$ and $q^{-1}(c)\cong SO(2)/\{\pm I\}$.

LEMMA 4. *Let $a_1, \cdots, a_n$ be positive real numbers. We put*

$$A_i=\int_{x_1^2+\cdots+x_n^2=1}\frac{a_i x_i^2}{\sqrt{a_1x_1^2+\cdots+a_nx_n^2}}dS^{n-1} \text{ (not summed with respect to } i).$$

If $A_1=\cdots=A_n$ holds, we have $a_1=\cdots=a_n$.

Proof of Lemma 4). By an elementary calculus we get

$$A_1-A_2=(a_1-a_2)\int_{S^{n-1}}(\text{positive function on } S^{n-1})dS^{n-1}.$$

Now we shall return to the proof of Theorem A. We put $g(0)=g$, $g'(0)=h$ and assume that g is a critical point of f. Then for any digerentiable one parameter family $g(t)$ in $\mathfrak{M}$ with $g(0)=g$, we have by (2.1)

$$(2.3)\qquad d/dt(f(g(t)))|_{t=0}=(1/(2\operatorname{vol}G))\{c(g)\}^{-n-1}\Big[\int_G\Big\{\int_0^\pi\|\dot c(s)\|_g ds\Big\}\nu_G$$

$$\int_{P^n}\operatorname{trace}_g h\nu_g-n\operatorname{vol}(P^n,g)\int_G\Big\{\int_0^\pi\frac{h(\dot c(s),\dot c(s))}{\|\dot c(s)\|_g}ds\Big\}\nu_G\Big]=0.$$

Let $S^2(M)$ denote the space of symmetric covariant 2-tensor fields on M. So g is a critical point of f if and only if

$$\frac{\int_G\Big\{\int_0^\pi\frac{h(\dot c(s),\dot c(s))}{\|\dot c(s)\|_g}ds\Big\}\nu_G}{\int_{pn}\operatorname{trace}_g h\nu_g}=\text{constant}$$

for any $h\in S^2(P^n)$. On the other hand by (2.2) and Lemma 3, using the integration along the fiber of riemannian submersion the left hand side of (2.4) takes the form

$$\frac{\omega_{n-1}\int_{Pn}\Big\{\int_{U_m^{(g_0)}P^n}\frac{h(x,x)}{\|x\|_g}\nu_{p-1_{(m)}}\Big\}\nu_{g_0}}{n\int_{Pn}\Big\{\int_{U_m^{(g)}P^n}h(x,x)\nu_{p-1_{(m)}}\Big\}\nu_g},$$

where $U_m^{(g)}P^n$ denote the space of unit tangent vectors at m with respect to the metric g. Note that $\nu_g(m)=\sqrt{\det(g_{ij}(m))/\det((g_0)_{ij}(m))}\,\nu_{g_0}(m)$ holds. So if g is a critical point of f, then there exists a C^∞-function $k(m)$ on P^n such that

$$(2.5)\qquad \int_{Pn}\Big\{\int_{U_m^{(g_0)}P^n}\frac{h(x,x)}{\|x\|_g}\nu_{p-1(m)}-k(m)\int_{U_m^{(g)}P^n}h(x,x)\nu_{p-1(m)}\Big\}\nu_{g_0}=0$$

holds for any $h\in S^2(P^n)$. Then at every point $m\in P^n$, we have

$$(2.6)\qquad \alpha_h(m)\equiv\int_{x\in U_m^{(g_0)}P^n}\frac{h_m(x,x)}{\|x\|_{g(m)}}\nu_{p-1(m)}-k(m)\int_{U_m^{(g)}P^n}h_m(x,x)\nu_{p-1(m)}=0$$

for any $h\in S^2(P^n)$. In fact, assume that (2.6) is not satisfied for some $m\in P^n$ and $h\in S^2(M)$. Then we may assume that $m\to\alpha_h(m)$ is positive on some neighborhood U of m. Choose a C^{∞}-function φ ($\varphi\geqq 0$) on P^n such that $\varphi=1$ on some $V(\subset U)$ and $\varphi=0$ outside of U. Then $\int_{P^n}\varphi\alpha_h\nu_{g_0}=\int_{P^n}\alpha_{\varphi h}\nu_{g_0}>0$. This contradicts (2.5). Now take an orthonormal frame relative to g_0 in T_mP^n so that g takes the form

$$g(x, x)=a_1x_1^2+\cdots+a_nx_n^2 \qquad (a_1, \cdots, a_n>0)\,,$$

where x_i's are the components of x with respect to this orthonormal basis. In (2.6) take especially $h(x, x)=x_i^2$. So we have

$$\int_{x_1^2+\cdots+x_n^2=1}\frac{x_i^2}{\sqrt{a_1x_1^2+\cdots+a_nx_n^2}}\,dS^{n-1}=k(m)\int_{y_1^2+\cdots+y_n^2=1}\frac{y_i^2}{a_i}\,dS^{n-1}$$

where we have put $y_i^2=a_ix_i^2$ (not summed). That is,

$$A_i=\int_{x_1^2+\cdots+x_n^2=1}\frac{a_ix_i^2}{\sqrt{a_1x_1^2+\cdots+a_nx_n^2}}\,dS^{n-1}=k(m)\omega_{n-1}/n \quad (=\text{constant})\,.$$

So by Lemma 4 we have $a_1=\cdots=a_n$, that is, g $(=ag_0)$ is conformally related to the canonical riemannian structure on P^n. Finally we show that this positive C^{∞}-function a on P^n must reduce to a constant. Let $h=\varphi g_0$ where φ is any C^{∞}-function, then from (2.4) we have

$$\int_{P^n}\Big\{\Big(\int_{P^n}a^n\nu_{g_0}\Big)\frac{1}{a}-a^{n-2}\Big(\int_{P^n}a\nu_{g_0}\Big)\Big\}\varphi\nu_{g_0}=0$$

and consequently a must be a constant. q. e. d.

3. *Proof of Theorem* B. Let $g=g_0$ be the riemannian structure of symmetric space of rank one on P_i^a. Let $g(t)$ be any differentiable one parameter family of riemannian structures on P_i^a with $g(0)=g_0$. Then it is known that $d/dt(f(g(t)))|_{t=0}=0$. But g_0 is never a minimum value of f. In fact, φ_t be a one parameter family of diffeomorphism of P_i^a and set $g(t)=\varphi_t^*g_0$. Then we have $f(g(t))\geqq f(g_0)$. Now we shall calculate the second variation of f at g, i. e. $d^2/dt^2(f(g(t)))|_{t=0}$. The usefull tool is the integration along the fiber of the following two riemannian submersions. Let $V_i^{a,b}$ be the set of K_i-subspaces of real dimension bi (tangent spaces to the b-dimensional projective subspaces of P_i^a) and $p: V_i^{a,b}\to P_i^a$ be the map which associate to $V\in V_i^{a,b}$ the point $m\in P_i^a$ such that $V\subset T_mP_i^a$. Let $q: V_i^{a,b}\to G$ be the map which associate to $V\in V_i^{a,b}$ the K_i-projective subspace $Y(\in G)$ tangent to V at $p(V)$. Then there are natural riemannian structures h_0, k_0 on $V_i^{a,b}$ and G respectively such that $p:(V_i^{a,b}, h_0)\to(P_i^a, g_0)$, $q:(V_i^{a,b}, h_0)\to(G, k_0)$ are riemannian submersions (For the proof see Berger [1] pp. 26-31). In the following we put $g'(0)=h$, $g''(0)=k$, $\operatorname{vol}P=\operatorname{vol}(P_i^a, g_0)$, and $\operatorname{vol}P'=\operatorname{vol}(P_i^b, g_0)$. Then by Lemma 1 (See the proof of Lemma 1) we get

(3.1) $$\{\mathrm{vol}\,(Y, g(t)_{|Y})\}''|_{t=0}=-1/2\langle h, h\rangle_{P_i^a}+1/2\int_{P_i^a}\mathrm{trace}_g k\nu_g$$
$$+1/4\int_{P_i^a}(\mathrm{trace}_g h)^2\nu_g\,,$$

where $\langle\,,\rangle_{P_i^a}$ denotes the global inner product by the integration.

(3.2) $$\mathrm{vol}\,G\{c(g(t))\}''|_{t=0}=-1/2\int_{Y\in G}\langle h_{|Y}, h_{|Y}\rangle_Y\nu_G$$
$$+1/2\frac{b}{a}\,\frac{\mathrm{vol}\,P'\cdot\mathrm{vol}\,G}{\mathrm{vol}\,P}\int_{P_i^a}\mathrm{trace}_g k\nu_g+\frac{1}{4}\int_G\Big\{\int_Y(\mathrm{trace}_{g|Y}k_{|Y})^2\nu_{g|Y}\Big\}\nu_G\,.$$

From (3.1) and (3.2) we have

(3.3) $$(\mathrm{vol}\,P)^{-b+2}(\mathrm{vol}\,P')^{a+2}d^2/dt^2(f(g(t)))|_{t=0}$$
$$=-\{b(a-b)/4a\}(\mathrm{vol}\,P')^2\Big\{\int_{P_i^a}\mathrm{trace}_g h\nu_g\Big\}^2+(b\,\mathrm{vol}\,P(\mathrm{vol}\,P')^2/4)$$
$$\times\int_{P_i^a}(\mathrm{trace}_g h)^2\nu_g-b\,\mathrm{vol}\,P(\mathrm{vol}\,P')^2/2\cdot\langle h, h\rangle_{P_i^a}$$
$$+a(\mathrm{vol}\,P)^2\,\mathrm{vol}\,P'/(2\,\mathrm{vol}\,G)\int_{Y\in G}\langle h_{|Y}, h_{|Y}\rangle_Y\nu_G$$
$$-a(\mathrm{vol}\,P)^2\,\mathrm{vol}\,P'/(4\,\mathrm{vol}\,G)\int_{Y\in G}\nu_G\int_Y(\mathrm{trace}_{g|Y}h_{|Y})^2\nu_{g|Y}\,.$$

There are many $h=g'(0)$ which makes (3.3) negative. But such an h doesn't give a counter example to Berger's problem because $f(g)\geqq\mathrm{quot}_b(P_i^a, g)$ holds. On the other hand it seems interesting to find the class of $g'(0)$ which make (3.3) positive, because $f(g(t))$ takes the strictly minimal value at $t=0$ for such $g'(0)$.

Next we shall give an example of such $g'(0)$. We assume that $h=g'(0)=\lambda g_0$ holds, where λ is any C^∞-function over P_i^a. In this case (3.3) takes the form

(3.4) $$(\mathrm{vol}\,P)^{-b+2}(\mathrm{vol}\,P)^{a+2}\,d^2/dt^2(f(g(t)))|_{t=0}$$
$$=-\{ab(a-b)i^2\}/4a\cdot(\mathrm{vol}\,P')^2\Big(\int_{P_i^a}\lambda\nu_g\Big)^2+(a^2bi^2)/4\cdot\mathrm{vol}\,P$$
$$\times(\mathrm{vol}\,P')^2\Big(\int_{P_i^a}\lambda\nu_g\Big)-abi/2\cdot\mathrm{vol}\,P(\mathrm{vol}\,P')^2\Big(\int_{P_i^a}\lambda^2\nu_g\Big)$$
$$+abi/2\cdot\mathrm{vol}\,P(\mathrm{vol}\,P')^2\Big(\int_{P_i^a}\lambda^2\nu_g\Big)-(ab^2i^2)/4\cdot\mathrm{vol}\,P(\mathrm{vol}\,P')^2\Big(\int_{P_i^a}\lambda^2\nu_g\Big)$$
$$=\{ab(a-b)i^2\}/4\,(\mathrm{vol}\,P')^2\Big\{\mathrm{vol}\,P\int_{P_i^a}\lambda^2\nu_g-\Big(\int_{P_i^a}\lambda\nu_g\Big)^2\Big\}\geqq0\,,$$

by virtue of Cauchy-Schwarz inequality, and equality holds if and only if λ is a constant function. This completes the proof of Theorem B.

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