A NOTE ON MITCHELL'S FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS

By Masayoshi Eguchi and Yōichi Kijima

1. Introduction.

Let X be a nonempty compact convex subset of a Banach space, and let S be a semigroup of nonexpansive mappings of X info itself. Mitchell [3] proved that if S is left reversible, then X contains a common fixed point of S. In this note, we shall show that Mitchell's theorem can be further extended.

2. Fixed point theorem.

Let $C(X)$ be the Banach space of all continuous real functions on X, and let $B(S)$ be the Banach space of all bounded real functions on S. For each $x \in X$ and $f \in C(X)$, we define $x \otimes f \in B(S)$ by $(x \otimes f)(s) = f(sx)$. Let $L(S)$ be the linear span of $\{x \otimes f : x \in X, f \in C(X)\}$ in $B(S)$. A linear functional m on $L(S)$ is called a left invariant mean on $L(S)$ if $\|m\| = m(1) = 1$, $m(\varphi) \geq 0$ whenever $\varphi \geq 0$, and $m(_s\varphi) = m(\varphi)$ for every $s \in S$ and $\varphi \in L(S)$ where $_s\varphi$ denotes the left translation of φ by s.

Takahashi [5] proved that if there exists a left invariant mean on $B(S)$, then X contains a common fixed point of S. The following is an extension of Takahashi's theorem.

THEOREM 1. *If there exists a left invariant mean on $L(S)$, then X contains a common fixed point of S.*

Proof. Let K be a minimal S-invariant nonempty compact convex subset of X, and let M be a minimal S-invariant nonempty compact subset of K. Let x_0 be an element in M, and let m be a left invariant mean on $L(S)$. We define $\mu \in C(X)^*$ by $\mu(f) = m(x_0 \otimes f)$, then $\|\mu\| = \mu(1) = 1$, $\mu \geq 0$, and $\mu(f \circ s) = \mu(f)$ for every $s \in S$ and $f \in C(X)$ where $(f \circ s)(x) = f(sx)$.

Let F be the support of μ, that is, the unique nonempty closed subset of X such that $f \geq 0$ vanishes on F if and only if $\mu(f) = 0$. It is easy to see that $F \subset M$. We shall show that F is S-invariant. Suppose, on the contrary, that $sx \in M - F$ for some $x \in F$ and $s \in S$. Then there exists $f \geq 0$ such that $f(sx) > 0$ and $f(F) = 0$. Since $x \in F$ and $(f \circ s)(x) = f(sx) > 0$, we have $\mu(f \circ s) > 0$. But this

contradicts $\mu(f \circ s) = \mu(f) = 0$. We conclude from minimality of M that $F = M$.

We shall show that $sM = M$ for all $s \in S$. Suppose, on the contrary, that $x \in M - sM$ for some $x \in M$ and $s \in S$. Then there exists $f \geq 0$ such that $f(x) > 0$ and $f(sM) = 0$. Since $x \in M (= F)$ and $f(x) > 0$, we have $\mu(f) > 0$. But $f(sM) = 0$ implies that $\mu(f) = \mu(f \circ s) = 0$. This is a contradiction. We have shown that $sM = M$ for all $s \in S$. The remainder of the proof is due to De Marr [2].

THEOREM 2. *If S is left reversible, then there exists a left invariant mean on $L(S)$.*

Proof. Let Σ be the semigroup of all nonexpansive mappings of X into itself. For each pair of s and t in Σ, we define $d(s, t) = \sup \{ \|sx - tx\| : x \in X \}$, then d is a compact metric on Σ (see Takahashi [6]). The inequalities $d(s_1 t, s_2 t) \leq d(s_1, s_2)$ and $d(st_1, st_2) \leq d(t_1, t_2)$ hold, and they imply that the multiplication in Σ is jointly continuous. Thus Σ can be considered as a compact topological semigroup.

Let $\bar{S}$ be the closure of S in Σ, then $\bar{S}$ is also left reversible. By Rosen [4], there exists a left invariant mean $\bar{m}$ on $C(\bar{S})$ where $C(\bar{S})$ denotes the Banach space of all continuous real functions on $\bar{S}$. Since each $\varphi \in L(S)$ is uniformly continuous on S with respect to the metric d, φ is uniquely extended to $\bar{\varphi} \in C(\bar{S})$. Let $m(\varphi) = \bar{m}(\bar{\varphi})$, then m is a left invariant mean on $L(S)$.

REFERENCES

[1] DAY, M. M., Amenable semigroup. Illinois J. Math. 1 (1957), 509–544.
[2] DE MARR, R., Common fixed points for commuting contraction mappings. Pacific J. Math. 13 (1963), 1139–1141.
[3] MITCHELL, T., Fixed points of reversible semigroups of nonexpansive mappings. Kōdai Math. Sem. Rep. 21 (1970), 322–323.
[4] ROSEN, W. G., On invariant means over compact semigroups. Proc. Amer. Math. Soc. 7 (1956), 1076–1082.
[5] TAKAHASHI, W., Fixed point theorem for amenable semigroups of nonexpansive mappings. Kōdai Math. Sem. Rep. 21 (1969), 383–386.
[6] TAKAHASHI, W., A convexity in metric space and nonexpansive mappings, I. Kōdai Math. Sem. Rep. 22 (1970), 142–149.

CENTRAL RESEARCH INSTITUTE OF ELECTRIC POWER INDUSTRY, AND
TOKYO INSTITUTE OF TECHNOLOGY.