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ON AN ISOMETRY OF RIEMANNIAN MANIFOLDS OF NEGATIVE CURVATURE

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Let M be an $n(\geqq 2)$-dimensional connected complete Riemannian manifold. We say that a continuous function $f: M\to R$ is convex if its restriction to any geodesic of M is convex and a nonempty subset A of M is totally convex if it contains every geodesic segment of M whose endpoints are in A. The following facts were proved by Bishop and O'Neill [1].

Fact 1. Let f be a convex function on M. Then, for each number c, the set $M^c=\{m\in M;\ f(m)\leqq c\}$ is totally convex.

Fact 2. Supposing that M is simply connected and of nonpositive sectional curvature, let φ be a fixed-point-free isometry of M. Then $d(p, \varphi(p))$, $p\in M$, is a convex function on M and it has no minimum if and only if no geodesic of M is translated by φ, where d is the distance function of M.

In this note we will obtain another condition that $d(p, \varphi(p))$, $p\in M$, has no minimum when $\dim M=2$. In the following, let M be an $n(\geqq 2)$-*dimensional simply connected complete Riemannian manifold of nonpositive sectional curvature.*

As is well known, for any two points p, q of M there exists a unique geodesic segment from p to q. Let $\sigma: [0, 1]\to M$ be the geodesic segment such that $\sigma(0)=p$ and $\sigma(1)=q$, which we denote by $\overline{p, q}$. First of all, we shall show the following

PROPOSITION 1. *Let φ be a fixed-point-free isometry of M. Then, for any positive integer k, $\varphi^k=\underbrace{\varphi\circ\cdots\circ\varphi}_{k}$ is also fixed-point-free.*

Proof. Suppose that φ^2 has a fixed point $p\in M$. Then φ must fix the middle point of the geodesic segment $\overline{p, \varphi(p)}$ but this contradicts the assumption for φ. Hence φ^2 is fixed-point-free. Now, suppose that $k\geqq 3$ and φ^i, $1\leqq i\leqq k-1$, is fixed-point-free and φ^k has a fixed point $p\in M$. We consider a closed ball $B=B(p, r)=\{q\in M;\ d(p, q)\leqq r\}$ such that B contains the set $\{p, \varphi(p), \cdots, \varphi^{k-1}(p)\}$. Then $d(p, q)$, $q\in M$, is a convex function on M [1]. By virtue of Fact 1 the closed ball B is totally convex, so that geodesic segments $\overline{\varphi^i(p), \varphi^{i+1}(p)}$, $1\leqq i\leqq k-1$, are contained in B. Now we consider the subset $K:=\{q\in B;\ \varphi^j(q)\in B, j=1, 2, \cdots\}$ of B. Then we see that K is nonempty and compact and for each point $q\in K$, $\overline{q, \varphi(q)}\subset K$. Restricting $f(q)=d(q, \varphi(q))$, $q\in M$, to K, it attains

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its minimum at a point $q_0 \in K$. Since φ^2 is a fixed-point-free, $\overline{q_0, \varphi(q_0)}$ and $\overline{\varphi(q_0), \varphi^2(q_0)}$ do not overlap each other. Now we shall show that the angle between $\overline{\varphi(q_0), q_0}$ and $\overline{\varphi(q_0), \varphi^2(q_0)}$ is π. In fact, suppose that it is less than π. Let q_1 is an interior point of $\overline{q_0, \varphi(q_0)}$, then $q_1 \in K$ and we have

$$\begin{aligned} d(q_1, \varphi(q_1)) &< d(q_1, \varphi(q_0)) + d(\varphi(q_0), \varphi(q_1)) \\ &= d(q_1, \varphi(q_0)) + d(q_0, q_1) = d(q_0, \varphi(q_0)), \end{aligned}$$

which contradicts the supposition that $f|K$ takes its minimum at q_0. Thus three points q_0, $\varphi(q_0)$, $\varphi^2(q_0)$, in this order, are on the geodesic σ passing through q_0 and $\varphi(q_0)$, so that φ translates σ. Since any geodesic ray of M diverges, $\varphi^j(q_0) \in M - B$ for a sufficiently large positive integer j. This is a contradiction since $q_0 \in K$. Therefore, by the induction, φ^k must be fixed-point-free.

Using the same way as Proposition 1, we can prove the following.

COROLLARY. *In Proposition* 1, *for each point* $p \in M$, *the sequence* $\{d(p, \varphi^k(p))\}$, $k \in N$, *is unbounded.*

For any geodesic segment σ of M, we denote by σ^* the geodesic extention of σ in the both sides.

LEMMA 1. *Under the same assumption as Proposition* 1, *if φ does not translate any geodesic of M, then we have the following: For each point p of M,*

$$\begin{array}{llll} p \notin \varphi\tau^*, & p \notin \varphi^2\sigma^*, & \varphi(p) \notin \tau^*, & \varphi(p) \notin \varphi^2\sigma^*, \\ \varphi^2(p) \notin \sigma^*, & \varphi^2(p) \notin \varphi\tau^*, & \varphi^3(p) \notin \sigma^*, & \varphi^3(p) \notin \tau^*, \end{array}$$

where σ, τ are the geodesic segments $\overline{p, \varphi(p)}$ and $\overline{p, \varphi^2(p)}$, respectively.

Proof. We shall show $p \notin \varphi\tau^*$. Suppose that $p \in \varphi\tau^*$. Then we easly see that $\sigma = \overline{p, \varphi(p)}$ is contained in $\varphi\tau^*$. Hence exactly one of the following holds:

$$(1)\ \ \varphi(p) \in \overline{p, \varphi^3(p)} \qquad (2)\ \ p \in \overline{\varphi(p), \varphi^3(p)} \qquad (3)\ \ \varphi^3(p) \in \overline{\varphi(p), p}\,.$$

In the case (1), considering the geodesic triangle $\Delta(p, \varphi^2(p), \varphi^3(p))$, we have

$$\begin{aligned} d(p, \varphi^3(p)) &= d(p, \varphi(p)) + d(\varphi(p), \varphi^3(p)) \\ &= d(\varphi^2(p), \varphi^3(p)) + d(p, \varphi^2(p)), \end{aligned}$$

which implies $\varphi^2(p) \in \overline{p, \varphi^3(p)}$. Then either $\varphi^2(p) \in \overline{p, \varphi(p)}$ or $\varphi^2(p) \in \overline{\varphi(p), \varphi^3(p)}$ holds. In the former case, it is clear that $\varphi^2(p) = p$ must hold. This contradicts Proposition 1. In the latter case, φ translates $\varphi\tau^*$, which contradicts the assumption for φ. Hence the case (1) never arise. We can also prove the cases (2), (3) never arise by the same way. Thus we have $p \notin \varphi\tau^*$. We can also prove the other facts similarly.

PROPOSITION 2. *In Proposition* 1, *if* $\dim M=2$ *and* φ *is orientation preserving, then the following conditions* (a), (b) *are equivalent:*

(a) *any geodesic of* M *is not translated by* φ.

(b) *for each point* p *of* M, $\overline{p, \varphi^2(p)}$ *and* $\overline{\varphi(p), \varphi^3(p)}$ *or* $\overline{p, \varphi(p)}$ *and* $\overline{\varphi^2(p), \varphi^3(p)}$ *intersect at an interior point of these geodesic segments.*

Proof. We shall deduce (b) from (a). Suppose that there exists a point p of M such that (b) does not hold for p. By Proposition 1, four points $p, \varphi(p)$, $\varphi^2(p)$, and $\varphi^3(p)$ are all distinct and by Lemma 1, above any three points are not on a same geodesic. Note that M is homeomorphic to R^2. Since φ is orientation preserving, the following two cases are possible:

(1) $\varphi^3(p)$ is in the geodesic triangle $\Delta(p, \varphi(p), \varphi^2(p))$.

(2) p is in the geodesic triangle $\Delta(\varphi(p), \varphi^2(p), \varphi^3(p))$.

Then $\varphi(\Delta(p, \varphi(p), \varphi^2(p))=\Delta(\varphi(p), \varphi^2(p), \varphi^3(p))$. In the case (1) since $\Delta(\varphi(p), \varphi^2(p), \varphi^3(p))\subset\Delta(p, \varphi(p), \varphi^2(p))$, it contradicts that φ is an isometry. In the case (2), we get also a contradiction. The converse is clear.

REMARK. In Proposition 2, the curvature of M is not zero identically.

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