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ON THE TOTAL CURVATURE OF NONCOMPACT RIEMANNIAN MANIFOLDS

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Let M be a 2-dimensional complete connected noncompact Riemannian manifold with positive Gaussian curvature K. Then Cohn-Vossen proved in [2] that M is diffeomorphic to a 2-dimensional Euclidean spase E^2 and its total curvature satisfies

$$(*) \qquad \iint_M K\,dv \leqq 2\pi\,,$$

where dv is the area element of M. The purpose of this paper is to show the inequality (*) is still true for manifolds of nonnegative Gaussian curvature. That is,

THEOREM. *Let M be a 2-dimensional complete connected noncompact Riemannian manifold with nonnegative Gaussian curvature K. Then*

$$\iint_M K\,dv \leqq 2\pi\,.$$

The auther dose not know whether this Theorem had been proved by anyone or not.

Throughout this paper, let M be a complete connected Riemannian manifold and every geodesic parametrized with respect to arc length. A geodesic $c:[0, \infty)\to M$ (or $(-\infty, \infty)$) is called a ray (or a line) if each segment of c is minimal. d denotes the metric distance of M. A subset A of M will be called totally convex if for any $p, q\in A$ and any geodesic $c:[0, s]\to M$ from p to q, we have $c([0, s])\subset A$. Let C be a non-empty closed totally convex subset of M. Then C is an imbedded topological submanifold of M with totally geodesic interior and possibly nonsmooth boundary ∂C, which might be empty, see [1]. Let M be a noncompact manifold of nonnegative sectional curvature. Then the following facts were also proved in [1]. Let C be a closed totally convex subset of M. If $\partial C\neq\phi$, we set

$$C^a := \{p\in C : d(p, \partial C)\geqq a\}\,,$$

$$C^{\max} = \bigcap_{C^a\neq\phi} C^a\,.$$

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Then for any $a \geqq 0$, C^a is totally convex and therefore $C^{\max}$ is totally convex and $\dim C^{\max} < \dim C$. For any $p \in M$, there exists a family of compact totally convex subsets C_t, $t \geqq 0$ of M such that

1) $t_1 \leqq t_2$ implies $C_{t_1} \subset C_{t_2}$ and $C_{t_1} = \{q \in C_{t_2};\ d(q, \partial C_{t_2}) \geqq t_2 - t_1\}$, in particular, $\partial C_{t_1} = \{q \in C_{t_2} : d(q, \partial C_{t_2}) = t_2 - t_1\}$,
2) $\bigcup_{t \geqq 0} C_t = M$,
3) $p \in C_0$ and if $\partial C_0 \neq \phi$, then $p \in \partial C_0$.

We set $C_0 =: C(0)$ and if $\partial C(0) \neq \phi$, we set $C(1) := C(0)^{\max}$. Inductively we set $C(i+1) := C(i)^{\max}$ if $\partial C(i) \neq \phi$. Then there exists an integer $k \geqq 0$ such that $\partial C(k) = \phi$. $C(k)$ will be called a soul of M and denoted by S.

LEMMA 1. *Let M be a noncompact Riemannian manifold with nonnegative sectional curvature and S be a soul of M. Then for any point $q_0 \in S$, there exist at least two rays starting from q_0.*

Proof. Since M is noncompact there exists a ray $\sigma : [0, \infty) \to M$ starting from q_0. We set $v := -\dot{\sigma}(0)$. Let $\{C_t\}_{t0 \geqq 0}$ be the family of compact totally convex sets from which S was constructed as above. Choose an $s_0 > 0$. Let $c : [0, L] \to M$ be the geodesic such that $L < s_0$ i.e. $c([0, L]) \subset \operatorname{int} C_{s_0}$ and $\dot{c}(0) = v$. Let $\{t_i\}$ be a sequence such that $t_i \in (0, L]$ and $t_i \to 0$ and $\{s_i\}$ be a seqence such that $s_i \to \infty$ and $s_i \geqq s_0$. Let $q_i \in \partial C_{s_i}$ be a point such that $d(c(t_i), q_i) = d(c(t_i), \partial C_{s_i})$ and $c_i ; [0, d(c(t_i), q_i)] \to M$ be a minimal geodesic from $c(t_i)$ to q_i. Then for all i

$$\measuredangle(\dot{c}(t_i), \dot{c}_i(0)) \leqq \frac{\pi}{2},$$

where $\measuredangle(\dot{c}(t_i), \dot{c}_i(0))$ is the angle between $\dot{c}(t_i)$ and $\dot{c}_i(0)$. To see this, we use the fact that the function $\psi : [0, L] \to R$ defined by $\psi(s) := d(c(s), \partial C_{s_i})$ is concave i.e.

$$\psi(at_1 + bt_2) \geqq a\psi(t_1) + b\psi(t_2)$$

where $a, b \geqq 0$, $a + b = 1$, see Theorem 1.10 in [1]. Since $q = c(0) \in S$, ψ takes a maximum at 0 and hence ψ is monotone decreasing. But if $\measuredangle(\dot{c}(t_i), \dot{c}_i(0)) > \frac{\pi}{2}$, then we can find $t_i' < t_i$ such that $d(c(t_i'), q_i) < d(c(t_i), q_i)$. Hence $\psi(t_i') \leqq d(c(t_i'), q_i) < d(c(t_i), q_i) \leqq \psi(t_i)$. This is a contradiction. We choose a convergent subsequence $\{\dot{c}_{i_j}(0)\}$ of $\{\dot{c}_i(0)\}$ such that $\dot{c}_{i_j}(0) \to w$. By the construction, the geodesic $\tau : [0, \infty) \to M$ such that $\dot{\tau}(0) = w$ is a ray which satisfies

$$\measuredangle(v, \dot{\tau}(0)) \leqq \frac{\pi}{2}. \qquad \text{q.e.d.}$$

Proof of Theorem. By the Classification Theorem in [1], M must be isometric to a cylinder or a Möbius band or a P_2 which is diffeomorphic to E^2. So we may assume that M is diffeomorphic to E^2 and not flat. Let S be a soul of M and $\{C_t\}_{t \geqq 0}$ the family of compact totally convex subsets of M which determines S. For a fixed point $q_0 \in S$, by Lemma 1, there exist two rays σ, $\tau : [0, \infty) \to M$ starting from q_0 and $\dot{\sigma}(0) \neq \dot{\tau}(0)$. Since M is diffeomorphic to E^2, by the broken geodesic $\tau^{-1} \circ \sigma : (-\infty, \infty) \to M$ defined by

$$\tau^{-1} \circ \sigma(t) : = \begin{cases} \tau(-t) & \text{if } t \leqq 0 \\ \sigma(t) & \text{if } t \geqq 0, \end{cases}$$

M is decomposed into two domains D_1, D_2 such that $D_1 \cap D_2 = \phi$, $\bar{D}_1 \cup \bar{D}_2 = M$ and $\partial \bar{D}_1 = \partial \bar{D}_2 = \tau^{-1} \circ \sigma$. For each $t > 0$, ∂C_t is homeomorphic to a circle and σ (or τ) meets ∂C_t uniquely at σ_t (or τ_t). For each $t > 0$ and $i = 1, 2$, we set

$$D_t^i : = D_i \cap C_t,$$

$$E_t^i : = D_i \cap \partial C_t,$$

and

$$B_t^i : = \left\{ \begin{array}{l} \tilde{q} \in E_t^i : \text{there exist minimal geodesics from } \sigma_t \text{ to } \tilde{q} \text{ and} \\ \qquad \text{from } \tau_t \text{ to } \tilde{q} \text{ which are contained in } \bar{D}_t^i \end{array} \right\}$$

B_t^i is nonempty subset of E_t^i. We show this for $i = 1$. We set

$$N_\sigma : = \left\{ \begin{array}{l} \tilde{q} \in E_t^1 : \text{there exists a minimal geodesic from } \sigma_t \text{ to } \tilde{q} \\ \qquad \text{which is contained in } \bar{D}_t^1 \end{array} \right\}$$

$$N_\tau : = \left\{ \begin{array}{l} \tilde{q} \in E_t^1 : \text{there exists a minimal geodesic from } \tau_t \text{ to } \tilde{q} \\ \qquad \text{which is contained in } \bar{D}_t^1 \end{array} \right\}$$

Let $\tilde{q} \in E_t^1$. We will show if $\tilde{q} \notin N_\tau$, then $\tilde{q} \in N_\sigma$. By the assumption there exists no minimal geodesic from τ_t to $\tilde{q}$ which is contained in $\bar{D}_t^1$. First of all, we note that any minimal geodesics from τ_t to $\tilde{q}$ and σ_t to $\tilde{q}$ are contained completely ∂C_t or do not interset ∂C_t except the end points. So we may assume that any minimal geodesic from τ_t to $\tilde{q}$ is not contained in ∂C_t. Let $a_t : [0, d(\tau_t, \tilde{q})] \to M$ be a minimal geodesic from τ_t to $\tilde{q}$. Then a_t dose not meet $\tau | [0, d(\tau(0), \tau_t))$. For, if it dose not so, then $a_t([0, d(\tau(0), \tau_t)]) = \tau([0, d(\tau(0), \tau_t)])$, because τ is a minimal geodesic. From the assumption $a_t \notin \bar{D}_t^1$, $a_t([0, d(\tau_t, \tilde{q})]) \cap D_t^2 \neq \phi$. Hence a_t must meet σ at $\sigma(s_0)$, $s_0 > 0$. So $a_t([d(\tau(0), \tau_t), d(\tau_t, \tilde{q})]) \subset \sigma([0, \infty))$, because σ is a minimal geodesic. This contradicts $\tilde{q} \notin \sigma([0, \infty))$. Let $\delta : = \min \{d(q_0, \tau_t), d(\tilde{q}, \tau_t)\}$. Then $a_t([0, \delta]) \subset D_t^1$ or $a_t([0, \delta]) \subset D_t^2$. In the first case, we get the same contradiction by the analogous argument above. It $a_t([0, \delta]) \subset D_t^2$, then $a_t([0, d(\tau_t, \tilde{q})]) \cup \{$restriction of E_t^1 from τ_t to $\tilde{q}\}$ is a Jordan curve and contains q_0 in its interior, because τ and σ are rays. If $\tilde{q} \notin N_\sigma$, then by the same argument above, we see that if $b_t : [0, d(\sigma_t, \tilde{q})] \to M$ be a minimal geodesic from σ_t to $\tilde{q}$, then $b_t([0, d(\sigma_t, \tilde{q})]) \cup \{$restriction of E_t^1 from σ_t to $\tilde{q}\}$ is a Jordan curve and contains q_0 in its interior. Then by the topological consideration, we see that a_t must intersect b_t at $a_t(s')$, $s' > 0$. So $a_t = b_t$ because a_t and b_t are minimal geodesics. This is a contradiction. So $\tilde{q} \in N_\sigma$. Similarly if $\tilde{q} \notin N_\sigma$, then $\tilde{q} \in N_\tau$.

That is, every point of E_t^1 is contained in N_σ or N_τ. If $\tilde{q}\in E_t^1$ is contained in a convex neighborhood of σ_t (or τ_t), then $\tilde{q}\in N_\sigma$ (or N_τ). So N_σ and N_τ are non-empty. By considering limits of geodesics, we see N_σ and N_τ are closed subsets of E_t^1. Thus, if $N_\sigma\cap N_\tau=\phi$, then N_σ and N_τ are non-empty open and closed subsets of E_t^1. This is a contradiction. So there exists a point $q\in N_\sigma\cap N_\tau\subset B_t^1$.

We choose $q_t^i\in B_t^i$ and let $a_t^i:[0, m_t^i]\to M$ and $b_t^i:[0, n_t^i]\to M$ are minimal geodesics from τ_t to q_t^i and from σ_t to q_t^i such that $a_t^i([0, m_t^i])$, $b_t^i([0, n_t^i])\subset\bar{D}_t^i$, $i=1, 2$. We denote by Q_t the closed bounded domain with the boundary consisting of four geodesic segments a_t^1, b_t^1, b_t^2 and a_t^2.

LEMMA 2. *For any point $q\in M$, there exists a positive number $t(q)$ such that for all $t\geqq t(q)$, $q\in Q_t$.*

Proof. We may assume that $q\in D_1$. We assume Lemma 2 dose not hold. Then there exists a sequence $\{t_i\}$ such that $\lim t_i=\infty$ and $q\notin Q_{t_i}$ for all i. Let $c:[0, b]\to M$ be a minimal geodesic from q_0 to q. Then $c((0, b])\subset D_1$. Since every Q_t contains q_0, $a_{t_i}^1$ or $b_{t_i}^1$ meets $c([0, b])$. Without loss of generality, we may assume $a_{t_i}^1$ meets $c([0, b])$ at $a_{t_i}^1(s_{t_i})$. By the triangle inequality,

$$\begin{aligned} d(a_{t_i}^1(s_{t_i}), q_{t_i}^1) &\geqq d(q_{t_i}^1, q_0)-d(q_0, a_{t_i}^1(s_{t_i})) \\ &\geqq d(q_{t_i}^1, q_0)-d(q_0, q) \\ &\geqq t_i-d(q_0, q)\,, \\ d(a_{t_i}^1(s_{t_i}), \tau_{t_i}) &\geqq d(\tau_{t_i}, q_0)-d(q_0, a_{t_i}^1(s_{t_i})) \\ &\geqq d(\tau_{t_i}, q_0)-d(q_0, q) \\ &\geqq t_i-d(q_0, q)\,, \end{aligned}$$

since q_0 is a point of the soul S which is made from the family of totally convex sets $\{C_t\}_{t\geqq 0}$. Hence $\lim_{i\to\infty} d(a_{t_i}^1(s_{t_i}), q_{t_i}^1)=\infty$ and $\lim_{i\to\infty} d(a_{t_i}^1(s_{t_i}), \tau_{t_i})=\infty$. By the compactness of $c([0, b])$, we can choose a convergent subsequence of $\{\dot{a}_{t_i}^1(s_{t_i})\}$. Let v be its limit vector. Then the geodesic $\gamma:(-\infty, \infty)\to M$ such that $\dot{\gamma}(0)=v$ is a line by the above fact. Then by the Toponogov's splitting Theorem (see [1]), M must be isometric to E^2. This is a contradiction. q.e.d.

Taking a positive number r_1, for $i=1, 2, \cdots$, we set $r_{i+1}:=\max\{t(q_{r_i}^1), t(q_{r_i}^2), r_i\}+1$. Then $Q_{r_i}\subset Q_{r_{i+1}}$, because $q_{r_i}^k\in Q_{r_{i+1}}$ by Lemma 2 and $a_{r_i}^k$, $b_{r_i}^k$, $a_{r_{i+1}}^k$, $b_{r_{i+1}}^k$ are minimal geodesics, for $k=1, 2$. Since $r_i\uparrow\infty$, for any point $q\in M$, by Lemma 2 there exists r_i such that $q\in Q_{r_i}$. Hence $\bigcup_i Q_{r_i}=M$. The vertical angles of Q_{r_i} are not larger than π, because C_{r_i} is totally convex. Hence applying the Gauss-Bonnet's Theorem to Q_{r_i}, we get

$$\iint_{Q_{r_i}} K\,dv\leqq 2\pi\,.$$

The sequence $\left\{\iint_{Q_{ri}} K\,dv\right\}$ are monotone increasing, so there exists the limit value and

$$\lim \iint_{Q_{ri}} K\,dv = \iint_{M} K\,dv \leqq 2\pi$$

q. e. d.

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