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ON INTEGRAL INEQUALITIES IN RIEMANNIAN MANIFOLDS ADMITTING A ONE-PARAMETER CONFORMAL TRANSFORMATION GROUP

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1. Introduction.

Let $\mathfrak{B}$ be a connected (C^∞-) differentiable Riemannian manifold of dimension n and g_{ji}, ∇_i, $K_{kji}{}^h$, K_{ji} and K, respectively, the components of the metric tensor field, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor field, the Ricci tensor field and the scalar curvature field, here and hereafter the indices $a, b, c, \cdots, i, j, k, \cdots$ run over the range $1, 2, \cdots, n$. We shall denote $g^{ja}\nabla_a$ by ∇^j and the Laplace-Beltrami operator by Δ. Throughout this paper, we assume that Riemannian manifolds are connected and differentiable and functions are also differentiable.

An infinitesimal transformation $\mathcal{V}^h$ is said to be conformal if it satisfies

$$\mathcal{L}g_{ji}=\nabla_j\mathcal{V}_i+\nabla_i\mathcal{V}_j=2\rho g_{ji} \tag{1.1}$$

for some function ρ on $\mathfrak{B}$, where $\mathcal{L}$ denotes the operator of Lie derivation with respect to $\mathcal{V}^h$ and $\mathcal{V}_i=g_{ia}\mathcal{V}^a$. The ρ satisfies

$$\rho=\frac{1}{n}\nabla_a\mathcal{V}^a\,. \tag{1.2}$$

If ρ in (1.1) is a constant, the transformation is said to be homothetic, and if $\rho=0$, the transformation is called to be isometric. Hereafter we shall denote the gradient of ρ by $\rho_i=\nabla_i\rho$.

We now put

$$G_{ji}=K_{ji}-\frac{K}{n}g_{ji} \tag{1.3}$$

and

$$Z_{kjih}=K_{kjih}-\frac{K}{n(n-1)}(g_{kh}g_{ji}-g_{jh}g_{ki})\,. \tag{1.4}$$

Then we have

$$G_{ji}g^{ji}=0\,,\qquad Z_{aji}{}^a=G_{ji}\,, \tag{1.5}$$

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$$G_{ji}G^{ji}=K_{ji}K^{ji}-\frac{K^2}{n} \tag{1.6}$$

and

$$Z_{kjih}Z^{kjih}=K_{kjih}K^{kjih}-\frac{2K^2}{n(n-1)}\,. \tag{1.7}$$

In 1969, K. Yano [3] proved

THEOREM A. *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n>2$ admits an infinitesimal non-isometric conformal transformation $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, $\rho\neq 0$, then*

$$n(n-1)\int_{\mathfrak{B}} K_{ji}\rho^j\rho^i dV \leqq K^2\int_{\mathfrak{B}}\rho^2 dV\,, \tag{1.8}$$

where dV is the volume element of $\mathfrak{B}$, equality holding if and only if $\mathfrak{B}$ is isometric to a sphere.

The purpose of the present paper is to prove the following theorems and corollary.

THEOREM 1.1. *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n>2$ admits an infinitesimal non-isometric conformal transformation $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, $\rho\neq 0$, then*

$$\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}\Delta(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})dV \leqq \frac{n(n+2)}{2}\int_{\mathfrak{B}}\rho^2\Delta(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})dV \tag{1.9}$$

for any nonnegative constants α and β not both zero, equality holding if and only if $\mathfrak{B}$ is isometric to a sphere.

THEOREM 1.2. *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n>2$ admits an infinitesimal non-isometric conformal transformation $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, $\rho\neq 0$, then for any non-negative constants α and β not both zero and for any constant k such that $k\geqq -4n$,*

$$\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})dV + k\int_{\mathfrak{B}}\rho^2(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})dV \geqq 0\,, \tag{1.10}$$

equality holding if and only if $\mathfrak{B}$ is isometric to a sphere.

THEOREM 1.3. *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n>2$ admits an infinitesimal non-isometric conformal transformation $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, $\rho\neq 0$, then for any non-negative constants α and β not bath zero and for any constant k such that $k\geqq -4n$,*

(1.11)
$$(2n(n-2)\alpha+8n\beta)\int_{\mathfrak{B}} K_{ji}\rho^j\rho^i d\mathcal{V}$$
$$\leqq -\Big[\Big(\frac{2n}{n-1}+\frac{k}{n}\Big)\alpha+\frac{2k\beta}{n(n-1)}\Big]K^2\int_{\mathfrak{B}}\rho^2 d\mathcal{V}$$
$$+(4n+k)\int_{\mathfrak{B}}\rho^2(\alpha K_{ji}K^{ji}+\beta K_{kjih}K^{kjih})d\mathcal{V}\,,$$

equality holding if and only if $\mathfrak{B}$ *is isometric to a sphere.*

COROLLARY. *Under the same assumption as in Theorem* 1.3, *we have*

(1.12)
$$n\int_{\mathfrak{B}} K_{ji}\rho^j\rho^i d\mathcal{V}\leqq\frac{K^2}{n-1}\int_{\mathfrak{B}}\rho^2 d\mathcal{V}\,,$$

(1.13)
$$\frac{K^2}{n-1}\int_{\mathfrak{B}}\rho^2 d\mathcal{V}\leqq -n\int_{\mathfrak{B}} K_{ji}\rho^j\rho^i d\mathcal{V}$$
$$+\frac{2n}{(n-1)\alpha+2\beta}\int_{\mathfrak{B}}\rho^2(\alpha K_{ji}K^{ji}+\beta K_{kjih}K^{kjih})d\mathcal{V}$$

and

(1.14)
$$\int_{\mathfrak{B}} K_{ji}\rho^j\rho^i d\mathcal{V}$$
$$\leqq\frac{1}{(n-1)\alpha+2\beta}\int_{\mathfrak{B}}\rho^2(\alpha K_{ji}K^{ji}+\beta K_{kjih}K^{kjih})d\mathcal{V}\,,$$

in each of (1.12), (1.13) *and* (1.14), *equality holding if an only if* $\mathfrak{B}$ *is isometric to a sphere.*

Inequality (1.12) is nothing else than Theorem A, and so both Theorems 1.2 and 1.3 are generalizations of Theorem A, because, as will be seen in the proof of Theorem 1.3 in section 4, inequalities (1.10) and (1.11) are equivalent.

Here we state the following two important remarks.

REMARK 1. By using Lemma 2.1 which will be proved in section 2, we can easily prove that any infinitesimal homothetic transformation of a compact orientable Riemannian manifold is necessarily an infinitesimal isometric transformation.

REMARK 2. K. Yano [2] proved that if a compact orientable Riemannian manifold with constant scalar curvature field K and of dimension $n\geqq 2$ admits an infinitesimal nonisometric conformal transformation then K is necessarily a positive constant.

2. Preliminaries

In a Riemannian manifold $\mathfrak{B}$, for an infinitesimal conformal transformation $\mathcal{V}^h : \mathcal{L}g_{ji}=2\rho g_{ji}$, we have [1]

(2.1) $$\mathcal{L}K_{kjih}=-g_{kh}\nabla_j\rho_i+g_{jh}\nabla_k\rho_i-(\nabla_k\rho_h)g_{ji}+(\nabla_j\rho_h)g_{ki}+2\rho K_{kjih},$$

(2.2) $$\mathcal{L}K_{ji}=-(n-2)\nabla_j\rho_i-\Delta\rho g_{ji}$$

and

(2.3) $$\mathcal{L}K=-2(n-1)\Delta\rho-2\rho K.$$

Thus, for $\mathfrak{B}$ with constant scalar curvature field K,

(2.4) $$\Delta\rho=-\frac{K}{n-1}\rho$$

and

(2.5) $$\nabla^j G_{ji}=\frac{n-2}{2n}\nabla_i K=0.$$

We have, from (1.3) and (1.4),

(2.6) $$\mathcal{L}G_{ji}=-(n-2)\left(\nabla_j\rho_i-\frac{1}{n}\Delta\rho g_{ji}\right)$$

and

(2.7) $$\mathcal{L}Z_{kjih}=-g_{kh}\nabla_j\rho_i+g_{jh}\nabla_k\rho_i-(\nabla_k\rho_h)g_{ji}+(\nabla_j\rho_h)g_{ki}+\frac{2}{n}\Delta\rho(g_{kh}g_{ji}-g_{jh}g_{ki})+2\rho Z_{kjih}$$

respectively.

By straightforward calculations, we have, from (2.6) and (2.7), respectively,

(2.8) $$(\mathcal{L}G_{ji})G^{ji}=-(n-2)(\nabla^j\rho^i)G_{ji}$$

and

(2.9) $$(\mathcal{L}Z_{kjih})Z^{kjih}=-4(\nabla^j\rho^i)G_{ji}+2\rho Z_{kjih}Z^{kjih}.$$

On the other hand, we have

(2.10) $$\mathcal{L}G^{ji}=(\mathcal{L}G_{ba})g^{jb}g^{ia}-4\rho G^{ji}$$

and

(2.11) $$\mathcal{L}Z^{kjih}=(\mathcal{L}Z_{dcba})g^{kd}g^{jc}g^{ib}g^{ha}-8\rho Z^{kjih}.$$

Therefore, we have

(2.12) $$(\mathcal{L}G^{ji})G_{ji}=(\mathcal{L}G_{ji})G^{ji}-4\rho G_{ji}G^{ji}$$

and

(2.13) $$(\mathcal{L}Z^{kjih})Z_{kjih}=(\mathcal{L}Z_{kjih})Z^{kjih}-8\rho Z_{kjih}Z^{kjih}.$$

Thus, from (2.8) and (2.12),

$$\mathcal{L}(G_{ji}G^{ji})=-2(n-2)(\nabla^j\rho^i)G_{ji}-4\rho G_{ji}G^{ji} \tag{2.14}$$

and similarly, from (2.9) and (2.13),

$$\mathcal{L}(Z_{kjih}Z^{kjih})=-8(\nabla^j\rho^i)G_{ji}-4\rho Z_{kjih}Z^{kjih}\,. \tag{2.15}$$

LEMMA 2.1. *If a compact orientable Remannian manifold* $\mathfrak{B}$ *of dimension* n *admits an infinitesimal conformal transformation* $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, *then*

$$\int_{\mathfrak{B}}\rho f dV=-\frac{1}{n}\int_{\mathfrak{B}}\mathcal{L}f dV \tag{2.16}$$

for any function f *on* $\mathfrak{B}$.

Proof. Since $\rho=\frac{1}{n}\nabla_a v^a$, we have, by using Green's theorem,

$$\int_{\mathfrak{B}}\nabla_a(f v^a)dV=\int_{\mathfrak{B}}\mathcal{L}f dV+n\int_{\mathfrak{B}}\rho f dV=0$$

which gives (2.16).

LEMMA 2.2. *If a compact orientable Riemannian manifold* $\mathfrak{B}$ *with constant scalar curvature field* K *and of dimension* n *admits an infinitesimal conformal transformation* $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, *then*

$$\int_{\mathfrak{B}}\rho(\nabla^j\rho^i)G_{ji}dV=-\int_{\mathfrak{B}}G_{ji}\rho^j\rho^i dV\,. \tag{2.17}$$

Proof. By using Green's theorem, we have

$$\int_{\mathfrak{B}}\nabla^j(G_{ji}\rho^i\rho)dV=\int_{\mathfrak{B}}(\nabla^j G_{ji})\rho^i\rho dV$$
$$+\int_{\mathfrak{B}}G_{ji}(\nabla^j\rho^i)\rho dV+\int_{\mathfrak{B}}G_{ji}\rho^j\rho^i dV=0\,,$$

which reduces to (2.17) by virtue of (2.5).

LEMMA 2.3. *If a compact orientable Riemannian manifold* $\mathfrak{B}$ *with constant scalar curvature* K *and of dimension* n *admits an infinitesimal conformal transformation* $v^h:\mathcal{L}g_{ji}=2\rho g_{ji}$, *then*

$$\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(G_{ji}G^{ji})dV=-2n(n-2)\int_{\mathfrak{B}}G_{ji}\rho^j\rho^i dV+4n\int_{\mathfrak{B}}\rho^2 G_{ji}G^{ji}dV \tag{2.18}$$

and

$$\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(Z_{kjih}Z^{kjih})dV=-8n\int_{\mathfrak{B}}G_{ji}\rho^j\rho^i dV+4n\int_{\mathfrak{B}}\rho^2 Z_{kjih}Z^{kjih}dV\,. \tag{2.19}$$

Proof. By using (2.14) and Lemmas 2.1 and 2.2, we have

$$\int_{\mathfrak{B}} \mathcal{L}\mathcal{L}(G_{ji}G^{ji})d\mathcal{V} = -n\int_{\mathfrak{B}} \rho\mathcal{L}(G_{ji}G^{ji})d\mathcal{V}$$

$$= 2n(n-2)\int_{\mathfrak{B}} \rho(\nabla^j\rho^i)G_{ji}d\mathcal{V} + 4n\int_{\mathfrak{B}} \rho^2 G_{ji}G^{ji}d\mathcal{V}$$

$$= -2n(n-2)\int_{\mathfrak{B}} G_{ji}\rho^j\rho^i d\mathcal{V} + 4n\int_{\mathfrak{B}} \rho^2 G_{ji}G^{ji}d\mathcal{V}\,.$$

Similarly, (2.19) follows from (2.15) and Lemmas 2.1 and 2.2.

3. Lemmas.

In this section, we shall prove some lemmas which will be used in the next section.

LEMMA 3.1. *In a compact orientable Riemannian manifold* $\mathfrak{B}$

$$\int_{\mathfrak{B}} (\nabla_i f)(\nabla^i h)d\mathcal{V} = -\int_{\mathfrak{B}} (\Delta h) f d\mathcal{V} = -\int_{\mathfrak{B}} h\Delta f d\mathcal{V} \tag{3.1}$$

for any functions f and h on $\mathfrak{B}$.

Proof. This follows from

$$\int_{\mathfrak{B}} \nabla_i(f\nabla^i h)d\mathcal{V} = \int_{\mathfrak{B}} (\nabla_i f)(\nabla^i h)d\mathcal{V} + \int_{\mathfrak{B}} f\Delta h d\mathcal{V} = 0$$

and

$$\int_{\mathfrak{B}} \nabla_i(h\nabla^i f)d\mathcal{V} = \int_{\mathfrak{B}} (\nabla_i h)(\nabla^i f)d\mathcal{V} + \int_{\mathfrak{B}} h\Delta f d\mathcal{V} = 0\,.$$

LEMMA 3.2. *If a compact arientable Riemannian manifold* $\mathfrak{B}$ *admits an infinitesimal conformal transformation* $\mathcal{V}^h : \mathcal{L}g_{ji} = 2\rho g_{ji}$, *then*

$$\int_{\mathfrak{B}} \rho^2 \Delta f d\mathcal{V} = -2\int_{\mathfrak{B}} \rho\rho_i \nabla^i f d\mathcal{V} \tag{3.2}$$

for any function f on $\mathfrak{B}$.

Proof. This follows from

$$\int_{\mathfrak{B}} \nabla_i(\rho^2\nabla^i f)d\mathcal{V} = 2\int_{\mathfrak{B}} \rho\rho_i\nabla^i f d\mathcal{V} + \int_{\mathfrak{B}} \rho^2\Delta f d\mathcal{V} = 0\,.$$

LEMMA 3.3 (K. Yano and S. Sawaki [4]). *If a Riemannian manifold* $\mathfrak{B}$ *of dimension n admits an infinitesimal conformal transformation* $\mathcal{V}^h : \mathcal{L}g_{ji} = 2\rho g_{ji}$, *then*

$$\Delta\mathcal{L}f = \mathcal{L}\Delta f + 2\rho\Delta f - (n-2)\rho_i\nabla^i f \tag{3.3}$$

for any function f on $\mathfrak{B}$.

LEMMA 3.4. *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n\geqq 2$ admits an infinitesimal conformal transformation $\mathcal{V}^h$: $\mathcal{L}g_{ji}=2\rho g_{ji}$, then*

$$(3.4)\qquad \int_{\mathfrak{B}}\mathcal{L}\mathcal{L}\Delta f d\mathcal{V}=-\frac{K}{n-1}\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}f d\mathcal{V}+\frac{n(n+2)}{2}\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}$$

for any function f on $\mathfrak{B}$.

Proof. By using (2.4), Lemma 2.1, Lemmas 3.1, 3.2 and 3.3, we have

$$\begin{aligned}\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}\Delta f d\mathcal{V}&=-n\int_{\mathfrak{B}}\rho\mathcal{L}\Delta f d\mathcal{V}\\ &=-n\int_{\mathfrak{B}}\rho(\Delta\mathcal{L}f-2\rho\Delta f+(n-2)\rho_i\nabla^i f)d\mathcal{V}\\ &=-n\int_{\mathfrak{B}}\rho\Delta\mathcal{L}f d\mathcal{V}+2n\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}-n(n-2)\int_{\mathfrak{B}}\rho\rho_i\nabla^i f d\mathcal{V}\\ &=-n\int_{\mathfrak{B}}(\Delta\rho)\mathcal{L}f d\mathcal{V}+2n\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}+\frac{n(n-2)}{2}\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}\\ &=\frac{nK}{n-1}\int_{\mathfrak{B}}\rho\mathcal{L}f d\mathcal{V}+2n\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}+\frac{n(n-2)}{2}\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}\\ &=-\frac{K}{n-1}\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}f d\mathcal{V}+\frac{n(n+2)}{2}\int_{\mathfrak{B}}\rho^2\Delta f d\mathcal{V}\,.\end{aligned}$$

LEMMA 3.5 (for instance, K. Yano [3]). *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n>2$ admits an infinitesimal nonisometric conformal transformation $\mathcal{V}^h$: $\mathcal{L}g_{ji}=2\rho g_{ji}$, $\rho\neq 0$, then*

$$(3.5)\qquad \int_{\mathfrak{B}}G_{ji}\rho^j\rho^i d\mathcal{V}\leqq 0\,,$$

equality holding if and only if $\mathfrak{B}$ is isometric to a sphere.

LEMMA 3.6. *If a compact orientable Riemannian manifold $\mathfrak{B}$ with constant scalar curvature field K and of dimension $n\geqq 2$ admits an infinitesimal conformal transformation $\mathcal{V}^h$: $\mathcal{L}g_{ji}=2\rho g_{ji}$, then*

$$(3.6)\qquad \begin{aligned}\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(K_{ji}K^{ji})d\mathcal{V}=&-2n(n-2)\int_{\mathfrak{B}}K_{ji}\rho^j\rho^i d\mathcal{V}\\ &+4n\int_{\mathfrak{B}}\rho^2K_{ji}K^{ji}d\mathcal{V}-\frac{2nK^2}{n-1}\int_{\mathfrak{B}}\rho^2 d\mathcal{V}\end{aligned}$$

and

$$(3.7)\qquad \begin{aligned}\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(K_{kjih}K^{kjih})d\mathcal{V}=&-8n\int_{\mathfrak{B}}K_{ji}\rho^j\rho^i d\mathcal{V}\\ &+4n\int_{\mathfrak{B}}\rho^2K_{kjih}K^{kjih}d\mathcal{V}\,.\end{aligned}$$

Proof. By using (1.3), (1.6) and (2.18), we have

$$\int_{\mathfrak{B}} \mathcal{L}\mathcal{L}(K_{ji}K_{ji})d\mathcal{V}=\int_{\mathfrak{B}} \mathcal{L}\mathcal{L}(G_{ji}G^{ji})d\mathcal{V}$$

$$=-2n(n-2)\int_{\mathfrak{B}} G_{ji}\rho^j\rho^i d\mathcal{V}+4n\int_{\mathfrak{B}} \rho^2 G_{ji}G^{ji}d\mathcal{V}$$

$$=-2n(n-2)\int_{\mathfrak{B}} \Big(K_{ji}-\frac{K}{n}g_{ji}\Big)\rho^j\rho^i d\mathcal{V}$$

$$+4n\int_{\mathfrak{B}} \rho^2\Big(K_{ji}K^{ji}-\frac{K^2}{n}\Big)d\mathcal{V}$$

$$=-2n(n-2)\int_{\mathfrak{B}} K_{ji}\rho^j\rho^i d\mathcal{V}+2(n-2)K\int_{\mathfrak{B}} \rho_i\rho^i d\mathcal{V}$$

$$+4n\int_{\mathfrak{B}} \rho^2 K_{ji}K^{ji}d\mathcal{V}-4K^2\int_{\mathfrak{B}} \rho^2 d\mathcal{V}\,.$$

But, from (2.4) and Lemma 3.1,

$$\int_{\mathfrak{B}} \rho_i\rho^i d\mathcal{V}=-\int_{\mathfrak{B}} \rho\Delta\rho\, d\mathcal{V}=\frac{K}{n-1}\int_{\mathfrak{B}} \rho^2 d\mathcal{V}\,.$$

Therefore we have (3.6). Similarly, from (1.3), (1.7) and (2.19), we can get (3.7).

4. Proofs of theorems.

We shall prove our theorems and corollary stated in section 1.

Proof of Theorem 1.1. Applying Lemma 3.4 to functions $G_{ji}G^{ji}$ and $Z_{kjih}Z^{kjih}$ and using (2.18) and (2.19), we have

$$\int_{\mathfrak{B}} \mathcal{L}\mathcal{L}\Delta(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}$$

$$-\frac{n(n+2)}{2}\int_{\mathfrak{B}} \rho^2\Delta(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}$$

$$=-\frac{K}{n-1}\int_{\mathfrak{B}} \mathcal{L}\mathcal{L}(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}$$

$$=[2n(n-2)\alpha+8n\beta]\frac{K}{n-1}\int_{\mathfrak{B}} G_{ji}\rho^j\rho^i d\mathcal{V}$$

$$-\frac{4nK}{n-1}\int_{\mathfrak{B}} \rho^2(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}\,.$$

From our assumption, $K>0$ and Lemma 3.5, the right hand side of the above relation is nonpositive, and consequently (1.9) holds. If the equality in (1.9) holds, we must have

$$\int_{\mathfrak{B}} G_{ji}\rho^j\rho^i d\mathcal{V}=0 \tag{4.1}$$

and hence $\mathfrak{B}$ is isometric to a sphere by virtue of Lemma 3.5. Conversely, if $\mathfrak{B}$ is isometric to a sphere, we have $G_{ji}=0$ and $Z_{kjih}=0$ and consequently the equality holds in (1.9).

Proof of Theorem 1.2. From (2.18) and (2.19), we have

$$\begin{aligned}&\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}\\&\qquad+k\int_{\mathfrak{B}}\rho^2(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}\\&\quad=-[2n(n-2)\alpha+8n\beta]\int_{\mathfrak{B}}G_{ji}\rho^j\rho^i d\mathcal{V}\\&\qquad+(4n+k)\int_{\mathfrak{B}}\rho^2(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}\,,\end{aligned}$$

from which, by using Lemma 3.5 and our assumption, we have (1.10). If the equality in (1.10) holds, we must have (4.1) and consequently $\mathfrak{B}$ is isometric to a sphere. Coversely, if $\mathfrak{B}$ is isometric to a sphere, we must have $G_{ji}=0$ and $Z_{kjih}=0$ and the equality in (1.10) holds.

Proof of Theorem 1.3. From (1.6), (1.7), (3.6) and (3.7), we have

$$\begin{aligned}&\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}\\&\qquad+k\int_{\mathfrak{B}}\rho^2(\alpha G_{ji}G^{ji}+\beta Z_{kjih}Z^{kjih})d\mathcal{V}\\&\quad=\int_{\mathfrak{B}}\mathcal{L}\mathcal{L}(\alpha K_{ji}K^{ji}+\beta K_{kjih}K^{kjih})d\mathcal{V}\\&\qquad+k\alpha\int_{\mathfrak{B}}\rho^2\Big(K_{ji}K^{ji}-\frac{K^2}{n}\Big)d\mathcal{V}\\&\qquad+k\beta\int_{\mathfrak{B}}\rho^2\Big(K_{kjih}K^{kjih}-\frac{2K^2}{n(n-1)}\Big)d\mathcal{V}\\&\quad=-[2n(n-2)\alpha+8n\beta]\int_{\mathfrak{B}}K_{ji}\rho^j\rho^i d\mathcal{V}\\&\qquad+(4n+k)\int_{\mathfrak{B}}\rho^2(\alpha K_{ji}K^{ji}+\beta K_{kjih}K^{kjih})d\mathcal{V}\\&\qquad-\Big[\Big(\frac{2n}{n-1}+\frac{k}{n}\Big)\alpha+\frac{2k\beta}{n(n-1)}\Big]K^2\int_{\mathfrak{B}}\rho^2 d\mathcal{V}\,.\end{aligned}$$

Thus, we can easily see that (1.10) and (1.11) are equivalent, and consequently the result follows from Theorem 1.2.

Proof of Corollary. If we put $k=-4n$,

$$k=\frac{-4n\alpha+8n\beta}{(n-1)\alpha+2\beta} \quad (>-4n) \qquad \text{and} \qquad k=\frac{-2n^2\alpha}{(n-1)\alpha+2\beta} \quad (>-4n)$$

in (1.11), then, from our assumption, we have (1.12), (1.13) and (1.14) respectively. The last inequality (1.14) follows also from (1.12) and (1.13).

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