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ON THE RADIAL DISTRIBUTION OF ZEROS AND POLES OF A MEROMORPHIC FUNCTION

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1. Introduction.

Edrei, Fuchs and Hellerstein [1] proved the following

THEOREM A. *Let $F(z)$ be a meromorphic function with positive zeros (a_n) and negative poles (b_n). Assume that*

$$\sum \frac{1}{|a_n|} + \sum \frac{1}{|b_n|} = +\infty$$

and that

$$\sum \frac{1}{|a_n|^s} + \sum \frac{1}{|b_n|^s} < +\infty$$

for some finite positive value of s. Then

$$K(F) = \overline{\lim_{r\to\infty}} \frac{N(r, 0, F) + N(r, \infty, F)}{T(r, F)} \leqq \frac{1}{1+A},$$

where A (>0) is an absolute constant.

By a rough estimation their constant A is less than 0.0017, and of course, far from the best.

Recently, Ozawa [4] gave an improved form of Theorem A in the case of a canonical product of finite genus having only negative zeros. His result is the following

THEOREM B. *Let $G(z)$ be a canonical product of genus q, having only negative zeros. If $q \geqq 2$, then*

$$\delta(0, G) \geqq \frac{A(q)}{1+A(q)},$$

where

$$A(q) \geqq \frac{1}{12\pi}.$$

If q tends to infinity, then $A(q)$ tends to $1/2\pi^2$.

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In this note we shall prove the following results.

THEOREM 1. *The assumptions of Theorem A imply*

$$K(F)\leqq\frac{1}{1+A^*},$$

where

$$A^*=\frac{1}{\pi}-\frac{1}{4}>0.068.$$

It should be remarked that our A^* is larger than $40A$, where A is defined in Theorem A.

The next theorem is an improvement of Theorem B.

THEOREM 2. *Let $G(z)$ be an infinite product such that*

$$G(z)=\prod E\left(\frac{z}{a_n}, q\right),$$

where $q\geqq 1$, and the sequence (a_n) satisfies the following conditions:

1) $a_n<0$ *for any* n,

2) $\sum\frac{1}{|a_n|^{q+1}}<+\infty$.

Then for any positive r

$$T(r, G)\geqq(1+A^*(q))N(r, 0, G),$$

where

$$A^*(q)\geqq\frac{1}{\pi}-\frac{1}{4}.$$

Further

$$\varliminf_{q\to\infty} A^*(q)\geqq\frac{1}{2\pi}-\frac{1}{2\pi^2}>0.1084.$$

COROLLARY 1. *Let $G(z)$ be a canonical product of finite genus having only negative zeros. If its genus is sufficiently large, then*

$$\delta(0, G)>\frac{1}{11}.$$

THEOREM 3. *Let $F(z)$ be a meromorphic function of order λ, lower order μ, genus q and whose zeros (a_n) and poles (b_n) satisfy the following conditions:*

$$|\arg a_n-\pi|\leqq\beta, \qquad |\arg b_n|\leqq\beta$$

for some β $(0\leqq\beta<\pi/6q)$. Then

$$2\left[\frac{q-1}{2}\right]+1\leqq\mu\leqq\lambda\leqq q+1.$$

COROLLAY 2. *Let $F(z)$ be a meromorphic function of order λ, lower order μ,* *genus q and having only negative zeros and positive poles. Then*

$$2\left[\frac{q-1}{2}\right]+1\leqq\mu\leqq\lambda\leqq q+1\,.$$

This Corollary 2 is not new, since J. Williamson pointed out this fact in his paper [6]. Further, he mentioned that this inequality is best possible. Professor N. Suita also made an example which gives the best possibility. The author expresses his heartiest thanks to Professor M. Ozawa for his tender support in preparing this note.

2. To prove Theorem 1, we need the following lemmas.

LEMMA 1. *Let $g(z)$ be a meromorphic function defined by*

$$g(z)=\Pi E\left(\frac{z}{a_n},2\right)\Big/\Pi E\left(\frac{z}{b_n},2\right),$$

where (a_n) and (b_n) satisfy

1) $a_n<0$, $b_n>0$ *for any n*,

2) $\sum\frac{1}{|a_n|^3}+\sum\frac{1}{|b_n|^3}<+\infty$.

Then for any positive r

$$T(r,g)\geqq(1+A^*)N(r)\,,$$

where A^* is the absolute constant in Theorem 1, *and*

$$N(r)=N(r,0,g)+N(r,\infty,g)\,.$$

Proof. According to [5; Lemma 3], for any a, b $(0<a\leqq b<\pi)$ and r (>0)

$$\begin{aligned}m(r,g)&\geqq\frac{1}{\pi}\int_a^b\log|g(re^{iu})|\,du\\&=\int_0^\infty N(t,0)[K_2(t,r,b)-K_2(t,r,a)]dt\\&\quad+\int_0^\infty N(t,\infty)[K_2(t,r,\pi-b)-K_2(t,r,\pi-a)]dt\,,\end{aligned}$$

where

$$K_2(t,r,x)=\frac{1}{\pi}\left(\frac{r}{t}\right)^3\frac{t\sin 3x+r\sin 2x}{t^2+2tr\cos x+r^2}\,.$$

Putting $a=\pi/2$ and $b\to\pi-0$, then

$$m(r,g)\geqq N(r,0)-\int_0^\infty N(t)K_2\left(t,r,\frac{\pi}{2}\right)dt\,.$$

Hence

$$T(r,g)\geqq N(r)+\frac{1}{\pi}\int_0^\infty N(t)\frac{r^3}{t^2(t^2+r^2)}\,dt$$

$$\geqq N(r)+\frac{1}{\pi}\int_1^\infty N(tr)\frac{1}{t^4+t^2}\,dt$$

$$\geqq(1+A^*)N(r)\,.$$

This is the desired result.

LEMMA 2. *Let $f(z)$ be a meromorphic function of genus one or two, having only negative zeros (a_n) and positive poles (b_n). If*

$$\Sigma\frac{1}{|a_n|}+\Sigma\frac{1}{|b_n|}=+\infty\,,$$

then

$$K(f)\leqq\frac{1}{1+A^*}\,.$$

Further, this inequality still holds if $f(z)$ is replaced by $F(z)$:

$$F(z)=e^{S(z)}f(z)$$

where $S(z)$ is an entire function.

Proof. Let $g(z)$ be a meromorphic function defined by

$$g(z)=\Pi E\left(\frac{z}{a_n},\,2\right)\Big/\Pi E\left(\frac{z}{b_n},\,2\right).$$

Of course, this function is well defined. By $T(r,f)=o(r^3)$ and $T(r,g)=o(r^3)$ [3; p. 235],

$$f(z)=e^{az^2+bz+c}g(z)\,.$$

Since $f(z)/f(-z)=e^{2bz}g(z)/g(-z)$, we have

$$2T(r,f)+2|b|r+0(1)\geqq T(r,G)\,,$$

where

$$G(z)=g(z)/g(-z)\,.$$

Applying Lemma 1 to $G(z)$, we obtain

$$T(r,G)\geqq 2(1+A^*)N(r)$$

for any positive r, where $N(r)=N(r,0,f)+N(r,\infty,f)$. On the other hand, by the assumption and

$$T(r,G)\geqq\frac{2r}{\pi}\int_0^r\frac{N(t)}{t^2}\,\frac{r^2}{r^2+t^2}\,dt$$

$$\geqq\frac{r}{\pi}\int_0^r\frac{N(t)}{t^2}\,dt\,,$$

we have

$$\lim_{r\to\infty}T(r,G)/r=+\infty\,.$$

Hence

$$(1+o(1))T(r, f)\geqq(1+A^*)N(r).$$

This means that

$$K(f)\leqq\frac{1}{1+A^*}.$$

The second part of Lemma 2 is an immediate consequence of the first part.

Proof of Theorem 1. By the assumptions, there exists the integer q which satisfies

$$\sum\frac{1}{|a_n|^q}+\sum\frac{1}{|b_n|^q}=+\infty,$$

$$\sum\frac{1}{|a_n|^{q+1}}+\sum\frac{1}{|b_n|^{q+1}}<+\infty.$$

In the case that q is less than three, Theorem 1 is obvious by Lemma 2. Hence we may assume that q is greater than two.

Consider the auxiliary meromorphic function $G(z)$ defined by

$$G(z)=\prod_{n=1}^{k} F(w^n \sqrt[k]{z}),$$

where k is an odd integer satisfying $(q+1)/3\leqq k\leqq q$, and

$$w=\exp\left(i\frac{2\pi}{k}\right).$$

Evidently we have

$$N(r^k, \infty, G)=kN(r, \infty, F), \qquad N(r^k, 0, G)=kN(r, 0, F),$$

$$T(r^k, G)\leqq kT(r, F).$$

Therefore,

$$K(G)\geqq K(F).$$

Since $(q+1)/3\leqq k\leqq q$, we obtain

$$\sum\frac{1}{|a_n^k|}+\sum\frac{1}{|b_n^k|}=+\infty,$$

$$\sum\frac{1}{|a_n^k|^3}+\sum\frac{1}{|b_n^k|^3}<+\infty.$$

Thus Lemma 2 yields

$$K(G)\leqq\frac{1}{1+A^*}.$$

Hence, the proof of Theorem 1 is completed.

3. In this section, we shall show Theorem 2. The following lemma upon which our method of proof depends heavily, is due to Hellerstein and Williamson [2].

LEMMA 3. *Let* $H_q(t, r, s_1, s_2, \cdots, s_{q+1})$ *be*

$$\sum_{n=0}^{[(q+1)/2]}(K_q(t, r, s_{2n+1})-K_q(t, r, s_{2n})),$$

where

$$K_q(t, r, x)=\frac{(-1)^q}{\pi}\left(\frac{r}{t}\right)^{q+1}\frac{t\sin(q+1)x+r\sin qx}{t^2+2tr\cos x+r^2}$$

If

$$\frac{2n-1}{2q+2}\pi\leqq s_n\leqq\frac{2n-1}{2q}\pi; \qquad n=1, 2, \cdots, q$$

$$\frac{2q+1}{2q+2}\pi\leqq s_{q+1}\leqq\pi, \qquad s_0=s_{q+2}=0$$

then for any $r>0$ *and* $t>0$,

$$(-1)^q H_q(t, r, s_1\, s_2 \cdots s_{q+1})\geqq 0.$$

Further, for any $t\geqq 1$,

$$(-1)^q H_q(t, 1, s_1\, s_2 \cdots s_{q+1})\geqq\frac{q}{\pi t^{q+1}(1+t^{2q})}.$$

In the first place, we assume that q is even. Put

$$s_{2n+1}=\frac{4n+1}{2q}\pi \qquad n=0, 1, \cdots, (q-2)/2$$

$$s_{2n}=\frac{4n-1}{2q+2}\pi \qquad n=1, 2, \cdots, q/2$$

and

$$s_0=0, \qquad s_{q+1}=\pi.$$

By Shea's representation, we have

$$T(r, G)\geqq\frac{1}{\pi}\int_{I_q}\log|G(re^{iu})|\,du$$

$$=N(r)+\int_0^\infty N(t)H_q(t, r, s_1\, s_2 \cdots s_{q+1})dt,$$

where

$$I_q=\sum_{n=0}^{q/2}[s_{2n}, s_{2n+1}], \qquad N(t)=N(t, 0, G).$$

According to Lemma 3,

$$T(r, G)\geqq N(r)\left(1+\int_1^\infty H_q(t, 1, s_1\, s_2 \cdots s_{q+1})dt\right)$$

for any positive r.

Here, we define $A^*(q)$ such that

$$A^*(q)=\int_1^\infty H_q(t, 1, s_1\, s_2 \cdots s_{q+1})dt.$$

Since

$$H_q(t,\,1,\,s_1\,s_2\cdots s_{q+1})\geqq\frac{q}{\pi t^{q+1}(1+t^{2q})}$$

for any $t\geqq 1$,

$$A^*(q)\geqq\frac{q}{\pi}\int_1^\infty\frac{1}{t^{q+1}(1+t^{2q})}dt$$

$$=\frac{1}{\pi}\int_1^\infty\frac{1}{u^2(1+u^2)}du$$

$$=\frac{1}{\pi}-\frac{1}{4}.$$

By an easy calculation,

$$\pi t^{q+1}H_q(t,\,1,\,s_1,\,s_2,\,\cdots,\,s_{q+1})=\sum_{n=0}^{(q-2)/2}\frac{1+t\cos s_{2n+1}}{t^2+2t\cos s_{2n+1}+1}+\sum_{n=1}^{q/2}\frac{t+\cos s_{2n}}{t^2+2t\cos s_{2n}+1}.$$

The following elementary relations

$$\mathcal{R}_e\left(\frac{1}{t+e^{iu}}\right)=\frac{t+\cos u}{t^2+2t\cos u+1},$$

$$\mathcal{R}_e\left(\frac{1}{1+te^{iu}}\right)=\frac{1+t\cos u}{t^2+2t\cos u+1},$$

$$\frac{t}{t+e^{iu}}=\sum_{n=0}^\infty(-1)^n\frac{e^{inu}}{t^n}\qquad(1<t,\ |u|\leqq\pi),$$

$$\frac{1}{1+te^{iu}}=\sum_{n=1}^\infty(-1)^{n+1}\frac{1}{t^n e^{inu}}\qquad(1<t,\ |u|\leqq\pi),$$

yield

$$\pi t^{q+1}H_q(t,\,1,\,s_1,\,s_2,\,\cdots,\,s_{q+1})=\frac{q}{2(1+t^{2q})}+\frac{q\,t^{2q+1}}{2(1+t^{2q+2})}+A_q(t)$$

for any $t>1$, where

$$A_q(t)=\sum_{n=1}^\infty\frac{t}{2t^{2n}\cos\dfrac{2n-1}{2q}\pi}-\sum_{\substack{n\geqq 2\\ 2\mid n\\ (q+1)\nmid n}}\frac{1}{2t^{n+1}\cos\dfrac{n}{2q+2}\pi}.$$

Since

$$\sum_{n=1}^\infty\frac{1}{t^{2n}\cos\dfrac{2n-1}{2q}\pi}=\frac{-1}{1+t^{2q}}\sum_{n=0}^{q-1}\frac{t^{2n}}{\cos\dfrac{2n+1}{2q}\pi},$$

$$\sum_{\substack{n\geqq 2\\ 2\mid n\\ (q+1)\nmid n}}\frac{1}{t^{n+1}\cos\dfrac{n}{2q+2}\pi}=\frac{-1}{1+t^{2q+2}}\sum_{n=0}^{q-1}\frac{t^{2n+1}}{\cos\dfrac{n+1}{q+1}\pi},$$

we obtain

$$2(1+t^{2q})(1+t^{2q+2})A_q(t)=-A_{q-1}t^{4q+1}+\sum_{n=1}^{q-1}(B_n-A_{n-1})t^{2q+2n+1}+B_0t^{2q+1}+\sum_{n=0}^{q-1}(B_n-A_n)t^{2n+1},$$

where

$$A_n=\frac{1}{\cos\frac{2n+1}{2q}\pi},\quad B_n=\frac{1}{\cos\frac{n+1}{q+1}\pi},\qquad n=0,1,\cdots,q-1.$$

Clearly

$$A_{q-1}<0,\qquad B_0>0,$$

$$B_n-A_n>0\qquad\text{for}\quad n;\ 0\leqq n\leqq\frac{q}{2}-1,$$

$$B_n-A_n<0\qquad\text{for}\quad n;\ \frac{q}{2}\leqq n\leqq q-1,$$

$$B_n-A_{n-1}>0\qquad\text{for}\quad n;\ 0\leqq n\leqq q-1,\ n\neq\frac{q}{2}.$$

Hence

$$2(1+t^{2q})(1+t^{2q+2})A_q(t)\geqq\sum_{n=q/2}^{q-1}(B_n-A_{n-1})t^{2q+2n+1}+\sum_{n=q/2}^{q-1}(B_n-A_n)t^{2n+1}$$

$$\geqq C_qt^{3q+1}+D_qt^{q+1}+\sum_{n=(q+2)/2}^{q-1}(2B_n-A_{n-1}-A_n)t^{2n+1},$$

where

$$C_q=B_{\frac{q}{2}}-A_{\frac{q}{2}-1},\qquad D_q=B_{\frac{q}{2}}-A_{\frac{q}{2}}.$$

From the concavity of $\cos x$ for $0\leqq x\leqq\frac{\pi}{2}$,

$$2B_n\geqq A_{n-1}+A_n,\qquad n=\frac{q}{2}+1,\cdots,q-1.$$

Thus, we have

$$A_q(t)\geqq\frac{C_qt^{3q+1}+D_qt^{q+1}}{2(1+t^{2q})(1+t^{2q+2})}.$$

Therefore

$$A^*(q)\geqq\frac{q}{2\pi}\int_1^\infty\frac{1}{t^{q+1}(1+t^{2q})}dt+\frac{q}{2\pi}\int_1^\infty\frac{t^q}{1+t^{2q+2}}dt+\frac{1}{\pi}\int_1^\infty\frac{A_q(t)}{t^{q+1}}dt$$

$$\geqq\frac{1}{2\pi}-\frac{1}{8(q+1)}+\frac{C_q}{2\pi}\int_1^\infty\frac{t^{2q}}{(1+t^{2q+1})^2}dt+\frac{D_q}{2\pi}\int_1^\infty\frac{1}{t^{4q+2}}dt$$

$$=\frac{1}{2\pi}-\frac{1}{(8q+1)}+\frac{C_q}{4(2q+1)\pi}+\frac{D_q}{2(4q+1)\pi}.$$

Further

$$\lim_{q\to\infty}\frac{C_q}{q}=-\frac{4}{\pi},\qquad\lim_{q\to\infty}\frac{D_q}{q}=0.$$

Hence

$$\varliminf_{\substack{q\to\infty\\2|q}}A^*(q)\geqq\frac{1}{2\pi}-\frac{1}{2\pi^2}.$$

Next consider the case that q is odd. In this case we select (s_n) such that

$$s_{2n}=\frac{4n-1}{2q}\pi \qquad n=1, 2, \cdots, \frac{q-1}{2}$$

$$s_{2n+1}=\frac{4n+1}{2q+1}\pi \qquad n=0, 1, \cdots, \frac{q-1}{2}$$

$$s_{q+1}=\pi .$$

By Shea's representation, we obtain

$$T(r, G)\geqq\frac{1}{\pi}\int_{I_q}\log|G(re^{iu})|\,du$$

$$=N(r)-\int_0^\infty N(t)H_q(t, r, s_1\, s_2 \cdots s_{q+1})dt$$

$$\geqq N(r)\Big(1-\int_1^\infty H_q(t, 1, s_1\, s_2 \cdots s_{q+1})dt\Big),$$

where

$$I_q=\sum_{n=1}^{(q+1)/2}[s_{2n-1}, s_{2n}].$$

Put $A^*(q)$ such that

$$A^*(q)=-\int_1^\infty(H_q(t, 1, s_1\, s_2 \cdots s_{q+1})dt.$$

Then, Lemma 3 yields

$$A^*(q)\geqq\frac{1}{\pi}-\frac{1}{4}.$$

The same process leads

$$-\pi t^{q+1}H_q(t, 1, s_1\, s_2 \cdots s_{q+1})$$

$$\geqq\frac{q-1}{2(1+t^{2q})}+\frac{(q+1)t^{2q+1}}{2(1+t^{2q+2})}+\frac{C_q t^{3q+1}+D_q t^{q+1}}{2(1+t^{2q})(1+t^{2q+2})},$$

where

$$C_q=\frac{1}{\cos\frac{q+1}{2q}\pi}-\frac{1}{\cos\frac{q}{2q+2}\pi},$$

$$D_q=\frac{1}{\cos\frac{q-1}{2q}\pi}-\frac{1}{\cos\frac{q}{2q+2}\pi}.$$

Hence

$$\varliminf_{\substack{q\to\infty\\ 2\nmid q}} A^*(q)\geqq\frac{1}{2\pi}-\frac{1}{2\pi^2}.$$

Therefore, we have the desired result.

4. Before proceeding with the proof of Theorem 3, we need the following lemmas.

LEMMA 4 [1; Lemma 2]. *Let $g(z)$ be an infinite product such that*

$$g(z)=\Pi E\left(\frac{z}{c_n}, 2\right),$$

where the sequence (c_n) *satisfies the following conditions:*

1) $\sum \frac{1}{|c_n|}=+\infty$ *and* $\sum\frac{1}{|c_n|^3}<+\infty$

2) $|\arg c_n|\leqq s$ *for some* $s(0\leqq s\leqq\pi/6)$.

Then

$$\lim_{r\to\infty} T(r, g(z)/g(-z))/r=+\infty .$$

LEMMA 5. *Let* $f(z)$ *be a meromorphic function of genus one or two, and whose zeros and poles are* (a_n) *and* (b_n), *respectively. If*

$$\sum\frac{1}{|a_n|}+\sum\frac{1}{|b_n|}=+\infty$$

and

$$|\arg a_n-\pi|\leqq s , \qquad |\arg b_n|\leqq s$$

for some s $(0\leqq s<\pi/6)$, *then*

$$\lim_{r\to\infty} T(r, f)/r=+\infty .$$

Proof. Put

$$A(z)=\Pi E\left(\frac{z}{a_n}, 2\right), \qquad B(z)=\Pi E\left(\frac{z}{b_n}, 2\right),$$

$$f(z)B(z)/A(z)=e^{P(z)} .$$

Since the genus of $f(z)$ is less than three, $P(z)$ must be a polynomial of degree at most two. Thus

$$f(z)/f(-z)=e^{cz}G(z)/G(-z) ,$$

where

$$G(z)=A(z)B(-z) .$$

Applying Lemma 4 to $G(z)$,

$$\lim_{r\to\infty} T(r, f(z)/f(-z))/r=+\infty .$$

Hence

$$\lim_{r\to\infty} T(r, f)/r=+\infty .$$

Proof of Theorem 3. If (a_n), (b_n) satisfy

$$\sum\frac{1}{|a_n|^q}+\sum\frac{1}{|b_n|^q}<+\infty .$$

then $F(z)$ is regular growth. So there is nothing to prove. Hence, we my assume that

$$\sum\frac{1}{|a_n|^q}+\sum\frac{1}{|b_n|^q}=+\infty .$$

Consider the auxiliary function

$$G(z)=\prod_{n=1}^{k} F(w^n \sqrt[k]{z}),$$

where

$$k=2\left[\frac{q-1}{2}\right]+1, \qquad w=\exp\left(i\frac{2\pi}{k}\right).$$

By the definition,

$$T(r^k, G)\leqq kT(r, F).$$

Thus, the order of $G(z)$ is not greater than λ/k. Further, from

$$\frac{\lambda}{k}\leqq\frac{q+1}{q-1}\leqq 2 \qquad (q\geqq 3),$$

the genus of $G(z)$ is at most two. The zeros and the poles of $G(z)$ are $(a_n{}^k)$ and $(b_n{}^k)$ respectively, and by $1\leqq k\leqq q$, we obtain

$$\sum\frac{1}{|a_n{}^k|}+\sum\frac{1}{|b_n{}^k|}=+\infty,$$

$$|\arg a_n{}^k-\pi|\leqq k\beta<\frac{\pi}{6}, \qquad |\arg b_n{}^k|\leqq k\beta.$$

Then, Lemma 5 yields

$$\lim_{r\to\infty} T(r, G)/r=+\infty.$$

This means that the lower order of $G(z)$ is at least one and hence that of $F(z)$ is at least k. Thus, we complete the proof.

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