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SOME PROPERTIES OF CANONICAL PRODUCTS OF FINITE GENUS

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Introduction. Let $f(z)$ be a canonical product of finite order with only negative zeros. If $\lambda>1$, then

$$\delta(0, f)>\frac{A}{1+A}$$

with an absolute constant $A>0$. This result is due to Edrei, Fuchs and Hellerstein [1]; for $\delta(0, f)$ and other standard terminology and notations used below, see [2].

Recently Ozawa obtained a fairly improved bound of the above constant A [3]. But it still remains open to find the best possible bound of A.

We now set

$$h(\lambda)=\inf \delta(0, f)$$

$$l(\lambda)=\sup \overline{\lim_{r\to\infty}} \frac{N(r, 1/f)}{\log M(r, f)},$$

where f ranges over all canonical products of finite order λ, with only negative zeros. Then the above problem reduces to get the exact value of $h(\lambda)$. In this note we shall prove first the following

THEOREM 1. *If* $1\leqq q\leqq\lambda<q+1$, *then we obtain*

$$h(\lambda)\leqq 1-\frac{1}{B(q)},$$

where

$$B(q)=2(2q+1)(2+\log (q+1)).$$

From the definitions it is clear that

$$1-h(\lambda)\geqq l(\lambda).$$

Hence Theorem 1 is contained in the following

THEOREM 2. *If* $1\leqq q\leqq\lambda<q+1$, *then*

$$l(\lambda)\geqq 1/B(q).$$

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Our proof of Theorem 2 depends on the construction of a canonical product $f(z)$ of order λ satisfying

$$\overline{\lim_{r\to\infty}}\frac{N(r, 1/f)}{\log M(r, f)}\geqq 1/B(q).$$

On the other hand Shea conjectured [4] that for entire functions of order $\lambda>1$

$$\overline{\lim_{r\to\infty}}\frac{N(r, 1/f)}{\log M(r, f)}\geqq\frac{|\sin\pi\lambda|}{\pi\lambda} \tag{1}$$

and Williamson showed [5] that for canonical products with only negative zeros (1) is valid under suitable hypotheses. In this connection Williamson asked if canonical products $f(re^{i\theta})$ of genus $q\geqq 2$ with only negative zeros asymptotically attain their maximum modulus for $|\theta|\leqq\pi/2$. It will be shown here that this is not in general the case. In fact for a canonical product $f(re^{i\theta})$ if we denote by $S(\alpha)$ a set of r such that the maximum modulus of $f(z)$ on $|z|=r$ only attains for $|\theta-\pi|<\alpha$, our third result is

THEOREM 3. *There exists a canonical product $f(re^{i\theta})$ of genus $q\geqq 1$ with only negative zeros such that for an arbitrarily given number $\varepsilon>0$ $S(\varepsilon)$ has upper density* 1.

1. Constructions of functions $n(r)$ and $N(r)$. Consider a decreasing sequence $\{\varepsilon_n\}$ $(n=1, 2, 3, \cdots)$ such that

$$\varepsilon_n\longrightarrow 0 \qquad \text{as} \qquad n\longrightarrow\infty$$

and define an increasing unbounded sequence $\{r_n\}$ $(n=1, 2, 3, \cdots)$ satisfying

$$nr_n^{\lambda+\varepsilon_n}<r_{n+1}^{\varepsilon_{n+1}}, \tag{1.1}$$

where λ is a positive constant. From the sequence $\{r_n\}$ we construct three sequences $\{t_n\}$, $\{s_n\}$ and $\{u_n\}$ by

$$u_n=\frac{r_n}{(\log r_n)^3},\qquad t_n=\frac{r_n}{(\log r_n)^2},\qquad s_n=\frac{r_n}{\log r_n}, \tag{1.2}$$

for $r_n>e$, respectively.
Denoting by $[X]$ the integral part of X, we define

$$L(n, p)=\left[\left(\frac{r_n}{u_n}\right)^{1+p}\right]=[(\log r_n)^{3(1+p)}]$$

with a positive constant p.
Let

$$u_{n,k}=u_n k^{\frac{1}{1+p}} \qquad (k=1, 2, 3, \cdots, L(n, p)) \tag{1.3}$$

and

$$m_{n,p}=\left[\frac{(1+p)r_n^{\lambda+\varepsilon_n}}{L(n,\,p)}\right]. \tag{1.4}$$

We may assume, by renumbering of $\{r_n\}$ if necessary, that the following relations are satisfied:

$$\begin{aligned} &u_{n,1}=u_n<t_n<s_n<u_{n,L(n,p)}\leqq r_n\,, \\ &r_n<u_{n+1}\,, \\ &L(n,\,p)\geqq 2\,, \\ &m_{n,p}\geqq 2 \end{aligned} \tag{1.5}$$

for $n=1, 2, 3, \cdots$.
We now put

$$n(r)=0\,, \qquad (0\leqq r<u_1)$$

$$n(r)=\begin{cases} km_{1,p} & (u_{1,k}\leqq r<u_{1,k+1}:\ k=1, 2, 3, \cdots, L(1,\,p)-1) \\ L(1,\,p)m_{1,p} & (u_{1,L(1,p)}\leqq r<u_2) \end{cases}$$

and for $n\geqq 2$

$$n(r)=\begin{cases} n(r_{n-1})+km_{n,p} & (u_{n,k}\leqq r<u_{n,k+1}:\ k=1, 2, 3, \cdots, L(n,\,p)-1) \\ n(r_{n-1})+L(n,\,p)m_{n,p} & (u_{n,L(n,p)}\leqq r<u_{n+1})\,. \end{cases} \tag{1.6}$$

Then we deduce from (1.1), (1.2) and (1.4) that

$$\frac{n(r_{n-1})}{n(r_n)}<\frac{(n-1)L(n-1,\,p)m_{n-1,p}}{L(n,\,p)m_{n,p}}\longrightarrow 0 \qquad (n\to\infty) \tag{1.7}$$

and

$$n(r_n)=n(r_{n-1})+L(n,\,p)m_{n,p}=(1+p)r_n^{\lambda+\varepsilon_n}(1+o(1)) \qquad (n\to\infty)\,. \tag{1.8}$$

We next notice that if $t_n\leqq r\leqq u_{n,L(n,p)}$, there is a k such that

$$u_{n,k}\leqq r<u_{n,k+1}$$

namely,

$$k\leqq\left(\frac{r}{u_n}\right)^{1+p}<k+1 \qquad (1\leqq k\leqq L(n,\,p))$$

and then, in view of (1.3) and (1.6),

$$\left[\left(\frac{r}{u_n}\right)^{1+p}\right]m_{n,p}+n(r_{n-1})\leqq n(r)<\left(\frac{r}{u_n}\right)^{1+p}m_{n,p}+n(r_{n-1})\,. \tag{1.9}$$

By (1.2), (1.4) and (1.6) we obtain for $t_n\leqq r\leqq r_n$

$$\left[\left(\frac{r}{u_n}\right)^{1+p}\right]m_{n,p}\Big/\left(\frac{r}{u_n}\right)^{1+p}m_{n,p}=1+o(1) \qquad (n\to\infty) \tag{1.10}$$

and by (1.4) and (1.8)

$$\left(\frac{r}{u_n}\right)^{1+p}m_{n,p}=n(r_n)\left(\frac{r}{r_n}\right)^{1+p}(1+o(1)) \qquad (n\to\infty)\,. \tag{1.11}$$

Hence, by (1.7) and (1.9)-(1.11)

$$n(r)=n(r_n)\left(\frac{r}{r_n}\right)^{1+p}(1+o(1)) \qquad (n\to\infty) \tag{1.12}$$

for $t_n \leqq r \leqq r_n$.
We now set

$$N(r)=\int_0^r \frac{n(t)}{t}\,dt\,.$$

We deduce from (1.6) and (1.7) that if $s_n \leqq r \leqq r_n$,

$$\begin{aligned} N(r)&=N(t_n)+\int_{t_n}^r \frac{n(t)}{t}\,dt \\ &=N(t_n)+(1+o(1))n(r_n)\int_{t_n}^r \left(\frac{t}{r_n}\right)^{1+p}\frac{dt}{t} \\ &=N(t_n)+(1+o(1))\frac{n(r_n)}{1+p}\left(\frac{r}{r_n}\right)^{1+p} \qquad (n\to\infty)\,. \end{aligned}$$

On the other hand, if $s_n \leqq r \leqq r_n$,

$$\begin{aligned} N(t_n)&=\int_0^{t_n}\frac{n(t)}{t}\,dt \leqq n(t_n)\log t_n \\ &=(1+o(1))n(r_n)\left(\frac{r}{r_n}\right)^{1+p}\left(\frac{t_n}{r}\right)^{1+p}\log t_n \qquad (n\to\infty) \\ &=o\left(n(r_n)\left(\frac{r}{r_n}\right)^{1+p}\right) \qquad (n\to\infty) \end{aligned}$$

by (1.2) and (1.12).
Hence we obtain

$$N(r)=(1+o(1))\frac{n(r_n)}{1+p}\left(\frac{r}{r_n}\right)^{1+p} \qquad (n\to\infty) \tag{1.13}$$

for $s_n \leqq r \leqq r_n$
and

$$N(r_n)=(1+o(1))\frac{n(r_n)}{1+p} \qquad (n\to\infty)\,. \tag{1.14}$$

Finally we notice that both $n(r)$ and $N(r)$ have the same order λ.

2. Proof of Theorem 2. Let $q(\geqq 1)$ be an integer. Put

$$E(u,\,q)=(1-u)\exp\left(u+\frac{u^2}{2}+\cdots+\frac{u^q}{q}\right).$$

Let $\lambda(\geqq 1)$ be a positive number and choose the integer q satisfying $q+1>\lambda\geqq q$.
We consider the canonical product

$$\prod_{n=1}^{\infty}\prod_{k=1}^{L(n,p)} E\left(-\frac{z}{u_{n,k}},\,q\right)^{m_{n,p}}=f(z)\,, \tag{2.1}$$

where p is a positive constant and $L(n, p)$, $u_{n,k}$ and $m_{n,p}$ are the ones defined in Section 1. By the construction of $n(r)$ we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{L(n,p)}\frac{m_{n,p}}{u_{n,k}^{q+1}}=\int_0^{\infty}\frac{dn(r)}{r^{q+1}}$$

$$=(q+1)\int_0^{\infty}\frac{n(r)}{r^{q+2}}dr<\infty$$

and, since $u_{n,L(n,p)}\leqq r_n$,

$$\sum_{k=1}^{L(n,p)}\frac{m_{n,p}}{u_{n,k}^{q}}\geqq L(n, p)\frac{m_{n,p}}{u_{n,L(n,p)}^{q}}$$

$$\geqq\frac{(1+p)r_n^{\lambda+\varepsilon_n}}{r_n^q}(1+o(1)) \qquad (n\to\infty).$$

In view of (1.1) these inequalities show that the product in (2.1) converges absolutely and uniformly in any bounded part of the plane to an integral function $f(z)$ having the order λ and the genus q. Further $f(z)$ satisfies an inequality

$$\log|f(z)|\leqq(q+1)A(q+1)\left\{r^q\int_0^r\frac{n(t)}{t^{q+1}}dt+r^{q+1}\int_r^{\infty}\frac{n(t)}{t^{q+2}}dt\right\} \tag{2.2}$$

where $A(q+1)=2(2+\log(q+1))$ and $|z|=r$ [2].

We now put $p=2q$ in (1.4) and $|z|=r=r_n$ in (2.2). To estimate the first integral part of (2.2) we set

$$r_n^q\int_0^{r_n}\frac{n(t)}{t^{q+1}}dt=r_n^q\int_0^{s_n}\frac{n(t)}{t^{q+1}}dt+r_n^q\int_{s_n}^{r_n}\frac{n(t)}{t^{q+1}}dt.$$

By (1.12) we find

$$\begin{aligned}r_n^q\int_{s_n}^{r_n}\frac{n(t)}{t^{q+1}}dt&=(1+o(1))r_n^q n(r_n)\int_{s_n}^{r_n}\left(\frac{t}{r_n}\right)^{1+2q}\frac{dt}{t^{q+1}}\\&=(1+o(1))\frac{n(r_n)}{q+1} \qquad (n\to\infty).\end{aligned} \tag{2.4}$$

Suppose that

$$r_n^q\int_0^{s_n}\frac{n(t)}{t^{q+1}}dt=r_n^q\int_0^{u_n}\frac{n(t)}{t^{q+1}}dt+r_n^q\int_{n_n}^{t_n}\frac{n(t)}{t^{q+1}}dt+r_n^q\int_{t_n}^{s_n}\frac{n(t)}{t^{q+1}}dt$$

$$=I_1+I_2+I_3, \qquad \text{say}.$$

Then, we have

$$\begin{aligned}I_1=r_n^q\int_0^{u_n}\frac{n(t)}{t^{q+1}}dt&\leqq r_n^q\int_0^{u_{n-1}}\frac{n(t)}{t^{q+1}}dt+r_n^q\int_{u_{n-1}}^{u_n}\frac{n(t)}{t^{q+1}}dt\\&=o(n(r_n)) \qquad (n\to\infty)\end{aligned} \tag{2.5}$$

in view of (1.1) and (1.8). Similarly, by (1.12)

(2.6)
$$I_2=r_n^q\int_{u_n}^{t_n}\frac{n(t)}{t^{q+1}}dt\leqq n(t_n)r_n^q\frac{1}{q}\Big(\frac{1}{u_n^q}-\frac{1}{t_n^q}\Big)$$
$$=o(n(r_n)) \qquad (n\to\infty)$$

and

(2.7)
$$I_3=r_n^q\int_{t_n}^{s_n}\frac{n(t)}{t^{q+1}}dt\leqq n(s_n)r_n^q\frac{1}{q}\Big(\frac{1}{t_n^q}-\frac{1}{s_n^q}\Big)$$
$$=o(n(r_n)) \qquad (n\to\infty)\,.$$

Hence we deduce from (2.4)-(2.7) that

$$r_n^q\int_0^{r_n}\frac{n(t)}{t^{q+1}}dt=(1+o(1))\frac{n(r_n)}{q+1} \qquad (n\to\infty)\,.$$

We find next that, since the order of $n(r)$ is λ, for all sufficiently large n

$$r_n^{q+1}\int_{r_n}^{\infty}\frac{n(t)}{t^{q+2}}dt=r_n^{q+1}\int_{r_n}^{u_{n+1}}\frac{n(t)}{t^{q+2}}dt+r_n^{q+1}\int_{u_{n+1}}^{\infty}\frac{n(t)}{t^{q+2}}dt$$
$$\leqq\frac{n(r_n)}{q+1}+\frac{r_n^{q+1}}{(q+1-(\lambda+\varepsilon))u_{n+1}^{q+1-(\lambda+\varepsilon)}}\,,$$

where ε $(<q+1-\lambda)$ is a positive constant. Hence, (1.1) yields

$$r_n^{q+1}\int_{r_n}^{\infty}\frac{n(t)}{t^{q+1}}dt=\frac{n(r_n)}{q+1}(1+o(1)) \qquad (n\to\infty)\,.$$

We now obtain from (2.2), (2.8), (2.9) and (1.14)

$$\log M(r_n, f)\leqq(q+1)A(q+1)(1+o(1))N(r_n)(2q+1)\frac{1}{q+1} \qquad (n\to\infty)$$

which leads to

$$\varlimsup_{r\to\infty}\frac{N(r, 1/f)}{\log M(R, f)}\geqq\frac{1}{2(2q+1)A(q+1)}\,.$$

The assertion of Theorem 2 now follows from this inequality.

3. Proof of Theorem 3. We shall adopt the functions and the notations of Section 2. For m sufficiently large we consider R such that

(3.1)
$$r_m\log r_m\leqq R\leqq r_m(\log r_m)^2<u_{n+1}\,.$$

Let

$$G_q(R)=\begin{cases}\displaystyle\sum_{n=1}^{m}m_{np}\sum_{k=1}^{L(n,p)}\Big\{\frac{1}{2}\Big(\frac{R}{u_{n,k}}\Big)^2+\cdots+\frac{1}{q}\Big(\frac{R}{u_{n,k}}\Big)^q\Big\} & (q\geqq 2)\\ 0 & (q=1)\end{cases}$$

and

$$F(R,\theta)=\sum_{n=1}^{m}m_{n,p}\sum_{k=1}^{L(n,p)}\Big\{\frac{1}{2}\log\Big|1+\Big(\frac{R}{u_{n,k}}\Big)^2+2\frac{R}{u_{n,k}}\cos\theta\Big|-\frac{R}{u_{n,k}}\cos\theta\Big\}.$$

Since, by (1.1), $R/u_{m+1}<1/2$ for sufficiently large R, we have

$$\sum_{n\succ m} m_{n,p}\sum_{k=1}^{L(n,p)}\left|\log E\left(-\frac{Re^{i\theta}}{u_{n,k}},\ q\right)\right|\leqq 2\sum_{n\succ m} m_{n,p}\sum_{k=1}^{L(n,p)}\left(\frac{R}{u_{n,k}}\right)^{q+1}$$

$$\leqq 4n(r_m)\ ,$$

which yields

$$\log|f(Re^{i\pi})|=\sum_{n=1}^{m} m_{n,p}\sum_{k=1}^{L(n,p)}\log\left|E\left(-\frac{Re^{i\pi}}{u_{n,k}},\ q\right)\right|$$

$$+\sum_{n>m} m_{n,p}\sum_{k=1}^{J(n,p)}\log\left|E\left(-\frac{Re^{i\pi}}{u_{n,k}},\ q\right)\right|$$

$$\geqq F(R,\ \pi)+G_q(R)-4n(r_m)\qquad (R>R_0) \tag{3.2}$$

and

$$\log|f(Re^{i\theta})|\leqq F(R,\ \theta)+G_q(R)+4n(r_m)\qquad (R>R_0)\ . \tag{3.3}$$

Now the construction of $n(r)$ implies that

$$G_q(R)=\sum_{l=2}^{q}\left\{R^l\int_0^{r_m}\frac{n(t)}{t^{l+1}}\,dt+n(r_m)\frac{1}{l}\left(\frac{R}{r_m}\right)^l\right\}$$

and hence, by (2.4)-(2.7)

$$G_q(R)=n(r_m)(1+o(1))\sum_{l=2}^{q}\left(\frac{1}{2q-l+1}+\frac{1}{l}\right)\left(\frac{R}{r_m}\right)^l\qquad (R\to\infty)\ . \tag{3.4}$$

On the other hand, since

$$F(R,\ \theta)=\sum_{n=1}^{m} m_{n,p}\sum_{k=1}^{L(n,p)}\left\{\log\frac{R}{u_{n,k}}+\frac{1}{2}\log\left|1+\frac{u_{n,k}}{R}\left(2\cos\theta+\frac{u_{n,k}}{R}\right)\right|\right.$$

$$\left.-\frac{R}{u_{n,k}}\cos\theta\right\}\ ,$$

we have

$$\left|F(R,\ \theta)-\left(\int_0^R\log\frac{R}{t}\,dn(t)-R\cos\theta\int_0^R\frac{dn(t)}{t}\right)\right|<n(R)\log 2\qquad (R>R_0)$$

which implies that

$$\left|F(R,\ \theta)-\left\{N(R)-\cos\theta\left(R\int_0^{r_m}\frac{n(t)}{t^2}dt+n(r_m)\frac{R}{r_m}\right)\right\}\right| \tag{3.5}$$

$$<n(r_m)\log 2\qquad (R>R_0).$$

If $q\geqq 2$, by (3.2), (3.4), (3.5) and (2.4)-(2.7) we obtain

$$\log|f(Re^{i\pi})|\geqq N(R)+G_q(R)+(1+o(1))\left(n(r_m)\frac{R}{r_m}+R\int_0^{r_m}\frac{n(t)}{t^2}dt\right)\quad (R\to\infty)\ . \tag{3.6}$$

Similarily (3.3) and (3.5) yield

$$\log|f(Re^{i\theta})|\leqq N(R)+G_q(R)-(\cos\theta+o(1))\left(n(r_m)\frac{R}{r_m}+R\int_0^{r_m}\frac{n(t)}{t^2}dt\right)\ (R\to\infty). \tag{3.7}$$

By (3.4)

$$n(r_m)\frac{R}{r_m}/G_q(R) \longrightarrow 0 \qquad (R\to\infty)$$

and hence, from (3.6) and (3.7) we deduce that

$$\frac{\log|f(Re^{i\theta})|}{\log|f(Re^{i\pi})|} \leqq 1-(1+\cos\theta+o(1))C(R) \qquad (R\to\infty)$$

where

$$C(R)=\frac{R\int_0^{r_m}\frac{n(t)}{t^2}dt+n(r_m)\frac{R}{r_m}}{N(R)+G_q(R)}<0.$$

This inequality holds for each R satisfying (3.1) if m is sufficiently large and the assertions of Theorem 3 become obvious for $q\geqq 2$.

If $q=1$, $G_q(R)=o$ in (3.6) and (3.7), and hence, since

$$N(R)/n(r_m)\frac{R}{r_m} \longrightarrow 0 \qquad (R\to\infty),$$

we obtain

$$\frac{\log|f(Re^{i\theta})|}{\log|f(Re^{i\pi})|} \leqq -\cos\theta+o(1) \qquad (R\to\infty).$$

This completes the proof of Theorem 3.

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References

[1] Edrei, A., W. H. J. Fuchs and S. Hellerstein, Radial distribution and deficiencies of the values of a meromorphic function, Pacific Journ. Math. **11** (1961), 135-151.

[2] Hayman, W. K., Meromorphic functions, Oxford Press, 1964.

[3] Ozawa, M., Radial distribution of zeros and deficiency of a canonical product of finite genus, Kōdai Math. Sem. Rep. **25** (1973), 502-512.

[4] Shea, D. F., On the relations between the growth of a meromorphic function and the angular distribution of two of values, Ph. D. thesis, Syracuse University, 1965.

[5] Williamson, J., Remarks on the maximum modulus of an entire function with negative zeros, Quart. J. Math. Oxford (2), **21** (1970), 497-512.

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