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ON THE DEFICIENCIES AND THE EXISTENCE OF PICARD'S EXCEPTIONAL VALUES OF ENTIRE ALGEBROID FUNCTIONS

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1. Introduction. Some characteristic properties of algebroid functions with more than two branches have been recently made clear by Niino and Ozawa. These concern with the relations between the sum of deficiencies and the number of Picard's exceptional values. Toda showed that those are intimate with the theory of systems of entire functions and then he solved the problem in the general case. On the problems of this type, see the summary note, Toda [6].

In the notes, Niino and Ozawa [3], Ozawa [4] and Suzuki [5], they showed the following fact: Let $f(z)$ be a transcendental entire algebroid function defined by

$$(1) \qquad F(z, f)=f^n+A_1(z)f^{n-1}+\cdots+A_n(z)\equiv 0,$$

where A_j, $j=1, 2, \cdots, n$, are entire functions and $n=3, 4, 5$. Let a_j, $j=0, 1, \cdots, n$ be distinct finite numbers such that arbitrary $n-1$ functions of $\{F(z, a_j)\}_{j=0,1,\cdots,n}$ are linearly independent and $\sum_{j=0}^{n}\delta(a_j, f)+\sum_{\nu=1}^{n-3}\delta(a_{j_\nu}, f)>2n-3$ for all $n-3$ numbers a_{j_ν}, $\nu=1, 2, \cdots, n-3$ of $\{a_j\}_{j=0,1,\cdots,n}$.

Then there exists at least one Picard's exceptional value in $\{a_j\}_{j=0,1,\cdots,n}$.

In this note we shall show that this result is available for all $n\geqq 2$ and in the case of $n=5$, we shall obtain a slightly better result.

2. Regular family and algebroid functions.

DEFINITION. Let $f_j(z)$, $j=1, 2, \cdots, l$, be entire functions and $F_\nu=\sum_{j=1}^{l}a_{\nu j}f_j$, $\nu=1, 2, \cdots, N$ ($l\leqq N\leqq\infty$) linear combinations of f_j, $j=1, 2, \cdots, l$. We say that $\mathfrak{F}=\{F_\nu\}_{\nu=1,2,\cdots,N}$ is a regular family of linear combinations of f_j, $j=1, 2, \cdots, l$ when the matrices $(a_{\nu_k j})_{1\leqq k,j\leqq l}$ are regular for all l integers ν_k, $k=1, 2, \cdots, l$, $1\leqq\nu_k\leqq N$.

And we say that the elements $G_i\in\mathfrak{F}$, $i=1, 2, \cdots, k$ form a basis of $\mathfrak{F}$ if and only if G_i, $i=1, 2, \cdots, k$ are linearly independent and all of $\mathfrak{F}$ can be represented as linear combinations of G_i, $i=1, 2, \cdots, k$.

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LEMMA 1. *Let $f(z)$ be an entire algebroid function defined by the equation (1) and $\mathfrak{F}=\{F_\nu\}_{\nu=1,2,\cdots,N}$, $n+1\leqq N\leqq\infty$, a regular family of linear combinations of $1, A_1, \cdots, A_n$. Suppose that $G_\mu\in\mathfrak{F}$, $\mu=1, 2, \cdots, l$, form a basis of $\mathfrak{F}$.*

Then we have

$$T(r, f)=\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq j\leqq n}\{\log^+|A_j(re^{i\theta})|\}d\theta+O(1)$$

$$=\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq \mu\leqq l}\{\log^+|G_\mu(re^{i\theta})|\}d\theta+O(1)$$

$$=\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq \mu\leqq l}\{\log|G_\mu(re^{i\theta})|\}d\theta+O(1)\,.$$

Proof. The first equality was shown in Valiron [7].

$1, A_1, \cdots, A_n$ can be represented as linear combinations of $G_1, \cdots, G_l$, so we have

$$\max\{1, |A_1(z)|, \cdots, |A_n(z)|\}\leqq O(1)\max_{1\leqq\mu\leqq l}\{|G_\mu(z)|\}\,,$$

(2) $$\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq\mu\leqq l}\{\log^+|A_j(re^{i\theta})|\}d\theta\leqq\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq\mu\leqq l}\{\log|G(re^{i\theta})|\}d\theta+O(1)\,.$$

On the other hand, from $G_j\in\mathfrak{F}$, we have

$$|G_\mu(z)|\leqq O(1)\max\{1, |A_1(z)|, \cdots, |A_n(z)|\}\,,$$

$$\max_{1\leqq\mu\leqq l}|G_\mu(z)|\leqq O(1)\max\{1, |A_1(z)|, \cdots, |A_n(z)|\}$$

and hence

(3) $$\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq\mu\leqq l}\{\log^+|G_\mu(re^{i\theta})|\}d\theta\leqq\frac{1}{2\pi n}\int_0^{2\pi}\max_{1\leqq j\leqq n}\{\log^+|A_j(re^{i\theta})|\}d\theta+O(1)\,.$$

By (2) and (3), we obtain the lemma. (Q.E.D.)

LEMMA 2 (Nevanlinna [2]). *Let $f_j(z)$, $j=1, 2, \cdots, l$ be entire functions, non constants, and linearly independent such that $f_1+\cdots+f_l=1$.*

Then we have

$$\frac{1}{2\pi}\int_0^{2\pi}\max_{1\leqq j\leqq l}\{\log^+|f_j(re^{i\theta})|\}d\theta\leqq\sum_{j=1}^{l}N(r, 0, f_j)+S(r)\,,$$

where $S(r)=O(\log T(r)+\log r)$, $T(r)=\max_{1\leqq j\leqq l}T(r, f_j)$ as $r\to\infty$ possibly outside a set of r of finite linear measure when the order of $T(r)$ is infinite.

Proof. By $f_1+\cdots+f_l=1$ and $f_1^{(\mu)}+\cdots+f_l^{(\mu)}=0$ for $\mu\geqq 1$, we have $f_j=\Delta_j/\Delta$, $j=1, 2, \cdots, l$, *where*

$$\Delta=\begin{vmatrix} 1 & \cdots\cdots\cdots & 1 \\ f_1'/f_1 & \cdots & f_l'/f_l \\ & \cdots\cdots\cdots & \\ f_1^{(l-1)}/f_1 & \cdots & f_l^{(l-1)}/f_l \end{vmatrix}$$

and Δ_j, $j=1, \cdots, l$ are $(1, j)$-minor determinants of Δ. Hence we have

$$\max_{1\leqq j\leqq l}\{\log^+|f_j|\}\leqq \max_{1\leqq j\leqq l}\left\{\log^+|\Delta_j|+\log^+\left|\frac{1}{\Delta}\right|\right\}$$

$$\leqq \sum_{j=1}^{l}\log^+|\Delta_j|+\log^+\left|\frac{1}{\Delta}\right|,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\max_{1\leqq j\leqq l}\{\log^+|f_j(re^{i\theta})|\}d\theta\leqq\sum_{j=1}^{l}m(r, \Delta_j)+m\left(r, \frac{1}{\Delta}\right)$$

$$\leqq\sum_{j=1}^{l}m(r, \Delta_j)+T\left(r, \frac{1}{\Delta}\right)=\sum_{j=1}^{l}m(r, \Delta_j)+m(r, \Delta)+N(r, \Delta)+O(1)\,.$$

$N(r, \Delta)\leqq\sum_{j=1}^{l}N(r, 0, f_j)$ because $f_1\cdots f_l\Delta$ is entire, and $\sum_{j=1}^{l}m(r, \Delta_j)+m(r, \Delta)=S(r)$. (Q. E. D.)

COROLLARY. *Let $f(z)$ be an entire algebroid function defined by the equation (1) and a_j, $j=0, 1, \cdots, n$ distinct finite numbers such that $g_j(z)=F(z, a_j)$, $j=0, 1, \cdots, n$ are linearly independent. Then we have $\sum_{j=0}^{n}\delta(a_j, f)\leqq n$.*

Proof. By the distinctness of a_j, $j=0, 1, \cdots, n$, $q_0g_0+\cdots q_ng_n=1$, $q_j\neq 0$, $j=0, 1, \cdots, n$. By the definition, $N(r, a_j, f)=N(r, 0, g_j)/n$. So we have by Lemma 1 and Lemma 2

$$T(r, f)<\sum_{j=0}^{n}N(r, a_j, f)+S(r)$$

and further $T(r)=\max T(r, g_j)\leqq nT(r, f)+O(1)$, then $S(r)=O(\log T(r)+\log r)=O(\log T(r, f)+\log r)$ as $r\to\infty$ possibly outside a set of r of finite linear measure when the order of $T(r, f)$ is infinite. Hence

$$\sum_{j=0}^{n}\delta(a_j, f)\leqq n\,. \qquad \text{(Q. E. D.)}$$

3. Existence of Picard's exceptional values.

THEOREM 1. *Let $f(z)$ be a transcendental entire algebroid function defined by the equation (1) with $n\geqq 2$. Let a_j, $j=0, 1, \cdots, n$, be distinct finite numbers and $g_j(z)=F(z, a_j)$, $j=0, 1, \cdots, n$ satisfy the following conditions:*

(i) *Arbitrary $n-1$ functions of $\{g_j\}_{j=0,1,\cdots,n}$ are linearly independent.*

(ii) $\sum_{j=0}^{n}\delta(a_j, f)+\sum_{\nu=1}^{n-3}\delta(a_{j_\nu}, f)>2n-3$ *for all $n-3$ numbers a_{j_ν}, $\nu=1, 2, \cdots, n-3$,* of $\{a_j\}_{j=0,1,\cdots,n}$.

In the case of $n=2$, *the condition* (ii) *is replaced by* $\sum_{j=0}^{2}\delta(a_j, f)>2$.

Then there exists at least one Picard's exceptional value in $\{a_j\}_{j=0,1,\cdots,n}$.

Proof. Assume that any g_j is not constant. We set λ the number of distinct non-trivial linear relations among $1, A_1, \cdots, A_n$. The condition (i) implies $0\leqq\lambda\leqq 2$ immediately. However $\lambda=1$ is the case here. We shall show this in the following.

If $\lambda=0$, g_j, $j=0, 1, \cdots, n$ are linearly independent. So by Corollary of Lemma 2, we have $\sum_{j=0}^{n}\delta(a_j, f)\leqq n$ and

$$\sum_{j=0}^{n}\delta(a_j, f)+\sum_{\nu=1}^{n-3}\delta(a_{j_\nu}, f)\leqq 2n-3\,.$$

This is a contradiction.

If $\lambda=2$, we can take $F_k\in\{1, A_1, \cdots, A_n\}$, $k=1, 2, \cdots, n-1$ so that they form a basis of $\{1, A_1, \cdots, A_n\}$. Represent g_j, $j=0, 1, \cdots, n$ by F_k, $k=1, 2, \cdots, n-1$, then $\{g_j\}_{j=0,1,\cdots,n}$ is a regular family of linear combinations of F_k, $k=1, 2, \cdots, n-1$ because of the condition (i). By Cartan [1] and Lemma 1,

$$\sum_{j=0}^{n}\delta(a_j, f)\leqq n-1\,.$$

This leads also to a contradiction. Now, $\lambda=1$ and so we can take n functions F_k, $k=1, 2, \cdots, n$ from $\{1, A_1, \cdots, A_n\}$ as a basis of $\{1, A_1, \cdots, A_n\}$.

Represent g_j, $j=0, 1, \cdots, n$ as linear combinations of F_k, $k=1, 2, \cdots, n$ and suppose that any n functions of $\{g_j\}_{j=0,1,\cdots,n}$ are linearly independent, then $\{g_j\}_{j=0,1,\cdots,n}$ is a regular family of linear combinations of F_k, $k=1, 2, \cdots, n$. So similarly to the above, we have

$$\sum_{j=0}^{n}\delta(a_j, f)\leqq n\,.$$

This is a contradiction.

Now we may assume that g_j, $j=0, 1, \cdots, n-1$ are linearly dependent;

$$\sum_{j=0}^{n-1}\beta_j g_j=0, \qquad \beta_j\neq 0, \qquad j=0, 1, \cdots, n-1\,, \tag{4}$$

by the condition (i). Since $\lambda=1$, n functions of $\{g_j\}_{j=0,1,\cdots,n}$, one of which is g_n, are linearly independent and form a basis.

Because of the distinctness of a_j, $j=0, 1, \cdots, n$, we have

$$q_0 g_0+q_1 g_1+\cdots+q_n g_n=1\,, \qquad q_j\neq 0,\ j=0, 1, \cdots, n\,. \tag{5}$$

Set $\beta_0=q_0$. From (4) and (5) it follows that

$$(q_1-\beta_1)g_1+\cdots+(q_{n-1}-\beta_{n-1})g_{n-1}+q_n g_n=1\,.$$

Hence we have

(6) $$\alpha_1 g_1 + \cdots + \alpha_n g_n = 1, \qquad \alpha_n \neq 0.$$

If all $\alpha_j \neq 0$, since g_j, $j=1, 2, \cdots, n$ form a basis of $\{g_j\}_{j=0,1,\cdots,n}$, using Lemma 1 and Lemma 2, we obtain

$$\sum_{j=1}^{n} \delta(a_j, f) \leqq n-1,$$

and this is a contradiction. Thus we may set $\alpha_1 = 0$.

In the case of $n=2$, we have $\alpha_2 g_2 = 1$, $\alpha_2 \neq 0$ and so at least one of $\{g_j\}_{j=0,1,2}$ is a constant, i.e., there exists at least one lacunary value and hence Picard's exceptional value in $\{a_j\}_{j=0,1,2}$.

We consider the case of $n \geqq 3$ in the rest. We may set that non-zero elements of $\{\alpha_2, \cdots, \alpha_{n-1}\}$ are $\alpha_k, \cdots, \alpha_{n-1}$, $2<k<n-1$. The equation (6) is reduced to

(6′) $$\alpha_k g_k + \cdots + \alpha_n g_n = 1, \ \alpha_j \neq 0.$$

Set $\beta_k = \alpha_k$. From (6′) and (4) we obtain

$$-\beta_0 g_0 - \beta_1 g_1 - \cdots - \beta_{k-1} g_{k-1} + (\alpha_{k+1} - \beta_{k+1}) g_{k+1} + \cdots + (\alpha_{n-1} - \beta_{n-1}) g_{n-1} + \alpha_n g_n = 1.$$

Since $g_0, \cdots, g_{k-1}, g_{k+1}, \cdots, g_n$ form a basis of $\{g_j\}_{j=0,1,\cdots,n}$, one of the coefficients is zero, say, $\alpha_{k+1} - \beta_{k+1} = 0$. Thus we have

(7) $$-\beta_0 g_0 - \cdots - \beta_{k-1} g_{k-1} + (\alpha_{k+2} - \beta_{k+2}) g_{k+2} + \cdots + (\alpha_{n-1} - \beta_{n-1}) g_{n-1} + \alpha_n g_n = 1.$$

Let g_{j_ν}, $\nu = 1, 2, \cdots, l$ be the functions of $\{g_j\}_{j=0,1,\cdots,n}$ which appear with non-zero coefficients in both equations (6′) and (7). Evidently $1 \leqq l \leqq n-k-1 \leqq n-3$. Applying Lemma 1 and Lemma 2 to the equations (6′) and (7), we have

$$\sum_{j=0}^{n} \delta(a_j, f) + \sum_{\nu=1}^{l} \delta(a_{j_\nu}, f) \leqq n+l.$$

Let a_{j_ν}, $\nu = l+1, \cdots, n-3$ be any $n-l-3$ numbers of $\{a_j\}_{j=0,1,\cdots,n} - \{a_{j_\nu}\}_{\nu=1,\cdots,l}$. Then

$$\sum_{j=0}^{n} \delta(a_j, f) + \sum_{\nu=1}^{n-3} \delta(a_{j_\nu}, f) \leqq 2n-3.$$

This is a contradiction.

COROLLARY. *If $T(r, f) = T(r, g_j)/n + O(\log r)$ for some g_j, then the condition* (ii) *can be replaced by a weaker one*·

$$\sum_{j=0}^{n} \delta(a_j, f) > n.$$

The proof is clear.

Now, we have obtained the above theorem, but it is not the best for all $n \geqq 2$. Really we can show the following theorem in the case of $n=5$.

THEOREM 2. *Let $f(z)$ be a five-valued transcendental entire algebroid function defined by*

$$F(z, f)=f^5+A_1(z)f^4+\cdots+A_5(z)\equiv 0,$$

where A_j, $j=1, 2, \cdots, 5$ are entire.

Let $a_0, \cdots, a_5$ be six distinct finite numbers and $g_j(z)=F(z, a_j)$, $j=0, 1, \cdots, 5$ satisfy the following conditions:

(i) *Any four functions of $\{g_j\}_{j=0,1,\cdots,5}$ are linearly independent,*

(ii) $\sum_{j=0}^{5}\delta(a_j, f)+\delta(a_k, f)>6$ *for all a_k.*

Then there exists at least one Picard's exceptional value in $\{a_j\}_{j=0,1,\cdots,5}$.

Proof. Assume that all g_j, $j=0, 1, \cdots, 5$ are not constants. By the similar process in the proof of Theorem 1, we obtain the equations:

$$\beta_0 g_0+\beta_1 g_1+\cdots+\beta_4 g_4=0, \quad \beta_j\neq 0, \quad j=0, 1, \cdots, 4, \tag{8}$$

$$\beta_2 g_2+\beta_3 g_3+\alpha_4 g_4+\alpha_5 g_5=1, \quad \alpha_5\neq 0. \tag{9}$$

From these equations, we have

$$-\beta_0 g_0-\beta_1 g_1+(\alpha_4-\beta_4)g_4+\alpha_5 g_5=1. \tag{10}$$

In the case of $\alpha_4(\alpha_4-\beta_4)=0$, applying Lemma 1 and Lemma 2 to the equations (9) and (10), we have

$$T(r, f)<\sum_{j=0}^{n}N(r, a_j, f)+N(r, a_5, f)+S(r).$$

Hence,

$$\sum_{j=0}^{5}\delta(a_j, f)+\delta(a_5, f)\leqq 6.$$

This is a contradiction. In the case of $\alpha_4(\alpha_4-\beta_4)\neq 0$, from the equations (8) and (9), we have

$$-\frac{\alpha_4}{\beta_4}\beta_0 g_0-\frac{\alpha_4}{\beta_4}\beta_1 g_1+\left(1-\frac{\alpha_4}{\beta_4}\right)\beta_2 g_2+\left(1-\frac{\alpha_4}{\beta_4}\right)\beta_3 g_3+\alpha_5 g_5=1. \tag{11}$$

The functions $g_0, \cdots, g_3, g_5$ form a basis of $\{g_j\}_{j=0,1,\cdots,5}$ and all the coefficients of g_j, $j=0, 1, \cdots, 5$ in the equation (11) are non-zero, so by Lemma 1 and Lemma 2, as in the above, we have

$$\sum_{\substack{j=0\\ j\neq 4}}^{5}\delta(a_j, f)\leqq 4.$$

This is also a contradiction. (Q. E. D.)

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