H. CHODA
KŌDAI MATH. SEM. REP.
26 (1974), 1-21

ON FREELY ACTING AUTOMORPHISMS OF OPERATOR ALGEBRAS

BY HISASHI CHODA

§ 1. **Introduction.** The free action for automorphisms of operator algebras plays an important role from the begining in the theory of operator algebras. Murray and von Neumann used namelessly the notion for abelian von Neumann algebras to construct their factors in [16]. Afterwards, von Neumann named it in [21].

Many authors have investigated an automorphism with a property equivalent to the condition to be freely acting for an automorphism due to von Neumann ([2], [8], [9], [18], [22], [34] and others).

Recently, in [13] R. Kallman generalized the notion due to von Neumann and defined a freely acting automorphism of (not necessarily abelian) von Neumann algebras.

Very recently, M. Choda, I. Kasahara and R. Nakamoto [7] found that the definition of free action due to Kallman is applicable to any C^*-algebra and extendable some theorems in [13]. They introduced a concept of a dependent element of an automorphism of a C^*-algebra in order to investigate properties of a freely acting automorphism.

In this paper, we shall treat a freely acting automorphism of a operator algebra.

In § 2, we shall restate the definition and show a theorem on an automorphism of a C^*-algebra in [7]. A few additional properties of freely acting automorphisms are discussed.

In § 3, we shall prove that the tensor product $\alpha\otimes\beta$ of automorphisms α and β of C^*-algebras is freely acting if and only if α is freely acting or β is freely acting.

In § 4, we shall treat automorphisms of von Neumann algebras. We shall prove that the n-th power of an ergodic automorphism of a von Neumann algebra without minimal projections is freely acting, for every nonzero integer n. We shall show also that an automorphism commuting with an ergodic automorphism group is either freely acting or inner. As an application, we shall show that an ergodic abelian automorphism group is, in some sense, maximally abelian in the full group determined by it.

Finally, in § 6, we shall consider a unitary operator with a transversal

Received Dec. 22, 1972.

group. Using properties of an automorphism of an abelian von Neumann algebra and a theorem of Arveson [1] we shall show that the spectrum of a unitary operator with a transversal group is the entire unit circle.

§ 2. **Freely acting automorphisms.** In this paper, we shall assume that a C^*-algebra contains always the identity. A *-automorphism of a C^*-algebra is called simply an automorphism. We shall use the terminologies and the notations due to [8] without further explanations.

For an automorphism α of a C^*-algebra $\mathcal{A}$, an element A in $\mathcal{A}$ is called a *dependent element* of α if

$$AB=\alpha(B)A \tag{2.1}$$

is satisfied for every B in $\mathcal{A}$ (cf. [7]). If there is no dependent element of α up to 0, then α is called a *freely acting automorphism* of $\mathcal{A}$ (cf. [7] and [13]). This definition is a generalization of the following von Neumann's definition for an automorphism of an abelian von Neumann algebra (cf. [13]): For an abelian von Neumann algebra $\mathcal{A}$, an automorphism α of $\mathcal{A}$ is *freely acting* if for any nonzero projection P in $\mathcal{A}$, there exists a nonzero projection Q in $\mathcal{A}$ dominated by P such that

$$\alpha(Q)Q=0\,. \tag{2.2}$$

An automorphism α of a C^*-algebra $\mathcal{A}$ is called to be *ergodic* if every A in $\mathcal{A}$ such as

$$\alpha(A)=A \tag{2.3}$$

is scalar.

Let G be a group of automorphisms of a C^*-algebra $\mathcal{A}$. G is called to be *freely acting* if any $g\neq 1$ in G is freely acting, where 1 is the unit of G. G is *ergodic* if A in $\mathcal{A}$ which satisfies (2.3) for every g in G is scalar.

The following theorem on dependent elements is obtained in [7]. For the sake of completeness, we shall give a proof which is partly simplified.

THEOREM 1. *Let $\mathcal{A}$ be a C^*-algebra, α an automorphism of $\mathcal{A}$ and $\mathcal{Z}$ the center of $\mathcal{A}$. If A is a dependent element of α, then A satisfies the following conditions;*

(1) $A^*A\in\mathcal{Z}$ *and* $AA^*\in\mathcal{Z}$,

(2) $A^*A=AA^*$,

(3) A^* *is a dependent element of* α^{-1},

and

(4) $\alpha(A)=A$.

Proof. By the equality (2.1), we have

$$BA^*=A^*\alpha(B) \tag{2.4}$$

for every B in $\mathcal{A}$. Multiplying A from the left in the both sides of (2.4), we have, for every B in $\mathcal{A}$,

$$AA^*\alpha(B)=ABA^*=\alpha(B)AA^*,$$

so that $AA^*\in\mathcal{Z}$. Similarly, we have $A^*A\in\mathcal{Z}$.

Using (1),

$$\begin{aligned}(AA^*-A^*A)^*(AA^*-A^*A)&=AA^*AA^*-AA^*A^*A-A^*AAA^*+A^*AA^*A\\&=A^*AAA^*-A^*AAA^*-A^*AAA^*+A^*AAA^*\\&=0.\end{aligned}$$

Therefore we have $A^*A=AA^*$.

In the equality (2.4), replacing B by $\alpha^{-1}(B)$,

$$A^*B=\alpha^{-1}(B)A^* \tag{2.5}$$

for every B in $\mathcal{A}$, that is, A^* is a dependent element of α^{-1}.

Using the relations (2.1), (2) and (2.4),

$$\begin{aligned}0&\leqq(\alpha(A)-A)^*(\alpha(A)-A)\\&=\alpha(A^*A)-\alpha(A^*)A-A^*\alpha(A)+A^*A\\&=\alpha(A^*A)-AA^*-AA^*+A^*A\\&=\alpha(A^*A)-A^*A.\end{aligned} \tag{2.6}$$

Hence we have

$$A^*A\leqq\alpha(A^*A). \tag{2.7}$$

On the other hand, in the above calculation (2.6), we can replace α by α^{-1} using the relation (2.5). Hence we have

$$A^*A\leqq\alpha^{-1}(A^*A), \tag{2.8}$$

it follows that

$$\alpha(A^*A)=A^*A. \tag{2.9}$$

Therefore, by (2.6), we conclude that $\alpha(A)=A$. This completes the proof.

In the special case, the free action of an automorphism of a C^*-algebra is characterized as the following:

PROPOSITION 1. *Let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space $\mathfrak{H}$*

and α an automorphism of $\mathcal{A}$ induced by a unitary operator U on $\mathfrak{H}$;

$$\alpha(A)=UAU^*$$

for each A in $\mathcal{A}$. Then α is freely acting on $\mathcal{A}$ if and only if U satisfies

$$U\mathcal{A}'\cap\mathcal{A}=(0)\,, \tag{2.1'}$$

where $\mathcal{A}'$ is the commutant of $\mathcal{A}$.

Proof. Assume that α is freely acting on $\mathcal{A}$. For every A in $U\mathcal{A}'\cap\mathcal{A}$, there exists an operator A' in $\mathcal{A}'$ such as

$$A=UA'.$$

For any B in $\mathcal{A}$,

$$AB=UA'B=UBA'=\alpha(B)UA'=\alpha(B)A.$$

By the assumption that α is freely acting on $\mathcal{A}$, we have $A=0$, that is, $U\mathcal{A}'\cap\mathcal{A}=(0)$.

Conversely, assume that $U\mathcal{A}'\cap\mathcal{A}=(0)$. Take any dependent element A of α and fix. For any B in $\mathcal{A}$, by (2.1),

$$AB=\alpha(B)A=UBU^*A.$$

So,

$$U^*AB=BU^*A,$$

for any B in $\mathcal{A}$, it follows that $U^*A\in\mathcal{A}'$. Therefore,

$$A\in U\mathcal{A}'\cap\mathcal{A}=(0),$$

that is, $A=0$. Hence α is freely acting on $\mathcal{A}$.

In the case that an automorphism of a C^*-algebra of operators on a Hilbert space can be extended to its weak closure, we have the following

PROPOSITION 2. *Let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space $\mathfrak{H}$, $\mathcal{B}$ the weak closure of $\mathcal{A}$ and α an automorphism of $\mathcal{A}$ which can be extended to an automorphism β of $\mathcal{B}$. If β is freely acting on $\mathcal{B}$, then α is freely acting on $\mathcal{A}$.*

Proof. Let A be a dependent element of α, then we have

$$AB=\alpha(B)A=\beta(B)A$$

for every B in $\mathcal{A}$. Since $\mathcal{A}$ is σ-weakly dense in $\mathcal{B}$ and β is σ-weakly continuous, we have

$$AB=\beta(B)A$$

for every B in $\mathcal{B}$, that is, A is a dependent element of β. By the assumption that β is freely acting on $\mathcal{B}$, it follows that $A=0$. Therefore α is freely acting on $\mathcal{A}$.

REMARK. The converse of Proposition 2 is not true. In fact, let $\mathfrak{H}$ be an infinite dimentional Hilbert space, $\mathcal{A}$ the C^*-algebra generated by all compact operators and the identity operator on $\mathfrak{H}$ and α an automorphism of $\mathcal{A}$ induced by a unitary operator U on $\mathfrak{H}$ such as $U \notin \mathcal{A}$. Then $\mathcal{A}'$ is the algebra of scalar multiples of the identity operator on $\mathfrak{H}$. Therefore

$$U\mathcal{A}' = \{\lambda U \,;\ \lambda \text{ is a scalar}\} .$$

Since $U \notin \mathcal{A}$, it follows that

$$U\mathcal{A}' \cap \mathcal{A} = (0) .$$

Hence, by Proposition 1, α is freely acting on $\mathcal{A}$.

On the other hand, since the weak closure of $\mathcal{A}$ is the von Neumann algebra of all bounded operators on $\mathfrak{H}$, α is inner on the weak closure of $\mathcal{A}$.

§ 3. Tensor product.

R. Kallman [13] has shown the following Theorem.

THEOREM 2 (Kallman). *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras, and α and β automorphisms of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the automorphism $\alpha \otimes \beta$ of the tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is freely acting if and only if either α or β is freely acting.*

The Kallman Theorem is a generalization of a theorem for factors of type II_1 by M. Nakamura and Z. Takeda [19]. The proof of Kallman depends on the matrix representation of elements of the tensor product.

In this section, we shall prove the Kallman Theorem for C^*-algebras using an expectation.

Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras, and $\mathcal{A} \odot \mathcal{B}$ the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$. Then, for an element X in $\mathcal{A} \odot \mathcal{B}$, define a norm by

$$\|X\| = \sup_{\pi, \rho} \|(\pi \otimes \rho)(X)\|$$

where π and ρ run over all representations of $\mathcal{A}$ and $\mathcal{B}$ respectively. Denote by $\mathcal{A} \otimes_\alpha \mathcal{B}$ the *C^*-tensor product* of $\mathcal{A}$ and $\mathcal{B}$, the completion of $\mathcal{A} \odot \mathcal{B}$ by this norm (cf. [30]).

Let α and β be automorphisms of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then by the definition of the norm of the C^*-tensor product, there exists an automorphism $\alpha \otimes \beta$ of $\mathcal{A} \otimes_\alpha \mathcal{B}$ which satisfies

$$(\alpha \otimes \beta)(A \otimes B) = \alpha(A) \otimes \beta(B) \tag{3.1}$$

for each A in $\mathcal{A}$ and B in $\mathcal{B}$. The automorphism $\alpha \otimes \beta$ is called the *tensor product automorphism* of α and β.

THEOREM 3. *Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a C^*-algebra and α (resp. β) an automorphism of $\mathcal{A}$ (resp. $\mathcal{B}$). Then the tensor product automorphism $\alpha\otimes\beta$ of $\mathcal{A}\otimes_\alpha\mathcal{B}$ is freely acting if and only if α or β is freely acting.*

In this place, we shall review briefly the theory of expectations developed by H. Umegaki and the others (cf. [17], [25] and [32]).

Let $\mathcal{C}$ be a C^*-algebra and $\mathcal{D}$ a C^*-subalgebra of $\mathcal{C}$. A positive linear mapping π of $\mathcal{C}$ onto $\mathcal{D}$ is called an *expectation* of $\mathcal{C}$ onto $\mathcal{D}$ if it satisfies the following conditions.

$$\pi(1)=1\,, \tag{3.2}$$

$$\pi(CD)=\pi(C)D \tag{3.3}$$

for every C in $\mathcal{C}$ and D in $\mathcal{D}$. An expectation π of $\mathcal{C}$ onto $\mathcal{D}$ satisfies

$$\pi(DC)=D\pi(C) \tag{3.4}$$

for every C in $\mathcal{C}$ and D in $\mathcal{D}$, by the positivity of π and (3.3).

Especially let $\mathcal{C}$ be a von Neumann algebra and $\mathcal{D}$ be a von Neumann subalgebra of $\mathcal{C}$, then an expectation π of $\mathcal{C}$ onto $\mathcal{D}$ is called *normal* if

$$C_\nu\uparrow C \quad \text{implies} \quad \pi(C_\nu)\uparrow\pi(C)\,, \tag{3.5}$$

where $C_\nu\uparrow C$ means that (C_ν) is a directed set of nondecreasing elements of $\mathcal{C}$ having C as the supremum.

The following theorem with respect to expectations is proved in [26] (also cf. [6], [27], [28] and [29]).

THEOREM A. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then there exist sufficiently many expectations of $\mathcal{A}\otimes_\alpha\mathcal{B}$ onto $\mathcal{A}$ and $\mathcal{B}$ identifying $\mathcal{A}$ with $\mathcal{A}\otimes I$ and $\mathcal{B}$ with $I\otimes\mathcal{B}$, namely for each nonzero element X, there exists an expectation π such as $\pi(X)\neq 0$. On the other hand, if $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, then there exist sufficiently many normal expectations of $\mathcal{A}\otimes\mathcal{B}$ onto $\mathcal{A}$ and $\mathcal{B}$ with same identification in $\mathcal{A}\otimes\mathcal{B}$.*

As we can prove two theorems by same technique, we shall give a proof of Theorem 3 in this place.

Necessity. Assume that neither α nor β is freely acting. Then there exist a nonzero dependent element A of α and a nonzero dependent element B of β. For $X=\sum_{i=1}^n A_i\otimes B_i\in\mathcal{A}\odot\mathcal{B}$,

$$\begin{aligned}(A\otimes B)X&=\sum_{i=1}^n AA_i\otimes BB_i\\ &=\sum_{i=1}^n \alpha(A_i)A\otimes\beta(B_i)B\\ &=(\sum_{i=1}^n \alpha(A_i)\otimes\beta(B_i))(A\otimes B)\end{aligned}$$

$$=(\alpha\otimes\beta)(X)(A\otimes B).$$

Since $\mathcal{A}\odot\mathcal{B}$ is uniformly dense in $\mathcal{A}\otimes_\alpha\mathcal{B}$, the mapping $X\rightarrow(A\otimes B)X$ is continuous and $\alpha\otimes\beta$ is continuous, it follows that for any X in $\mathcal{A}\otimes_\alpha\mathcal{B}$

$$(A\otimes B)X=(\alpha\otimes\beta)(X)(A\otimes B),$$

that is, $A\otimes B$ is a nonzero dependent element of $\alpha\otimes\beta$. Therefore $\alpha\otimes\beta$ is not free.

Sufficiency. Assume that α or β is freely acting. It is sufficient to prove that if α is freely acting, then $\alpha\otimes\beta$ is freely acting.

Take and fix any dependent element $X\in\mathcal{A}\otimes_\alpha\mathcal{B}$ of $\alpha\otimes\beta$, then

$$XY=(\alpha\otimes\beta)(Y)X$$

for every Y in $\mathcal{A}\otimes_\alpha\mathcal{B}$. Especially, we have

$$XA=\alpha(A)X \tag{3.6}$$

for every A in $\mathcal{A}$.

Let π be an expectation of $\mathcal{A}\otimes_\alpha\mathcal{B}$ onto $\mathcal{A}$, and operate π on the both sides of (3.6), then (3.3) and (3.4) imply

$$\pi(X)A=\alpha(A)\pi(X) \tag{3.7}$$

for every A in $\mathcal{A}$. Therefore $\pi(X)$ is a dependent element of α in $\mathcal{A}$. On the other hand, α is freely acting, and so $\pi(X)=0$. For any expectation π of $\mathcal{A}\otimes_\alpha\mathcal{B}$ onto $\mathcal{A}$, $\pi(X)=0$, so that by Theorem A, it follows that $X=0$. Thus $\alpha\otimes\beta$ is freely acting.

§ 4. **Ergodic automorphisms.** In this section, we shall show an improvement of the previous note [4] and its consequence.

Let $\mathcal{A}$ be a von Neumann algebra. Denote by $s(A)$ (resp. $c(A)$) the support (resp. central support) of an operator A in $\mathcal{A}$ which is the minimum projection E in $\mathcal{A}$ (resp. $\mathcal{A}\cap\mathcal{A}'$) with

$$AE=A. \tag{4.1}$$

At first, we shall show the following lemma on the support of a fixed point of an automorphism of a von Neumann algebra.

LEMMA 1. *Let α be an automorphism of a von Neumann algebra $\mathcal{A}$. If an element A in $\mathcal{A}$ satisfies*

$$\alpha(A)=A, \tag{4.2}$$

then A satisfies

$$\alpha(s(A))=s(A)$$

and

$$\alpha(c(A))=c(A)\,.$$

Proof. Since α is an automorphism of $\mathcal{A}$, $\alpha(s(A))$ is a projection in $\mathcal{A}$ and $\alpha(c(A))$ is a central projection. By the following calculation

$$A\alpha(s(A))=\alpha(As(A))=\alpha(A)=A\,,$$

$\alpha(s(A))$ satisfies the condition (4.1). Hence, by the definition of the support,

$$s(A)\leqq\alpha(s(A))\,.$$

On the other hand, since $\alpha(A)=A$, it follows that $\alpha^{-1}(A)=A$. Replacing α by α^{-1} in the above calculations, we have

$$s(A)\leqq\alpha^{-1}(s(A))\,.$$

Therefore we have

$$\alpha(s(A))=s(A)\,.$$

By the same argument, we have

$$\alpha(c(A))=c(A)\,.$$

Next, we shall give an another form of [11, Lemma 2] and [13, Theorem 1.1] (also cf. [7]).

LEMMA 2. *Let α be an automorphism of a von Neumann algebra $\mathcal{A}$ and A a dependent element of α. Then there exists a unitary operator U in $\mathcal{A}$ such that*

$$\alpha(Bc(A))=UBc(A)U^* \tag{4.3}$$

for each B in $\mathcal{A}$.

Proof. By the Theorem 1 (2) and (1), A has a polar representation;

$$A=U|A| \tag{4.4}$$

for a unitary operator U in $\mathcal{A}$ and $|A|$ in the center of $\mathcal{A}$. By the condition (2.1), we have

$$U|A|B=\alpha(B)U|A|$$

for every B in $\mathcal{A}$, that is,

$$\alpha(B)|A|=UBU^*|A|$$

for every B in $\mathcal{A}$.

On the other hand, since $|A|$ belongs to $\mathcal{A}\cap\mathcal{A}'$, it follows that the range projection of $|A|$ is the central support of $|A|$, which equals to the central support $c(A)$ of A.

Therefore we have

$$\alpha(B)c(A)=UBU^*c(A) \tag{4.5}$$

for every B in $\mathcal{A}$. By (4) of Theorem 1 and Lemma 1, the relation (4.5) equals to the following;

$$\alpha(Bc(A))=UBc(A)U^*$$

for every B in $\mathcal{A}$, which completes the proof of Lemma.

The following lemma may be known among specialists.

LEMMA 3. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. If $\mathcal{A}$ has no minimal projections, then every maximal abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$ is non-atomic.*

Proof. If $\mathcal{B}$ is not non-atomic, then there exists a minimal projection P in $\mathcal{B}$. The reduced von Neumann algebra $\mathcal{B}_P$ is the algebra $C_{P\mathfrak{H}}$ of scalar multiples of the identity operator on the Hilbert space $P\mathfrak{H}$.

Therefore, by $P\in\mathcal{A}\cap\mathcal{B}'$ and $\mathcal{A}\cap\mathcal{B}'=\mathcal{B}$, we have

$$\begin{aligned}\mathcal{A}_P&=\mathcal{A}_P\cap C'_{P\mathfrak{H}}=\mathcal{A}_P\cap\mathcal{B}'_P\\&=(\mathcal{A}\cap\mathcal{B}')_P=\mathcal{B}_P=C_{P\mathfrak{H}};\end{aligned}$$

which implies that P is minimal in $\mathcal{A}$. This contradicts the assumption.

It is known that all the powers of an ergodic measure preserving automorphism on a non-atomic probability measure space are freely acting [9]. As an analogous statement for factors of type II_1, Kallman proved in [14] that all the powers of an ergodic automorphism of a factor of type II_1 are outer.

In this place, we shall give a generalization of those results, which is an improvement of [4, Theorem 4].

THEOREM 4. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and α an ergodic automorphism of $\mathcal{A}$. If $\mathcal{A}$ has no minimal projections, then every nonzero power of α is freely acting on $\mathcal{A}$.*

Proof (cf. [14]). Assume that α^n is not freely acting on $\mathcal{A}$ for some positive integer n. Then there exists a nonzero dependent element A of α^n. Put $P=c(A)$, then $P\neq 0$ and by Lemma 2 there exists a unitary operator U in $\mathcal{A}$ which satisfies

$$\alpha^n(BP)=UBPU^*$$

for every B in $\mathcal{A}$. Let $\mathcal{B}$ be a maximal abelian von Neumann subalgebra of $\mathcal{A}$ containing U, then by (4.3), we have

$$\alpha^n(BP)=BP \tag{4.6}$$

for every B in $\mathcal{B}$. Take a nonzero projection Q in $\mathcal{B}$ dominated by P and put

$$R=Q+\alpha(Q)+\cdots+\alpha^{n-1}(Q). \tag{4.7}$$

Then by (4.6), we have

$$\alpha^n(Q)=\alpha^n(PQ)=PQ=Q,$$

which implies that $\alpha(R)=R$. Therefore R is some scalar multiple of the identity operator, say $R=\lambda I$. Since Q is a nonzero projection, it follows that $\lambda\geqq 1$.

Let $\mathcal{A}^*$ be the conjugate space of $\mathcal{A}$ and Σ the state space of $\mathcal{A}$, then Σ is a w^*-compact convex subset of $\mathcal{A}^*$. Therefore, by the Markov-Kakutani fixed point theorem, there exists a $\varphi\in\Sigma$ such as

$$\varphi(A)=\varphi(\alpha^k(A))$$

for every integer k and every A in $\mathcal{A}$.

On the other hand, by Lemma 3, there exists a nonzero projection Q in $\mathcal{B}$ such that

$$Q\leqq P$$

and

$$\varphi(Q)<1/n\,.$$

For this nonzero projection Q in $\mathcal{B}$, we have

$$\begin{aligned}\lambda=\varphi(R)&=\varphi(Q)+\varphi(\alpha(Q))+\cdots+\varphi(\alpha^{n-1}(Q))\\ &<1\,,\end{aligned}$$

which contradicts that $\lambda\geqq 1$.

R. Kallman has shown in [13] that every automorphism α of a von Neumann algebra $\mathcal{A}$ can be decomposed into the direct sum of the freely acting part and the inner part. That is, there exists a central projection P in $\mathcal{A}$ such that

$$\alpha(P)=P\,,$$

$$\alpha \text{ is inner on } \mathcal{A}_P$$

and

$$\alpha \text{ is freely acting on } \mathcal{A}_{I-P}\,.$$

Denote by F_α this central projection P which determines the inner part of α, then F_α satisfies the following

(4.9) $$F_\alpha=\sup\{c(A):\ A \text{ is a dependent element of } \alpha\}\,.$$

Let α and β be automorphisms of a von Neumann algebra $\mathcal{A}$. Put

(4.10) $$F(\alpha,\beta)=F_{\alpha^{-1}\beta}\,.$$

Let G be a group of automorphisms of $\mathcal{A}$ and α an automorphism of $\mathcal{A}$. Then, Y. Haga and Z. Takeda [11] have defined that α *depends* on G if

(4.11) $$\sup_{g\in G} F(\alpha, g)=I\,,$$

and denoted by $[G]$ the set of all automorphisms which depend on G and called $[G]$ the *full group* of G. This set $[G]$ is also a group [11, Lemma 3]. This definition is a generalization of a notion for an abelian von Neumann algebra due to H.A. Dye [9].

For an abelian ergodic group G of automorphisms of a von Neumann algebra $\mathcal{A}$, we shall discuss a property of an automorphism commuting with every element of G.

LEMMA 4 ([4, Lemma 1]). *Let $\mathcal{A}$ be a von Neumann algebra, G an ergodic group of automorphisms of $\mathcal{A}$ and α an automorphism of $\mathcal{A}$ such that*

$$\alpha g = g\alpha$$

for every $g \in G$. Then the automorphism α is either freely acting or inner.

Proof. By the definition of F_α, there exists a unitary U in $\mathcal{A}$ satisfying

$$\alpha(A)F_\alpha = UAF_\alpha U^* \tag{4.12}$$

for every A in $\mathcal{A}$. For every g in G, we have

$$\alpha(g(A))g(F_\alpha) = g(U)g(A)g(F_\alpha)g(U)^*$$

by the condition that $\alpha g = g\alpha$ and (4.12). Therefore, we have

$$\alpha(A)g(F_\alpha U) = g(F_\alpha U)A$$

for every A in $\mathcal{A}$, which implies that $g(F_\alpha U)$ is a dependent element of α. Hence, by (4.9), we have

$$g(F_\alpha) = F_\alpha$$

for any g in G.

On the other hand, by the assumption that G is ergodic, we have

$$F_\alpha = 0 \quad \text{or} \quad F_\alpha = I,$$

that is, α is either freely acting or inner.

THEOREM 5. *Let $\mathcal{A}$ be a von Neumann algebra, and G an abelian ergodic group of automorphisms of $\mathcal{A}$. If an automorphism α in $[G]$ satisfies*

$$\alpha g = g\alpha \tag{4.13}$$

for every g in G, then there exist a g in G and a unitary U in $\mathcal{A}$ such as

$$\alpha = g\cdot\phi_U,$$

where ϕ_U is an inner automorphism of $\mathcal{A}$ induced by U.

Proof. For every g in G, we have

$$(g^{-1}\alpha)h = h(g^{-1}\alpha)$$

for every h in G, because $h\alpha = \alpha h$ and G is abelian. Therefore, by Lemma 4, $g^{-1}\alpha$ is either freely acting or inner for each $g \in G$, that is,

$$F(\alpha, g) = 0 \quad \text{or} \quad I$$

for each $g \in G$.

On the other hand, by the condition that $\alpha \in [G]$, we have

$$\sup_{g \in G} F(\alpha, g) = I.$$

Therefore, we have

$$F(\alpha, g) = I$$

for some g in G, which implies that there exists a unitary U in $\mathcal{A}$ such as

$$g^{-1}\alpha = \phi_U.$$

This completes the proof.

As a consequence of Theorem 5, we have a theorem of Tam [24, Theorem (ii)], which is a sharpening of [5, Theorem 6];

COROLLARY. *Let $\mathcal{A}$ be an abelian von Neumann algebra and G an ergodic abelian group of automorphisms of $\mathcal{A}$. Then G is maximally abelian in $[G]$, that is, if $\alpha \in [G]$ satisfies*

$$\alpha g = g \alpha$$

for every g in G, then α belongs to G.

Proof. Such an automorphism α is written in a form

$$\alpha = g \cdot \phi_U$$

for some g in G and some unitary operator U in $\mathcal{A}$, by Theorem 5. On the other hand, $\mathcal{A}$ is abelian, so

$$\phi_U = 1.$$

Therefore we have $\alpha = g$ for some g in G.

We shall need subsequently the following theorem [4, Theorem 2], which is a generalization of Tam's Theorem [24, Theorem (i)]. For the sake of completeness, we shall give a proof.

THEOREM 6. *Let $\mathcal{A}$ be a von Neumann algebra and G an abelian ergodic group of outer automorphisms of $\mathcal{A}$. Then G is freely acting on $\mathcal{A}$.*

Proof. For any element g in G, we have

$$gh = hg$$

for every h in G. Then, by Lemma 4, g is either freely acting or inner. Since G is a group of outer automorphisms of $\mathcal{A}$, it follows that $g \neq 1$ (unit of G) is freely acting on $\mathcal{A}$, that is, G is freely acting on $\mathcal{A}$.

§ 5. Crossed product.

In [5], we have considered the extension of an automorphism of an abelian von Neumann algebra to the crossed product of the von Neumann algebra by an automorphism group, in several cases.

In this section, we shall generalize some theorems in [5].

In this note, we shall use the notions in [11] for the crossed product. According to [11], we shall give the definition of crossed product, briefly (also cf. [18], [23] and [31]).

Let $\mathcal{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$ and G a countable group of automorphisms of $\mathcal{A}$. Denote by $G\otimes\mathfrak{H}$ the product space in the sense of H. Umegaki [33] and by

$$\sum_{g\in G} g\otimes x_g \qquad (x_g\in\mathfrak{H})$$

an element of $G\otimes\mathfrak{H}$. Define an operator $g\otimes A$ ($g\in G$, $A\in\mathcal{A}$) on $G\otimes\mathfrak{H}$ by

$$(g\otimes A)(\sum_h h\otimes x_h)==\sum_h hg^{-1}\otimes h(A)x_h\,. \tag{5.1}$$

Then, we have

$$(g\otimes A)(h\otimes B)=gh\otimes h^{-1}(A)B \tag{5.2}$$

and

$$(g\otimes A)^*=g^{-1}\otimes g(A^*)\,. \tag{5.3}$$

The *crossed product* $G\otimes\mathcal{A}$ is defined as the von Neumann algebra on $G\otimes\mathfrak{H}$ generated by $\{g\otimes A : g\in G,\ A\in\mathcal{A}\}$. Since $\mathcal{A}$ is isomorphic to $\{1\otimes A : A\in\mathcal{A}\}$, we shall identify $A\in\mathcal{A}$ with $1\otimes A\in G\otimes\mathcal{A}$ and we consider $\mathcal{A}$ as a von Neumann subalgebra of $G\otimes\mathcal{A}$. $g\otimes I$ ($g\in G$) is a unitary operator in $G\otimes\mathcal{A}$, and it induces the automorphism g on $\mathcal{A}$;

$$(g\otimes I)(1\otimes A)(g\otimes I)^*=1\otimes g(A)\,. \tag{5.4}$$

We need the following theorem due to Y. Haga and Z. Takeda [11]:

THEOREM B. *Let $\mathcal{A}$ be a von Neumann algebra and G a countable freely acting group of automorphisms of $\mathcal{A}$. Assume that a unitary operator U in $G\otimes\mathcal{A}$ satisfies*

$$U\mathcal{A}U^*=\mathcal{A}\,.$$

Then U has a unique decomposition

$$U=\sum_{g\in G}(g\otimes Q_g)(1\otimes V)\,, \tag{5.5}$$

where V is a unitary operator in $\mathcal{A}$ and $\{Q_g\}$ (resp. $\{g(Q_g)\}$) is a family of mutually orthogonal central projections having sum I.

In Theorem B, denote by α an automorphism of $\mathcal{A}$ induced by U, then the relation (5.5) is equivalent to the following form;

$$U=\sum_{g\in G}(g\otimes F(\alpha, g))(1\otimes V)\,.$$

As a generalization of [5, Theorem 4], we shall show the following

THEOREM 7. *Let $\mathcal{A}$ be a von Neumann algebra, G a countable group of freely acting automorphisms of $\mathcal{A}$ and φ a faithful G-invariant state on $\mathcal{A}$. Assume that the set $I_g=\{hgh^{-1}: h\in G\}$ is infinite for each g $(\neq 1)$ in G. If an automorphism α of $\mathcal{A}$ is extended to an automorphism σ of $G\otimes\mathcal{A}$ such that*

$$\sigma(g\otimes I)=g\otimes I \tag{5.6}$$

for every g in G, then either σ is outer or α is inner.

Proof. Let σ is an inner automorphism induced by a unitary operator U in $G\otimes\mathcal{A}$. Since σ is an extension of an automorphism α of $\mathcal{A}$, it follows that

$$U\mathcal{A}U^*=\mathcal{A}\,.$$

Then, by Theorem B, U has a decomposition such as (5.5). On the other hand, by (5.6), we have

$$g\otimes I=\sigma(g\otimes I)=U(g\otimes I)U^*\,,$$

for every g in G, so that

$$U(g\otimes I)=(g\otimes I)U$$

for every g in G, or we have

$$\sum_{h\in G}hg\otimes g^{-1}(Q_h)g^{-1}(V)=\sum_{h\in G}gh\otimes Q_hV \tag{5.7}$$

for each g in G. Therefore, we have

$$\sum_{h\in G}h\otimes g^{-1}(Q_{hg^{-1}})g^{-1}(V)=\sum_{h\in G}h\otimes Q_{g^{-1}h}V$$

for each g in G, which implies

$$g^{-1}(Q_{hg^{-1}})g^{-1}(V)=Q_{g^{-1}h}V$$

for each $g\in G$ and $h\in G$. Replace $g^{-1}h$ to h, so we have

$$g^{-1}(Q_{ghg^{-1}}V)=Q_hV$$

for each $g\in G$ and $h\in G$. By the assumption, for each $h\neq 1$ in G, I_h is an infinite set, and so

$$1\geqq\varphi((\sum_{k\in I_h}Q_kV)^*(\sum_{k\in I_h}Q_kV))$$

$$\geqq n\varphi(V^*Q_hV)$$

for any positive integer n. Therefore we have

$$\varphi(V^*Q_hV)=0$$

for each $h\neq 1$ in G. Since φ is faithful, it follows that $Q_h=0$ for each $h\neq 1$ in G. Since

$$\sum_{h \in G} Q_h = I \,,$$

we have $Q_1 = I$, that is, $U = 1 \otimes V$. Therefore σ is induced by a unitary operator $1 \otimes V$ in $\mathcal{A}$, which implies that α is inner.

THEOREM 8. *Let $\mathcal{A}$ be a von Neumann algebra and G an abelian ergodic countable group of outer automorphisms of $\mathcal{A}$. If an automorphism α of $\mathcal{A}$ is extended to an inner automorphism σ of $G \otimes \mathcal{A}$ satisfying the condition* (5.6) *for every g in G, then σ is induced by $h \otimes I$ for some h in G. Especially $\alpha = h$ for some h in G.*

Proof. By Theorem 6, G is freely acting on $\mathcal{A}$. By Theorem B, a unitary operator U in $G \otimes \mathcal{A}$, which induces σ, has a decomposition such as (5.5). Since σ satisfies (5.6), we have

$$\sum_{h \in G} hg \otimes g^{-1}(Q_h V) = \sum_{h \in G} gh \otimes Q_h V \,,$$

by (5.7) in the proof of Theorem 7. Since G is abelian, it follows that

$$g^{-1}(Q_h V) = Q_h V$$

for each $g \in G$ and $h \in G$. Since G is ergodic, it follows that $Q_h V$ is a scalar for each h in G. Therefore we have, for each h in G,

$$Q_h = 0 \quad \text{or} \quad I \,.$$

On the other hand,

$$\sum_{h \in G} Q_h = I \,,$$

so we have that $Q_h = I$ for some h in G. It follows that

$$V = g^{-1}(V)$$

for all g in G. Therefore, by the ergodicity of G, V is a scalar λ with absolute value 1. So that, U has a form

$$U = (h \otimes I)\lambda \,.$$

Hence, the automorphism σ induced by U is equal to an automorphism induced by $h \otimes I$ for some $h \in G$.

§ 6. **Transversal group.**

In this section, we shall treat an automorphism induced by a unitary operator having a transversal group.

A strongly continuous unitary representation of the additive group of real numbers is called a *one-parameter group* of unitary operators. A one-parameter group of unitary operators $\{V_t\}$ on a Hilbert space $\mathfrak{H}$ is called a *transversal group* with λ for a unitary operator U on $\mathfrak{H}$ if $\{V_t\}$ satisfies the following commutation relation

(6.1) $$UV_t = V_{\lambda t}U \qquad (-\infty < t < +\infty)$$

for a nonzero real number λ. The notion of transversal groups of unitary operators is due to M. Kowada [15].

Assume that U has a transversal group $\{V_t\}$ with $|\lambda| \neq 1$. Let $\mathcal{A}$ be a von Neumann algebra generated by $\{V_t\}$. Put

(6.2) $$\alpha(A) = UAU^*$$

for each A in $\mathcal{A}$, then by (6.1), α is an automorphism of $\mathcal{A}$ and satisfies the following condition

(6.3) $$\alpha(V_t) = V_{\lambda t} \qquad (-\infty < t < +\infty).$$

LEMMA 5. *Let F_α be the central projection in $\mathcal{A}$ which determines the inner part of the above automorphism α, then*

(6.4) $$V_t F_\alpha = F_\alpha$$

for every real t.

Proof. Since $\mathcal{A}$ is abelian, α is the identity on $\mathcal{A}_{F_\alpha}$. Therefore, we have

$$\alpha(V_t F_\alpha) = V_t F_\alpha \qquad (-\infty < t < +\infty).$$

On the other hand, by (6.3) we have

$$V_t F_\alpha = \alpha(V_t F_\alpha) = \alpha(V_t)\alpha(F_\alpha) = V_{\lambda t} F_\alpha, \qquad (-\infty < t < +\infty).$$

So, for any integer n, we have

(6.5) $$V_{\lambda^n t} F_\alpha = V_t F_\alpha, \qquad (-\infty < t < +\infty).$$

Since, for λ with $|\lambda| < 1$ (resp. $|\lambda| > 1$), $\lambda^n t$ converges to 0 as $n \to +\infty$ (resp. $-\infty$), it follows that

$$V_{\lambda^n t} \longrightarrow I \qquad \text{(strongly)}.$$

Hence, by (6.5), we have

$$V_t F_\alpha = F_\alpha, \qquad (-\infty < t < +\infty).$$

THEOREM 9. *If U has a transversal group $\{V_t\}$ with $|\lambda| \neq 1$ satisfying the condition*

(6.6) $$V_t x = x \ \ (\textit{for all } t) \ \textit{implies} \ x = 0,$$

then the automorphism α defined by (6.2) is a freely acting automorphism of the von Neumann algebra $\mathcal{A}$ generated by $\{V_t\}$.

Proof. Let F_α be the central projection in $\mathcal{A}$ which determines the inner part of α. By Lemma 5, we have

$$V_t F_\alpha = F_\alpha \qquad (-\infty < t < +\infty).$$

On the other hand, V_t satisfies (6.6), so that $F_\alpha=0$. Therefore, α is freely acting on $\mathcal{A}$.

COROLLARY. *Let $\{V_t\}$, U be as same as in Theorem 9 and $\mathcal{B}$ a C^*-algebra generated by $\{V_t\}$. Then an automorphism β induced by U is freely acting on $\mathcal{B}$.*

Proof. Since β is induced by the unitary operator U, β is extended to the automorphism α of the weak closure $\mathcal{A}$ of $\mathcal{B}$. By Theorem 9, α is freely acting on $\mathcal{A}$. Therefore, by Proposition 2, β is freely acting on $\mathcal{B}$ too.

Let $\mathcal{G}$ be an abelian unitary group on a Hilbert space $\mathfrak{H}$. The action of $\mathcal{G}$ is said to be *nondegenerate* if, for every finite subset $\mathcal{F}$ of $\mathcal{G}$, there exists a nonzero vector x in $\mathfrak{H}$ such that Ux is orthogonal to Vx for every pair U, V of distinct elements of $\mathcal{F}$. This notion is introduced by W. Arveson [1] (also cf. [34, Theorem 5.5] and [20]) and he has proved

THEOREM C. *Let $\mathcal{G}$ be an abelian unitary group on a Hilber space $\mathfrak{H}$, $\mathcal{A}$ a C^*-algebra generated by $\mathcal{G}$ and $\sigma(\mathcal{A})$ the maximal ideal space of $\mathcal{A}$. Then the natural injection ρ of $\sigma(\mathcal{A})$ into the compact character group Γ of the discrete group $\mathcal{G}$ is onto if and only if the action of $\mathcal{G}$ is nondegenerate.*

Furthermore, as an appropriately translated version of Theorem C, Arveson has pointed out

THEOREM D. *The spectrum of a unitary operator U on a Hilbert space $\mathfrak{H}$ is the entire unit circle if and only if, for every integer $n\geqq 1$, there exists a nonzero vector x in $\mathfrak{H}$ such that $\{x, Ux, \cdots, U^n x\}$ are mutually orthogonal.*

Using Theorem D, we shall determine the spectrum of a unitary operator having a transversal group (cf. [10]).

THEOREM 10. *Let U be a unitary operator on a Hilbert space $\mathfrak{H}$. If U has, for $|\lambda|\neq 1$, a transversal group $\{V_t\}$ with*

$$V_t \neq I \tag{6.7}$$

for some real t, the spectrum $\sigma(U)$ of U is the entire unit circle.

Proof. Put

$$\mathfrak{N}=\{x\in\mathfrak{H}:\ V_t x=x \quad \text{for all } t\}\,,$$

then $\mathfrak{N}$ reduces $V_t(-\infty<t<+\infty)$. By (6.1), $\mathfrak{N}$ reduces U and by (6.7) $\mathfrak{N}^\perp\neq(0)$. Therefore, it is sufficient to assume that $\mathfrak{N}=(0)$. Hence $\{V_t\}$ satisfies the condition (6.6). Let α be the automorphism defined by (6.2) on the von Neumann algebra $\mathcal{A}$ generated by $\{V_t\}$, then α is freely acting on $\mathcal{A}$ by Theorem 9. Since $\{V_t\}$ is a transversal group for U, we have

$$U^n V_t = V_{\lambda^n t} U^n \qquad (-\infty<t<+\infty) \tag{6.8}$$

for any integer n, by the condition (6.1), which implies that the automorphism α^n induced by U^n is freely acting on $\mathcal{A}$.

Since α is freely acting and $\mathcal{A}$ is an abelian von Neumann algebra, there exists a nonzero projection P_1 in $\mathcal{A}$ such as

$$\alpha(P_1)P_1=0\,.$$

Since α^2 is freely acting on $\mathcal{A}$ too, there exists a nonzero projection P_2 in $\mathcal{A}$ dominated by P_1, such as

$$\alpha^2(P_2)P_2=0\,.$$

Since α^n is freely acting for any integer $n\geqq 1$, inductively, there exists a nonzero projection Q in $\mathcal{A}$ such as

$$\alpha^k(Q)Q=0 \qquad (1\leqq k\leqq n)\,. \tag{6.9}$$

Take a nonzero vector x in $Q\mathfrak{H}$, then for any k $(1\leqq k\leqq n)$, (6.9) implies that

$$\begin{aligned}(U^k x, x)&=(U^k Qx, x)=(QU^k Qx, x)\\ &=(Q\alpha^k(Q)U^k x, x)\\ &=0\,.\end{aligned}$$

Therefore, by Theorem D, the spectrum $\sigma(U)$ is the entire unit circle.

By the proof of the theorem, we have the following

COROLLARY. *Let $\mathcal{G}$ be a unitary group on a Hilbert space $\mathfrak{H}$, $\mathcal{A}$ an abelian von Neumann algebra on $\mathfrak{H}$ which is invariant under every element of $\mathcal{G}$ (that is, $U\mathcal{A}U^*=\mathcal{A}$ for every element U of $\mathcal{G}$) and G the group of automorphisms of $\mathcal{A}$ induced by elements of $\mathcal{G}$. If G is freely acting on $\mathcal{A}$, then, for every finite subset $\mathcal{F}$ of $\mathcal{G}$, there exists a nonzero vector x in $\mathfrak{H}$ such that Ux is orthogonal to Vx for every pair U, V of distinct elements of $\mathcal{F}$.*

This corollary is generalized as the following;

THEOREM 11. *Let $\mathcal{G}$ be a unitary group on a Hillbert space $\mathfrak{H}$ and $\mathcal{A}$ a von Neumann algebra on $\mathfrak{H}$ which is invariant under every element of $\mathcal{G}$. Assume that there exists an expectation π of $\mathcal{L}(\mathfrak{H})$, the von Neumann algebra of all bounded operators on $\mathfrak{H}$, onto $\mathcal{A}$. If the automorphism group G of $\mathcal{A}$ induced by elements of $\mathcal{G}$ is freely acting on $\mathcal{A}$, then there exists a state φ on $\mathcal{L}(\mathfrak{H})$ such as*

$$\varphi(U)=0$$

for every $U\neq I$ in $\mathcal{G}$.

Proof. Denote by ϕ_U an automorphism of $\mathcal{A}$ induced by a unitary operator U in $\mathcal{G}$, then we have

$$UA=\phi_U(A)U \tag{6.10}$$

for every A in $\mathcal{A}$, which implies

$$\pi(U)A=\phi_U(A)\pi(U) \tag{6.11}$$

for every A in $\mathcal{A}$. Therefore, $\pi(U)$ is a dependent element of ϕ_U. Since every ϕ_U $(U\neq I)$ is freely acting on $\mathcal{A}$, it follows that

$$\pi(U)=0 \tag{6.12}$$

for each $U\neq I$ in $\mathcal{G}$.

For a unit vector x in $\mathfrak{H}$, put

$$\varphi(T)=(\pi(T)x, x)$$

for every operator T in $\mathcal{L}(\mathfrak{H})$, φ is a state on $\mathcal{L}(\mathfrak{H})$ by the properties that π is positive linear and $\pi(I)=1$. And, by (6.12), we have

$$\varphi(U)=(\pi(U)x, x)=0$$

for all $U\neq I$ in $\mathcal{G}$. This completes the proof.

REMARK. Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$. $\mathcal{A}$ is said to have an *extension property* if there exists an expectation of $\mathcal{L}(\mathfrak{H})$ onto $\mathcal{A}$ ([12] and [29]). As examples of von Neumann algebras with an extension property, there are von Neumann algebras with *property* P due to J. Schwartz ([22]) and von Neumann algebras with *property* Q ([3]), that is, von Neumann algebras generated by amenable unitary groups. Especially, an abelian von Neumann algebra has an extension property.

Acknowledgment: Kindly enough, Prof. Y. Nakagami has taken the pains of reading the manuscript of this paper carefully. The author would like here to express his thanks to Prof. Y. Nakagami.

REFERENCES

[1] ARVESON, W. B., A theorem on the action of abelian unitary groups, Pacific J. Math., **16** (1966), 205-212.

[2] BURES, D., Abelian subalgebras of von Neumann algebras, Memoirs Amer. Math. Soc., **110** (1971).

[3] CHODA, H. and M. ECHIGO, A new algebraic property of certain von Neumann algebras, Proc. Japan Acad., 39 (1963), 651-655.

[4] CHODA, H., On ergodic and abelian automorphism groups of von Neumann algebras, Proc. Japan Acad., **47** (1971), 982-985.

[5] CHODA, M. and H. CHODA, On extensions of autamorphisms of abelian von Neumann algebras, Proc. Japan Acad., **43** (1967), 295-299.

[6] CHODA, M., A normal expectation and the tensor product of von Neumann algebras—On Takesaki's Theorem—, Memoirs Osaka Kyoiku Univ., **20** (1971), 117-120.

[7] CHODA, M., I. KASAHARA and R. NAKAMOTO, Dependent elements of an automorphism of a C^*-algebra, Proc. Japan Acad., **48** (1972), 561-565.

[8] DIXMIER, J., Les algèbres d'opérateurs dans l'espace Hilbertien, Gauthier-Villars, Paris (1957).

[9] DYE, H. A., On groups of measure preserving transformations I., Amer. J. Math., **81** (1959), 119-159.

[10] ENOMOTO, M. and K. TAMAKI, Freely acting automorphisms of abelian C^*-algebras,

[11] HAGA, Y. and Z. TAKEDA, Correspondence between subgroups and subalgebras in a cross product von Neumann algebra, Tôhoku Math. J., **24** (1972), 167-190.

[12] HAKEDA, J. and J. TOMIYAMA, On some extension property of von Neumann algebras, Tôhoku Math. J., **19** (1967), 315-323.

[13] KALLMAN, R. R., A generalization of free action, Duke Math. J., **36** (1969), 781-789.

[14] KALLMAN, R. R., A decomposition theorem of automorphisms of von Neumann algebras, Proc. of a symposium on Functional Analysis (edited by C. O. Wilde), Academic Press, New York, (1970), 33-35.

[15] KOWADA, M., Spectral type of one-parameter group of unitary operators with transversal group, Nagoya Math. J., **32** (1968), 141-153.

[16] MURRAY, F. J. and J. VON NEUMANN, On rings of operators, Ann. Math., **37** (1936), 116-229.

[17] NAKAMURA, M. and T. TURUMARU, Expectations in an operator algebra, Tôhoku Math. J., **6** (1954), 182-188.

[18] NAKAMURA, M. and Z. TAKEDA, On some elementary properties of the crossed products of von Neumann algebras, Proc. Japan Acad., **34** (1958), 489-494.

[19] NAKAMURA, M. and Z. TAKEDA, On outer automorphisms of certain factors, Proc. Japan Acad., **37** (1961), 215-216.

[20] NAKAMURA, M. and H. TAKAI, Nongeneracy and discrete models, Proc. Japan Acad., **48** (1972), 566-568.

[21] VON NEUMANN, J., On rings of operators III, Ann. Math., **41** (1940), 94-161.

[22] SCHWARTZ, J. T. W^*-algebras, Gordon-Breach, New York, (1967).

[23] SUZUKI, N., Crossed products of rings of operators, Tôhoku Math. J., **11** (1959), 113-124.

[24] TAM, P. K., On an ergodic abelian $\mathscr{M}$-group, Proc. Japan Acad., **47** (1971), 456-457.

[25] TOMIYAMA, J., On the projection of norm one in W^*-algebras, Proc. Japan Acad., **33** (1957), 608-612.

[26] TOMIYAMA, J., On the product projection of norm one in the direct product of operator algebras, Tôhoku Math. J., **11** (1959), 305-313.

[27] TOMIYAMA, J., On the tensor products of von Neumann algebras, Pacific J. Math., **30** (1969), 263-270.

[28] TOMIYAMA, J., Applications of Fubini type theorems to the tensor products of C^*-algebras, Tôhoku Math. J., **19** (1967), 213-226.

[29] TOMIYAMA, J., Tensor products and projections of norm one in von Neumann algebras, Lecture note at Math. inst. of Univ. of Copenhagen (1970).

[30] TURUMARU, T., On the direct product of operator algebras, I. Tôhoku Math. J., **4** (1952), 242-251.

[31] TURUMARU, T., Crossed product of operator algebra, Tôhoku Math. J., **10**

(1958), 355-365.
[32] UMEGAKI, H., Conditional expectation in an operator algebra, Tôhoku Math. J., 6 (1954), 177-181.
[33] UMEGAKI, H., Positive definite functions and direct product of Hilbert spaces, Tôhoku Math. J., 7 (1955), 206-211.
[34] ZELLER-MEIER, G., Produits croises d'une C^*-algèbre par un groupe d'automorphisms, J. de Math. Pures et Appl., 47 (1968), 101-239.

DEPARTMENT OF MATHEMATICS,
OSAKA KYOIKU UNIVERSITY