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PSEUDO-UMBILICAL SUBMANIFOLDS OF CODIMENSION 3 WITH CONSTANT MEAN CURVATURE

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Let M^n be an n-dimensional submanifold[1] of an m-dimensional euclidean space E^m $(n<m)$ with the mean curvature vector $H\neq 0$. If the second fundamental tensor in the normal direction H is proportional to the first fundamental tensor of the submanifold M^n, then M^n is said to be *pseudoumbilical*. The mean curvature vector H is said to be *parallel* if the covariant derivative of H along M^n has no normal component, and H is said to be *nonparallel* if the covariant derivative of H along M^n has nonzero normal component everywhere.

In previous papers [2], [3], the authors proved that if M^n is pseudo-umbilical in E^m and the mean curvature vector is nonzero and parallel, then M^n is contained in a hypersphere of E^m as a minimal hypersurface. It is easy to see that if the mean curvature vector H is parallel, then the mean curvature is constant. If the codimension $m-n$ is two, then the constancy of the mean curvature implies the parallelism of the mean curvature vector [1]. In [4], the authors studied submanifolds of codimension two which are umbilical with respect to a non-parallel normal direction and showed that such manifolds are the loci of moving $(n-1)$-spheres, (see also [5]).

In the present paper, we shall study pseudo-umbilical submanifolds of codimension 3 with constant mean curvature, the mean curvature vector of which is nonparallel.

§ 1. Preliminaries.

We consider a submanifold M^n of codimension 3 of an $(n+3)$-dimensional euclidean space E^{n+3} and represent it by

$$(1) \qquad X=X(\xi^1, \xi^2, \cdots, \xi^n),$$

where X is the position vector from the origin of E^{n+3} to a point of the submanifold M^n and $\{\xi^h\}$ is a local coordinate system on M^n where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

We put

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1) Manifolds, mappings, functions, ... are assumed to be sufficiently differentiable and we shall restrict ourselves only to manifolds of dimension $n\geqq 3$.

$$X_i = \partial_i X, \qquad \partial_i = \partial/\partial\xi^i, \tag{2}$$

then X_i are n linear independent vectors tangent to M^n. We denote by C, D, E three mutually orthogonal unit normals to M^n.

Now denoting by ∇_j the operator of covariant differentiation with respect to Riemannian metric $g_{ji} = X_j \cdot X_i$ of M^n, we have equations of Gauss

$$\begin{aligned} \nabla_j X_i &\equiv \partial_j X_i - \left\{ {h \atop j\ i} \right\} X_h \\ &= h_{ji} C + k_{ji} D + f_{ji} E, \end{aligned} \tag{3}$$

where $\{_j{}^h{}_i\}$ are Christoffel symbols formed with g_{ji} and h_{ji}, k_{ji} and f_{ji} the second fundamental tensors with respect to normals C, D and E respectively. The mean curvature vector is then given by

$$H = \frac{1}{n} g^{ji} \nabla_j X_i, \tag{4}$$

where g^{ji} are contravariant components of the metric tensor.

If there exist, on the submanifold M^n, two functions α, β and a unit vector field v_i such that

$$h_{ji} = \alpha g_{ji} + \beta v_j v_i, \tag{5}$$

then the submanifold M^n is said to be *quasi-umbilical* with respect to the normal direction C. In particular, if $\beta = 0$ identically, then M^n is said to be *umbilical* with respect to the normal direction C. If M^n is umbilical with respect to the mean curvature vector H, then the submanifold M^n is said to be *pseudo-umbilical*.

The equations of Weingarten are given by

$$\nabla_j C = -h_j{}^h X_h \qquad + l_j D + m_j E, \tag{6}$$

$$\nabla_j D = -k_j{}^h X_h - l_j C \qquad + n_j E, \tag{7}$$

$$\nabla_j E = -f_j{}^h X_h - m_j C - n_j D, \tag{8}$$

where $h_j{}^h = h_{jt} g^{th}$, $k_j{}^h = k_{jt} g^{th}$ and $f_j{}^h = f_{jt} g^{th}$ and l_j, m_j and n_j are the third fundamental tensors.

In the sequel, we denote the normal components of $\nabla_j C$, $\nabla_j D$ and $\nabla_j E$ by $\nabla_j{}^\perp C$, $\nabla_j{}^\perp D$ and $\nabla_j{}^\perp E$ respectively.

The normal vector field C is said to be *parallel* if we have $\nabla_j{}^\perp C = 0$, that is, l_j and m_j vanish identically and it is said to be *non-parallel* if $\nabla_j{}^\perp C$ never vanishes, that is, $l_t l^t + m_t m^t$ never vanishes, where $l^t = l_i g^{it}$ and $m^t = m_i g^{it}$.

We have equations of Gauss:

$$K_{kji}{}^h = h_k{}^h h_{ji} - h_j{}^h h_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki} + f_k{}^h f_{ji} - f_j{}^h f_{ki}, \tag{9}$$

where $K_{kji}{}^h$ is the Riemann-Christoffel curvature tensor, those of Codazzi:

$$\nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} - m_k f_{ji} + m_j f_{ki} = 0, \tag{10}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} - n_k f_{ji} + n_j f_{ki} = 0, \tag{11}$$

$$\nabla_k f_{ji} - \nabla_j f_{ki} + m_k h_{ji} - m_j h_{ki} + n_k k_{ji} - n_j k_{ki} = 0, \tag{12}$$

and those of Ricci:

$$\nabla_k l_j - \nabla_j l_k + h_k{}^t k_{jt} - h_j{}^t k_{kt} + m_k n_j - m_j n_k = 0, \tag{13}$$

$$\nabla_k m_j - \nabla_j m_k + h_k{}^t f_{jt} - h_j{}^t f_{kt} + n_k l_j - n_j l_k = 0, \tag{14}$$

$$\nabla_k n_j - \nabla_j n_k + k_k{}^t f_{jt} - k_j{}^t f_{kt} + l_k m_j - l_j m_k = 0. \tag{15}$$

Denoting by $K_{ji} = K_{tji}{}^t$ and $K = g^{ji} K_{ji}$ the Ricci tensor and the scalar curvature respectively, we define a tensor field L_{ji} of type (0, 2) by

$$L_{ji} = -\frac{K_{ji}}{n-2} + \frac{K g_{ji}}{2(n-1)(n-2)}. \tag{16}$$

The conformal curvature tensor $C_{kji}{}^h$ is then given by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki}, \tag{17}$$

where δ_k^h are Kronecker deltas and $L_k{}^h = L_{kt} g^{th}$.

A Riemannian manifold M^n is locally conformal to a euclidean space and is called a *conformally flat space* if and only if we have

$$C_{kji}{}^h = 0, \tag{18}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = 0. \tag{19}$$

It is well known known that (18) holds automatically for $n=3$ and (19) is a consequence of (18) for $n>3$.

§ 2. Pseudo-umbilical submanifolds of codimension 3.

Throughout the rest of this paper, we assume that M^n is a pseudo-umbilical submanifold of a euclidean $(n+3)$-space E^{n+3} with nonzero constant mean curvature. Since the mean curvature vector H is nowhere zero, we may choose the normal C in the direction of H, i.e.,

$$H = \alpha C, \qquad \alpha = |H|. \tag{20}$$

Then by the assumption we have

$$h_{ji} = \alpha g_{ji}, \qquad \alpha = \text{constant} \neq 0, \tag{21}$$

$$k_t{}^t = 0, \qquad f_t{}^t = 0. \tag{22}$$

In the sequel, we denote by H_2 and H_3 the symmetric $n\times n$ matrices given by $(k_j{}^h)$ and $(f_j{}^h)$ respectively.

LEMMA 1. *Let M^n be a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha\neq 0$. If the two matrices H_2 and H_3 commute at a point $p\in M^n$, then either the covariant derivative $\nabla_j C$ of C has no normal component or the two matrices H_2 and H_3 are proportional at p.*

Proof. Suppose that M^n is pseudo-umbilical in E^{n+3} and with constant mean curvature $\alpha\neq 0$. Then (21) holds. Hence from (10) and (21), we have

(23) $$l_k k_{ji} - l_j k_{ki} + m_k f_{ji} - m_j f_{ki} = 0,$$

that is,

(24) $$l_k k_j{}^h - l_j k_k{}^h + m_k f_j{}^h - m_j f_k{}^h = 0.$$

Now suppose that $H_2=(k_j{}^h)$ and $H_3=(f_j{}^h)$ commute at $p\in M^n$. Then H_2 and H_3 are simultaneously diagonalizable. Hence if we choose a local coordinate system $\{\xi^h\}$ around p in M^n such that X_h form an orthonormal basis of the tangent space $T_p(M^n)$ and are in the principal directions with respect to the normal direction D at p, then X_h are also in the principal directions with respect to the normal direction E. Thus, if we denote by λ_i and μ_i the eigenvalues of H_2 and H_3 respectively, then (24) reduces to

(25) $$l_k\lambda_j + m_k\mu_j = 0, \quad \text{for} \quad k\neq j.$$

Since we have $\lambda_1+\lambda_2+\cdots+\lambda_n=0$, $\mu_1+\mu_2+\cdots+\mu_n=0$, (25) implies

(26) $$l_k\lambda_j + m_k\mu_j = 0, \quad \text{for all} \quad k \quad \text{and}\ j.$$

If $\nabla_j{}^\perp C\neq 0$ at p, then, without loss of generality, we can assume that $l_1\neq 0$. Thus from (26), we see that

(27) $$\lambda_j = -\frac{m_1}{l_1}\mu_j.$$

This implies that the two matrices H_2 and H_3 are proportional. This completes the proof of the lemma.

LEMMA 2. *Let M^n be a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha\neq 0$. If ihe two matrices H_2 and H_3 commute and $\nabla_j{}^\perp C\neq 0$ at p, then we can suitably choose the normal directions D and E in such a way that we have*

(28) $$k_{ji}=0 \quad \textit{and} \quad m_j=0$$

at $p\in M^n$, unless H_2 and H_3 vanish simultaneously.

Proof. Under the hypothesis of the lemma, we see, from Lemma 1, that H_2

and H_3 are proportional. Hence we may assume that

(29) $$H_2=cH_3, \quad \text{at} \quad p,$$

for some real c. Put $c=-\tan\theta$ and

(30) $$\bar{D}=(\cos\theta)D+(\sin\theta)E,$$
$$\bar{E}=-(\sin\theta)D+(\cos\theta)E.$$

Then we see that the second fundamental tensor in the normal direction $\bar{D}$ vanishes. Hence we may assume that H_2 vanishes, i.e., $k_{ji}=0$ at p. Substituting this into (24), we obtain

(31) $$m_k f_j{}^h - m_j f_k{}^h = 0$$

at p. If we choose a local coordinate system $\{\xi^h\}$ around p in such a way that X_h are orthogonal and in the principal directions of the normal E, then we obtain

(32) $$m_k \mu_j = 0, \qquad k \neq j,$$

at p, where μ_j denote eigenvalues of H_3. Hence by applying (22), we have

(33) $$m_k \mu_j = 0, \quad \text{for all} \quad k \quad \text{and} \quad j,$$

at p. This implies that we have either $m_j=0$ or $\mu_j=0$. This shows that we have either $m_j=0$ or $H_3=0$ at p. This completes the proof of the lemma.

LEMMA 3. *Let M^n be a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha\neq 0$. If the two matrices H_2 and H_3 commute, $\nabla_j{}^\perp C\neq 0$ and $E\cdot\nabla_j{}^\perp C=0$ at p, then we have*

(34) $$k_{ji}=0 \quad (i.e., H_2=0), \quad m_j=0,$$

(35) $$n_j=\nu l_j, \qquad \nu=\frac{n_t l^t}{l^2}\neq 0, \qquad l^2=l_t l^t\neq 0,$$

and

(36) $$f_{ji}=\frac{\alpha}{\nu}\left(g_{ji}-\frac{n}{l^2}\,l_j l_i\right)$$

at p.

Proof. Under the hypothesis, we have $l_j\neq 0$, $m_j=0$ and, from (23),

(37) $$l_k k_{ji} - l_j k_{ki} = 0,$$

from which

(38) $$k_{ji}=\beta l_j l_i$$

for some β and consequently by

$$k_t{}^t=\beta l_t l^t=0,$$

from which $\beta=0$ and hence we obtain (34).

On the other hand, from (14), (21) and (34), we have

$$n_k l_j - n_j l_k = 0,$$

we find

$$n_j=\nu l_j, \tag{39}$$

where $\nu=n_t l^t/l^2$ and $l^2=l_t l^t$. If $\nu=0$, then $n_j=0$. Hence (11) and (34) give

$$l_k \alpha g_{ji} - l_j \alpha g_{ki}=0, \tag{40}$$

from which, transvecting g^{ji},

$$(n-1)\alpha l_k=0,$$

which is a contradiction. Thus we have (35).

From (15), (34), (35) and (39), we find

$$\nabla_k(\nu l_j)-\nabla_j(\nu l_k)=0. \tag{41}$$

From (13), (34), (35) and (41), we obtain

$$\nu_k l_j - \nu_j l_k = 0,$$

where $\nu_k=\nabla_k \nu$. Hence

$$\nu_j=\frac{\nu_t l^t}{l^2} l_j. \tag{42}$$

Now substituting (21), (34) and (35) into (11), we find

$$l_k(\alpha g_{ji}-\nu f_{ji})-l_j(\alpha g_{ki}-\nu f_{ki})=0,$$

from which, transvecting l^k,

$$l^2(\alpha g_{ji}-\nu f_{ji})=l_j v_i$$

for some v_i. Since the left hand side is symmetric in j and i, we have

$$l^2(\alpha g_{ji}-\nu f_{ji})=\rho l_j l_i \tag{43}$$

for some ρ. Transvecting g^{ji} to (43), we find

$$\alpha n=\rho. \tag{44}$$

Thus (43) becomes

$$\nu f_{ji}=\alpha g_{ji}-\frac{n\alpha}{l^2}l_j l_i. \tag{45}$$

This completes the proof of the lemma.

LEMMA 4. *Under the hypothesis of Lemma* 3, *we have*

$$\nabla_j l_i=\gamma g_{ji}+l_j v_i+l_i v_j, \tag{46}$$

where

$$\gamma=\frac{\nu_t l^t}{n\nu},\ v_i=\frac{2}{l}\nabla_i l-\left[\frac{3}{2l^2}(l^t\nabla_t l)+\frac{\nu_t l^t}{2n\nu l^2}\right]l_i. \tag{47}$$

Proof. From (12), (13), (21), (34) and (45), we find

$$\nu_k f_{ji}-\nu_j f_{ki}=-\left(\nabla_k\frac{n\alpha}{l^2}\right)l_j l_i+\left(\nabla_j\frac{n\alpha}{l^2}\right)l_k l_i$$
$$-\frac{n\alpha}{l^2}l_j(\nabla_k l_i)+\frac{n\alpha}{l^2}l_k(\nabla_j l_i),$$

from which, using (36) and (42),

$$\frac{\nu_t l^t}{\nu l^2}(l_k g_{ji}-l_j g_{ki})=-\left(\nabla_k\frac{n}{l^2}\right)l_j l_i+\left(\nabla_j\frac{n}{l^2}\right)l_k l_i$$
$$-\frac{n}{l^2}l_j(\nabla_k l_i)+\frac{n}{l^2}l_k(\nabla_j l_i).$$

Transvecting l^k to this equation, we find

$$\frac{\nu_t l^t}{\nu l^2}(l^2 g_{ji}-l_j l_i)=-\left(l^t\nabla_t\frac{n}{l^2}\right)l_j l_i+l^2\left(\nabla_j\frac{n}{l^2}\right)l_i$$
$$-\frac{n}{l^2}l_j(l^t\nabla_t l_i)+n(\nabla_j l_i),$$

from which

$$\nabla_j l_i=al^2 g_{ji}+\left(l^t\nabla_t\frac{1}{l^2}-a\right)l_j l_i$$
$$-l^2\left(\nabla_j\frac{1}{l^2}\right)l_i+\frac{1}{l^2}l_j(l^t\nabla_t l_i), \tag{48}$$

where

$$a=\frac{\nu_t l^t}{n\nu l^2}.$$

On the other hand, from (13) and (34), we find

$$\nabla_j l_i - \nabla_i l_j = 0. \tag{49}$$

From (48) and (49), we have

$$\left(l^4 \nabla_j \frac{1}{l^2} + l^t \nabla_t l_j\right) l_i = \left(l^4 \nabla_i \frac{1}{l^2} + l^t \nabla_t l_i\right) l_j,$$

or

$$(-2l \nabla_j l + l^t \nabla_t l_j) l_i = (-2l \nabla_i l + l^t \nabla_t l_i) l_j, \tag{50}$$

from which, transvecting l^i,

$$(-2l \nabla_j l + l^t \nabla_t l_j) l^2 = (-2l l^t \nabla_t l + l l^t \nabla_t l) l_j,$$

or

$$l^t \nabla_t l_j = 2l \nabla_j l - \frac{1}{l} (l^t \nabla_t l) l_j. \tag{51}$$

Substituting (51) into (48), we have

$$\nabla_j l_i = a l^2 g_{ji} + \left(-\frac{2}{l^3} l^t \nabla_t l - a\right) l_j l_i$$
$$+ \frac{2}{l} (\nabla_j l) l_i + \frac{1}{l^2} l_j \left[2l \nabla_i l - \frac{1}{l} (l^t \nabla_t l) l_i\right],$$

or

$$\nabla_j l_i = a l^2 g_{ji} + \frac{2}{l} [(\nabla_j l) l_i + (\nabla_i l) l_j] - \left[\frac{3}{l^2} (l^t \nabla_t l) + a\right] l_j l_i. \tag{52}$$

Put

$$v_i = \frac{2}{l} \nabla_i l - \left[\frac{3}{2l^2} (l^t \nabla_t l) + \frac{1}{2} a\right] l_i, \tag{53}$$

then (53) gives (46). This proves the lemma.

§ 3. Conformally flat spaces.

For a submanifold M^n of E^{n+3}, if the second fundamental tensors $H_1 = (h_j{}^h)$, $H_2 = (k_j{}^h)$ and $H_3 = (f_j{}^h)$ are simultaneously diagonalizable, then we say that the *normal connection* of M^n in E^{n+3} is *trivial*. It is easy to see that a pseudo-umbilical submanifold M^n of E^{n+3} with non-zero mean curvature has trivial normal connection if and only if H_2 and H_3 commute, where H_2 and H_3 are those given in the previous section. For a pseudo-umbilical submanifold of codimension 2 with non-zero mean curvature, the normal connection is always trivial.

THEOREM 1. *Let M^n be a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha \neq 0$. If the normal connection is trivial and the mean curvature vector is non-parallel, then the submanifold M^n is conformally flat for $n \geqq 3$.*

Proof. If M^n is a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha\neq 0$ such that the normal connection is trivial and the mean curvature vector is non-parallel, then by Lemmas 2 and 3, we can suitably choose D and E in such a way that (34), (35) and (36) hold. In particular, we have

$$h_{ji}=\alpha g_{ji},\quad k_{ji}=0,\quad f_{ji}=\lambda g_{ji}+\mu l_j l_i, \tag{54}$$
$$l_j\neq 0,\qquad m_j=0,$$

where

$$\lambda=\frac{\alpha}{\nu},\qquad \mu=-\frac{n\alpha}{\nu l^2}. \tag{55}$$

We consider the cases $n>3$ and $n=3$ separately.

Case I. $n>3$. By substituting (54) into (9), we find

$$K_{kji}{}^h=(\alpha^2+\lambda^2)(\delta_k^h g_{ji}-\delta_j^h g_{ki}) \tag{56}$$
$$+\lambda\mu[(\delta_k^h l_j-\delta_j^h l_k)l_i+(l_k g_{ji}-l_j g_{ki})l^h],$$

from which

$$K_{ji}=[(n-1)(\alpha^2+\lambda^2)+\lambda\mu l^2]g_{ji}+(n-2)\lambda\mu l_j l_i \tag{57}$$

and

$$K=n(n-1)(\alpha^2+\lambda^2)+2(n-1)\lambda\mu l^2 \tag{58}$$

Thus, from (16), (57), and (58), we have

$$L_{ji}=-\frac{1}{2}(\alpha^2+\lambda^2)g_{ji}-\lambda\mu l_j l_i. \tag{59}$$

Substituting (56) and (59) into (17), we easily find that the conformal curvature tensor $C_{kji}{}^h$ vanishes identically. This shows that the submanifold M^n is conformally flat for $n>3$.

Case II. $n=3$. Substituting (54) into (12), we find

$$\lambda_k g_{ji}-\lambda_j g_{ki}+\mu_k l_j l_i-\mu_j l_k l_i+\mu l_j(\nabla_k l_i)-\mu l_k(\nabla_j l_i)=0, \tag{60}$$

by virtue of (49), where $\lambda_k=\nabla_k\lambda$ and $\mu_k=\nabla_k\mu$.

Substituting (46) into (60) and using (47), we find

$$(\lambda_k-\mu\gamma l_k)g_{ji}-(\lambda_j-\mu\gamma l_j)g_{ki} \tag{61}$$
$$+\left[\mu_k l_j-\mu_j l_k-\frac{2\mu}{l}l_k\nabla_j l+\frac{2\mu}{l}l_j\nabla_k l\right]l_i=0,$$

from which we obtain

(62) $$\lambda_k=\mu\gamma l_k$$

and

(63) $$\mu_k l_j-\mu_j l_k-\frac{2\mu}{l}l_k\nabla_j l+\frac{2\mu}{l}l_j\nabla_k l=0.$$

From (63), we have

$$\left(\mu_k+\frac{2\mu}{l}\nabla_k l\right)l_j=\left(\mu_j+\frac{2\mu}{l}\nabla_j l\right)l_k,$$

from which

(64) $$\mu_k+\frac{2\mu}{l}\nabla_k l=\sigma l_k,$$

σ being a function.

Now, from (59), we have

$$\nabla_k L_{ji}=-\lambda\lambda_k g_{ji}-\lambda_k\mu l_j l_i-\lambda\mu_k l_j l_i$$
$$-\lambda\mu(\nabla_k l_j)l_i-\lambda\mu l_j(\nabla_k l_i),$$

or, using (46), (62) and (64),

$$\nabla_k L_{ji}=-\lambda\mu\gamma l_k g_{ji}-\mu^2\gamma l_k l_j l_i$$
$$-\lambda\left(-\frac{2\mu}{l}\nabla_k l+\sigma l_k\right)l_j l_i$$
$$-\lambda\mu(\nabla_k l_j)l_i-\lambda\mu l_j[\gamma g_{ki}+l_k v_i+l_i v_k],$$

from which

$$\nabla_k L_{ji}-\nabla_j L_{ki}=\frac{2\lambda\mu}{l}[(\nabla_k l)l_j-(\nabla_j l)l_k]l_i$$
$$-\lambda\mu[v_k l_j-v_j l_k]l_i,$$

that is,

$$\nabla_k L_{ji}-\nabla_j L_{ki}=0,$$

by virtue of (53). This shows that M^n is a conformally flat space. Consequently we have proved the theorem completely.

§ 4. Locus of $(n-1)$-spheres.

The purpose of this section is to prove the following:

THEOREM 2. *Let M^n be a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha\neq 0$. If the mean curvature vector is non-parallel and the normal connection is trivial, then the submanifold M^n is not contained in any hypersphere* *of E^{n+3} and it is the locus of moving $(n-1)$-spheres where an $(n-1)$-sphere means a hypersphere of a euclidean n-space.*

Proof. Let M^n be a pseudo-umbilical submanifold of E^{n+3} with constant mean curvature $\alpha\neq 0$, such that the mean curvature vector is non-parallel and the normal connection is trivial. Then by Lemmas 1, 2 and 3, we have

$$
\begin{aligned}
&\nabla_j X_i=\alpha g_{ji}C+\frac{\alpha}{\nu}\left(g_{ji}-\frac{n}{l^2}l_j l_i\right)E,\\
&\nabla_j C=-\alpha X_j+l_j D,\\
(65)\qquad &\nabla_j D=-l_j C+\nu l_j E,\\
&\nabla_j E=-\frac{\alpha}{\nu}X_j-\nu l_j D+\frac{n\alpha}{\nu l^2}l_j l^i X_i.
\end{aligned}
$$

Since $\nabla_j l_i-\nabla_i l_j=0$, $l_i dx^i=0$ is integrable. We represent one of integral manifolds M^{n-1} by

$$X=X(\xi^h(\eta^a))$$

and put

$$X_b=\partial_b X=B_b{}^i X_i,\quad B_b{}^i=\partial_b\xi^i,\quad \partial_b=\partial/\partial\eta^b,$$

$$N^h=\frac{1}{l}l^h,\qquad g_{cb}=B_c{}^j B_b{}^i g_{ji}$$

and

$$\nabla_c B_b{}^h=H_{cb}N^h,$$

$\nabla_c B_b{}^h$ denoting the van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ along M^{n-1}, where, here and in the sequel, indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n-1\}$.

From

$$l_i B_b{}^i=0$$

and Lemma 4, we have

$$\left(\frac{\nu_t l^t}{n\nu}g_{ji}+l_j v_i+l_i v_j\right)B_c{}^j B_b{}^i+lH_{cb}=0,$$

from which

$$H_{cb}=-\frac{\nu_t l^t}{nl\nu}g_{cb}=\beta g_{cb}, \tag{66}$$

with $\beta=-\nu_t l^t/nl\nu$. Thus, from (65), we have, along M^{n-1},

$$\nabla_c X_b = \nabla_c(B_b{}^i X_i) = H_{cb} N^i X_i + B_c{}^j B_b{}^i(\nabla_j X_i)$$
$$= \alpha g_{cb} C + \frac{\alpha}{\nu} g_{cb} E + \beta g_{cb} N,$$

where $N = N^i X_i$. This shows that the integral manifold M^{n-1} is totally umbilical in E^{n+3}. Thus M^{n-1} is contained in a hypersphere of a linear n-subspace of E^{n+3}. Therefore M^n is the locus of the moving $(n-1)$-spheres. The remaining part of the theorem follows immediately from Theorem 5 of [3]. This completes the proof of the theorem.

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