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ON AN EXTENSION THEOREM AND ITS APPLICATION FOR TURNING POINT PROBLEMS OF LARGE ORDER

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1. Introduction.

Our results in this paper are concerned with the asymptotic nature of solutions of second order ordinary differential equations of the form

$$\varepsilon^{2h}\frac{d^2y}{dx^2}=p(x,\varepsilon)y \tag{1.1}$$

as a small parameter ε tends to zero. Here h is a positive integer and $p(x,\varepsilon)$ is a holomorphic function of x and ε which has an asymptotic expansion is power series of ε with polynomial coefficients:

$$p(x,\varepsilon)\simeq\sum_{\nu=0}^{\infty}p_\nu(x)\varepsilon^\nu \tag{1.2}$$

in the region

$$|x|<\infty,\ 0<\varepsilon\leqq\varepsilon_0. \tag{1.3}$$

There has been much investigated by many authors about the above type of equations and in particular the turning point problems are the subject of many papers and monographs. According to the problems in physical applications, it is desired to find the asymptotic behavior of solutions in an unbounded region of x which may not contain the turning points, or in a given bounded region which may not contain the turning points, or in a small neighborhood of a turning point containing turning point itself. In these cases, it is difficult to construct uniformly valid asymptotic expansions in a region where it is needed for application except for particularly simple equations. For these problems, the so-called *W-K-B* approximation may be the most familiar among the scientists of many fields, and this method has been put on rigourous mathematical foundations recently by several authors, in particular the precise definition of the region of existence of asymptotic solution and the lateral connection formula around a simple turning point were given by Evgrafov and Fedoryuk [2], the connection formulas and their error bounds at a simple turning point were given by Fröman and Fröman [3]., Olver [9], and Wasow [11].

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Usually, the *W-K-B* approximation is correct in a certain sectorial region of complex x-plane which is unbounded and deleted the arbitrarilly small neighborhood of turning points. On the other hand, the comparison method or the related equation method is used to obtain uniformly valid asymptotic solutions in sufficiently small neighborhood of a turning point, but at present this is applicable to very special class of equations, to be more precise, the order of turning point is at most two. The one more alternative method in analyzing the local theory is the matched asymptotic approximation, we call it simply the matching method which has been used effectively by myself when the order of turning point is greater than two.

The purpose of this paper is to extend the region of existence of the *W-K-B* type approximation of the equation (1.1) to an unbounded sectorial region such that:

1]. the domain of influence of each turning point is deleted. The domain of influence is a neighborhood of turning point which shrinks to the turning point itself as ε tend to zero, and is determined from the characteristic polygon associated with the turning point. The characteristic polygon of (1.1) will be explained in section 2.

2]. the independent variable x goes to infinity as well as the parameter ε tends to zero in such a way that the quantity $|x\varepsilon^{\alpha}|$ remains bounded, where α is a positive constant given precisely in later.

The results presented here are of interest in several respects. At first, this is one of the example in which the Kaplun's extension theorem and matching principle of the asymptotic theory can be used rigorously. Roughly speaking, the extension theorem asserts that if an asymptotic approximation is uniformly valid in an interval of x, then it is uniformly valid in a wider interval depending on the parameter ε. The stretching and matching methods are frequently used in various problems of applied mathematics, in particular the boundary layer theory of fluid mechanics, but it is in general very difficult to obtain a wider interval depending on ε in which an asymptotic approximation is uniformly valid. Hence it is ambiguous to ascertain that the outer expansion and the inner expansion can be matched rigorously. In turning point problems of second order ordinary differential equations, this is overcome because that the extension theorem can be applied to obtain a wider region of x explicitly which enable us to prove that the regions of existence of an outer and an inner expansion are overlapped for all sufficiently small parameter ε. For the Kaplan's extension theorem, we refer the reader to the book Cole [1], or Van Dyke [10].

Secondly our results play an essential part when we analyze the turning point problems in local by the matching method under the condition that the characteristic polygon consists of at least two segments. We explain this point in details. The differential equation we consider is of the form (1.1) with $h=1$ or in vector form it becomes

$$\text{(A)} \qquad \varepsilon \frac{dy}{dx} = \begin{bmatrix} 0 & 1 \\ f(x, \varepsilon) & 0 \end{bmatrix} y.$$

Here the function $f(x, \varepsilon)$ is holomorphic in the region D

$$D: |x| \leqq x_0 < 1, \; 0 < \varepsilon \leqq \varepsilon_0,$$

and has a uniformly asymptotic expansion in power series of ε such that

$$f(x, \varepsilon) \simeq x^q + \sum_{\nu=1}^{\infty} \phi_\nu(x) \varepsilon^\nu \qquad (q = \text{positive integer}),$$

as ε tends to zero with holomorphic coefficients:

$$\phi_\nu(x) = \sum_{\mu=m_\nu}^{\infty} \phi_{\nu\mu} x^\mu, \; \phi_{\nu m_\nu} \neq 0 \qquad (m_\nu \geqq 0, \; \nu = 1, 2, \cdots).$$

The problem is to find asymptotic expansions of the equation (A) in the full neighborhood of the origin as ε tends to zero. The characteristic polygon associated with a turning point $x=0$ of (A) consists at most two segments, and when it satisfies the one segments condition the above problem was considered by myself in [6], [7] by using the matching method. Suppose that the one segment condition does not satisfied, then the problem becomes quite complicated because of appearing the secondary turning points. The simplest equation of such cases was treated recently by Nakano and Nishimoto [5] by using the results of [2]. The condition that the characteristic polygon consists of two segments is simply described by

$$(1.2) \qquad 2m_1 + 2 - q < 0.$$

According to the theory of Iwano and Sibuya [4], the region D is divided into four types of subregion in each of which the equation (A) has different principal part. In the following we write down these regions and corresponding differential equations.

(1) Outer domain D_1

$$D_1: M\varepsilon^{\rho_1} \leqq |x| \leqq x_0,$$

$$\text{(A}_1\text{)} \qquad \varepsilon \frac{dz_1}{dx} = \sqrt{x^q} A_1(x, \varepsilon) z_1, \qquad y = \begin{bmatrix} 1 & 0 \\ 0 & x^{q/2} \end{bmatrix} z,$$

$$A_1(x, \varepsilon) \simeq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sum_{\nu=1} \sum_{\mu=m_\nu} B_{\nu\mu} x^{(\nu-1)q+\mu-\nu r} [\varepsilon x^{-1/\rho_1}]^\nu,$$

where we put $r = m_1$ and $B_{\nu\mu}$ are constant 2-*by*-2 matrices.

(2) The first intermidiate domain D_2

$$D_2: m\varepsilon^{\rho_1} \leqq |x| \leqq M\varepsilon^{\rho_1},$$

$$\text{(A}_2\text{)} \qquad \varepsilon^{1-\rho_1-\gamma_1} \frac{dz_2}{ds} = A_2(s, \varepsilon) z_2, \qquad y = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{\gamma_1} \end{bmatrix} z_2, \; x = \varepsilon^{\rho_1} s,$$

$$A_2(s, \varepsilon)\simeq\begin{bmatrix} 0 & 1 \\ s^q+\phi_{1r}s^r & 0 \end{bmatrix}+\sum_{k=1} B_k^{(2)}(s)\varepsilon^{k/(q-r)},$$

where $B_k^{(2)}(s)$ are 2-by-2 matrices of polynomials of degree at most $(k+q)/(q-r)$.

(3) The second intermidiate domain D_3

$$D_3\colon\ M\varepsilon^{\rho_2}\leqq|x|\leqq m\varepsilon^{\rho_1},\qquad M\varepsilon^{\rho_2-\rho_1}\leqq|s|\leqq m,$$

(A₃)
$$[\varepsilon s^{1/(\rho_1-\rho_2)}]^{1-\rho_1-\gamma_1}s\frac{dz_3}{ds}=A_3(s, \varepsilon)z_3,\qquad y=\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{1/2}x^{r/2} \end{bmatrix}z_3,$$

$$A_3(s, \varepsilon)\simeq\begin{bmatrix} 0 & 1 \\ \phi_{1r}+s^{q-r} & 0 \end{bmatrix}+\sum_{k=1}^{\infty} B_k^{(3)}(s)[s^{1/(\rho_1-\rho_2)}\varepsilon]^{\rho_1 k},$$

where $B_k^{(3)}(s)$ are 2-by-2 matrices of holomorphic functions of $s^{1/(q-2r-2)}$.

(4) Inner domain D_4

$$D_4\colon\ |x|\leqq M\varepsilon^{\rho_2},$$

(A₄)
$$\frac{dz_4}{dt}=A_4(t, \varepsilon),\qquad y=\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{\gamma_2} \end{bmatrix}z_4,\qquad x=\varepsilon^{\rho_2}t,$$

$$A_4(t, \varepsilon)\simeq\begin{bmatrix} 0 & 1 \\ \phi_{1r}t^r & 0 \end{bmatrix}+\sum_{k=1}^{\infty} B_k^{(4)}(t)\varepsilon^{k/(r+2)},$$

where $B_k^{(4)}(t)$ are 2-by-2 matrices of polynomials of t of degree at most $(k+r)/(r+2)$. Here m and M are appropriate constants and

$$\rho_1=\frac{1}{q-r},\quad \rho_2=\frac{1}{r+2},\quad \gamma_1=\frac{q}{2(q-r)},\quad \gamma_2=\frac{r+1}{r+2}.$$

The problem is to know the asymptotic behavior of one outer solution in the full neighborhood of the origin. To do so, it will be used the matching method, that is, an asymptotic solution of each differential equation (A$_i$) ($i=1,2,3,4$) is to be constructed in some region D_i so that the overlapping region between two adjacent regions D_i and D_{i+1} ($i=1,2,3$) exists, and thereafter it will be calculated the connection matrix between the asymptotic solutions of (A$_i$) and (A$_{i+1}$) ($i=1,2,3$) at a suitable point of overlapping region. If we consider the differential equation (A_2) in a small neighborhood of the origin, it has a turning point at the origin of order r and it is easily seen that the characteristic polygon associated with it consists of only one segment. From this fact, the asymptotic solutions of (A₃) and (A₄) can be constructed and matched together in the neighborhood of the origin by applying the results of [6], [7]. Therefore the remaining task to be done is to find the regions $\tilde{D}_i$ ($i=1,2,3$). More precisely, our aim is to construct a certain sectorial region $\tilde{D}_2$ which not only contains D_2 and D_3 but also overlaps with $\tilde{D}_1$ in absolute value $|x|$ for all sufficiently small ε, and if this is done, the regions $\tilde{D}_i$ ($i=1,2,4$) are obtained which overlap with each two adjacent regions. This is one of the main applications of our theory, and the differential equation (A₂) has the just the same form as (1.1) if we replace ε by some fractional power of ε.

In section 2, we explane the domain of influence at each turning point by using the characteristic polygon, in section 3, the given differential equation is changed into an asymptotically diagonal equation by appropriate transformations, and their asymptotic properties when x tends to infinity or to turning point are studied in three lemmas. In section 4, it is defined the canonical region based on the definition of Evgrafov and Fedoryuk [2], and from a small deformation of this region we introduce in Lemma 4.1 the region $D[\gamma, \varepsilon]$ which we call the admissible region. The existence of asymptotic expansion of fundamental solution in the region $D[\gamma, \varepsilon]$ is proved in section 5. In section 6, we treat, as an example of application of our theory, a second order differential equation having a turning point at the origin of large order and solve the central connection problem at the origin.

The main results of this paper was published in [8] without proof. The present paper is then devoted to give the proof and the application. Some notations and symbols are different from the previous paper.

2. Characteristic polygon and domain of influence.

Let $p_\nu(x)$ of (1.2) be polynomials of x whose degree may depend on the indices and we assume that

$$p_0(x)=x^q+p_{0q-1}x^{q-1}+\cdots+p_{00},$$

(2.1)

$$p_\nu(x)=p_{\nu q_\nu}x^{q_\nu}+p_{\nu q_\nu-1}x^{q_\nu-1}+\cdots+p_{\nu 0}\quad(\nu=1,2,3,\cdots),$$

where p_{jk} $(j=0,1,2,\cdots, k=0,1,2,\cdots,q_\nu)$ are constants which may be zero and q_ν is an integer at most $\alpha\nu+\beta$ with nonnegative rational numbers α and β. We call in this paper the roots of $p_0(x)$ turning points of the given differential equation (1.1) Now we introduce for each turning point the notion of the characteristic polygon and the domain of influence. Suppose that $x=a_k$ is one of the turning point, then we rewrite the polynomials $p_i(x)$ $(i=0,1,2,\cdots)$ as a polynomials of $(x-a_k)$ such that

$$p_0(x)=(x-a_k)^q+\tilde{p}_{0q-1}(x-a_k)^{q-1}+\cdots+\tilde{p}_{0r}(x-a_k)^r\qquad(r\geqq 1),$$

$$p_\nu(x)=\tilde{p}_{\nu q_\nu}(x-a_k)^{q_\nu}+\tilde{p}_{\nu q_\nu-1}(x-a_k)^{q_\nu-1}+\cdots+\tilde{p}_{\nu 0}$$

In the $X-Y$ plane, we plot the points $P_{\nu\mu}=(\nu/2, \mu/2)$ for which the coefficients $\tilde{p}_{\nu\mu}$ of the above expressions are not zero and $R=(h, -1)$. The characteristic polygon associated with the turning point a_k is a polygon Γ_{a_k} beginning from P_{0r} and ending at R, convex downward, consists of finite number of segments connecting two points P_{ik} such that all of the other points $P_{\nu\mu}$ are above or on the polygon.

If the equation of the segment $L^{(a_k)}$ of the characteristic polygon, which is situated on the upper and left hand side and is between P_{0r} and some point P_{ts} $(t\neq 0, r>s)$ (Fig. 1) is given by

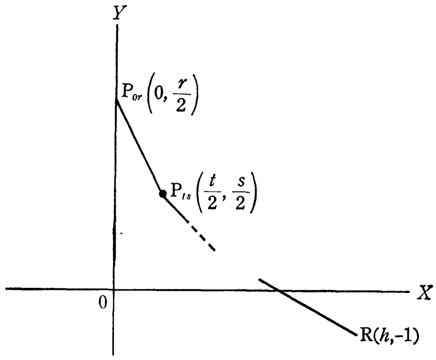

Fig. 1.

$$L^{(a_k)}: \quad Y=\frac{r}{2}-\frac{r-s}{t}X,$$

then the domain of influence N_{a_k} at the turning point a_k is roughly defined by

(2.2) $$N_{a_k}: \quad |x-a_k|\leqq N\varepsilon^{\lambda a_k},$$

with

$$\lambda_{a_k}=\left(\frac{h+1}{\rho_{a_k}}-\frac{r}{2}-1\right), \quad \rho_{a_k}=\frac{t}{r-s},$$

where N is some positive constant. Here it is to be noted that

$$\frac{h}{\rho_{a_k}}-\frac{r}{2}-1\geqq 0$$

because the point R above the line $L^{(a_k)}$, and the equality is correct if and only if the characteristic polygon consists of only one segment connecting P_{0r} and R and in this case $\lambda_{a_k}=\rho_{a_k}=2h/(r+2)$.

The precise definition of the domain of influence at a_k will be given at section 4 where we construct the admissible regions.

3. Lemma.

We state in this section a few lemmas that are necessary for our subsequent studies. The differential equation considered in this paper is written in vector form such that

(3.1) $$\varepsilon^h\frac{dy}{dx}=\begin{bmatrix} 0 & 1 \\ p(x,\varepsilon) & 0 \end{bmatrix}y.$$

We assume here that in the region D

$$D:\ |x|<\infty, \quad 0\leqq\varepsilon\leqq\varepsilon_0, \quad |\varepsilon x^{\alpha}|\leqq\delta_0<1,$$

the function $p(x,\varepsilon)$ can be asymptotically expanded in the sence that for each m, there exists constant M such that

$$\left|p(x,\varepsilon)-\sum_{\nu=0}^{m}p_{\nu}(x)\varepsilon^{\nu}\right|\leqq M(1+|x|^{\alpha(m+1)+\beta})\varepsilon^{m+1},$$

where the functions $p_{\nu}(x)$ and the constants α, β are as defined in the section 2.

For the equation (A_2) in the Introduction, the constants h, α and β correspond to the numbers $q-2r-2$, $[2(q-r)]^{-1}$ and $q(q-r)^{-1}$ respectively, if we replace ε by $\varepsilon^{1/2(q-r)}$.

At first, the equation (3.1) is changed by the transformation

$$y=\begin{bmatrix}1 & 1\\ \sqrt{p_0} & -\sqrt{p_0}\end{bmatrix}z$$

into

$$\varepsilon^{h}\frac{dz}{dx}=\sqrt{p_0}\left\{\sum_{\nu=0}^{\infty}A_{\nu}(x)\varepsilon^{\nu}\right\}z, \tag{3.2}$$

where

$$A_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad A_{\nu}(x)=\frac{1}{2}\frac{p_{\nu}}{p_0}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\qquad (\nu\geqq1,\ \nu\neq h),$$

$$A_h(x)=\frac{1}{2}\frac{p_h}{p_0}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}-\frac{p_0'}{4p_0\sqrt{p_0}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

Since the leading term is diagonalized, we can proceed as usual to make the equation (3.2) formally diagonal and then we have the following lemma.

LEMMA 3.1. *For every m, it can be constructed a linear transformation*

$$z=(E-\varepsilon Q_1)(E-\varepsilon^2Q^2)\cdots(E-\varepsilon^{m+h}Q_{m+h})z_m \tag{3.3}$$

such that the equation (3.2) *becomes*

$$\begin{aligned}&\varepsilon^{h}\frac{dz_m}{dx}=\sqrt{p_0}\left\{\sum_{\nu=0}^{m+h}G_{\nu}(x)\varepsilon^{\nu}+R_{m+h+1}(x,\varepsilon)\varepsilon^{m+h+1}\right\}z_m,\\ &G_0=A_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\end{aligned} \tag{3.4}$$

under the restriction that in a region defined in later the functions Q_{ν} must matisfy

$$||\varepsilon^{\nu}Q_{\nu}||<1 \qquad (\nu=1,2,\cdots,m+h). \tag{3.5}$$

Here E is the 2-by-2 unit matrix and for the norm of the matrix $Q=(q_{jk})$ we use the quantity $||Q||=\sum_{j,k=1}^{2}|q_{jk}|$.

The matrices Q_ν are antidiagonal and G_ν are diagonal, and their elements are determined from the elements of A_j $(0\leqq j\leqq \nu)$ and Q_j $(1\leqq j\leqq \nu-1)$ such as

(3.6)
$$Q_\nu,\ G_\nu=\mathcal{L}\{Q_1^{i_1}\cdots Q_{\nu-1}^{i_{\nu-1}}A_k;\ i_1+2i_2+\cdots+(\nu-1)i_{\nu-1}+k=\nu\}\quad(\nu\leqq h),$$
$$Q_{\nu+h},\ G_{\nu+h}=\mathcal{L}\begin{Bmatrix}Q_1^{i_1}\cdots Q_{\nu+h-1}^{i_{\nu+h-1}}A_k;\ i_1+2i_2+\cdots+(\nu+h-1)i_{\nu+h-1}+k=\nu+h\\ Q_1^{i_1}\cdots Q_\nu^{i_\nu}Q_l'/\sqrt{p_0};\ i_1+2i_2+\cdots+l+h=\nu+h\end{Bmatrix}$$
$$(1\leqq\nu\leqq m),$$

where $i_1, i_2, i_{\nu+h-1}$, k and l are nonnegative integers. Above expressions mean that the elements of Q_ν and other matrices in the left side of (3.6) are linear combinations of the elements of the matrices in the brackets of (3.6). And we assume that in these expressions the products specify only the matrices and their multiplicities, but do not indicate the order of performance of product. Lastly the term $R_{m+h+1}(x,\varepsilon)$ ε^{m+h+1} can be expanded formally

(3.7)
$$\varepsilon^{m+h+1}R_{m+h+1}(x,\varepsilon)\sim\sum_{\nu=m+h+1}^{\infty}R_{m+h+1,\nu}(x)\varepsilon^\nu,$$
$$R_{m+h+1,\nu}(x)=\mathcal{L}\begin{Bmatrix}Q_1^{i_1}\cdots Q_{m+h}^{i_{m+h}}A_k;\ i_1+2i_2+\cdots+(m+h)i_{m+h}+k=\nu\\ Q_1^{i_1}\cdots Q_{m+h}^{i_{m+h}}Q_l'/\sqrt{p_0};\ i_1+2i_2+\cdots+(m+h)i_{m+h}+l+h=\nu\end{Bmatrix},$$

where the symbol $\mathcal{L}$ denotes the same meaning as for Q_ν.

Proof. We can prove this lemma by the induction on m. Firstly this will be proved for $m=0$. The equation (3.2) becomes by the transformation $z=(E-\varepsilon Q_1)w_1$

$$\varepsilon^h\frac{dw_1}{dx}=\left\{(E-\varepsilon Q_1)^{-1}\sqrt{p_0}\left[\sum_{\nu=0}^{\infty}A_\nu(x)\varepsilon^\nu\right](E-\varepsilon Q_1)-(E-\varepsilon Q_1)^{-1}\varepsilon^{h+1}\frac{dQ_1}{dx}\right\}w_1.$$

Since $(E-\varepsilon Q_1)^{-1}$ can be expanded in power series of εQ_1 by virtue of (3.5), we have

$$\varepsilon^h\frac{dw_1}{dx}=\sqrt{p_0}\left\{\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\varepsilon\left(Q_1\begin{bmatrix}1&0\\0&-1\end{bmatrix}-\begin{bmatrix}1&0\\0&-1\end{bmatrix}Q_1+\frac{1}{2}\frac{p_1}{p_0}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\right)\right.$$
$$\left.+\varepsilon^2R_2(x,\varepsilon)\right\}w_1.$$

Here if we define the matrix Q_1 by

(3.8)
$$Q_1=-\frac{p_1}{4p_0}\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

then G_1 becomes

(3.9)
$$G_1=\frac{p_1}{4p_0}\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

Clearly the matrices Q_1 and G_1 have the properties mentioned in the Lemma, and $\varepsilon^2 R_2(x, \varepsilon)$ satisfy the same relation (3.7) with 1 in place of $m+h$. Assuming that the Lemma is true for $\nu=r<h$, we prove for $\nu=r+1$. Then there exists a transformation

$$z=(E-\varepsilon Q_1)\cdots(E-\varepsilon^r Q_r)w_r$$

from which we have

$$\varepsilon^h \frac{dw_r}{dx}=\sqrt{p_0}\left\{\sum_{\nu=0}^{r} G_\nu(x)\varepsilon^\nu+\varepsilon^{r+1}R_{r+1}(x, \varepsilon)\right\}w_r,$$

$$\varepsilon^{r+1}R_{r+1}(x, \varepsilon)\sim\sum_{\nu=r+1}^{\infty} R_{r+1,\nu}(x)\varepsilon^\nu,$$

where the matrices Q_ν, $G_\nu(\nu\leqq r)$ satisfy the properties of the Lemma and $R_{r+1,\nu}(x)$ satisfy the relation (3.7) with r in place of $m+h$. By the transformation

$$w_r=(E-\varepsilon^{r+1}Q_{r+1})w_{r+1},$$

the above equation becomes by the same calculation as for $\nu=1$,

$$\varepsilon^h \frac{dw_{r+1}}{dx}=\sqrt{p_0}\left\{\left(\sum_{\nu=0}^{r} G_\nu(x)\varepsilon^\nu+\left(R_{r+1,r+1}(x)+Q_{r+1}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right.\right.\right.$$
$$\left.\left.\left.-\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}Q_{r+1}\right)\varepsilon^{r+1}+\varepsilon^{r+2}R_{r+2}(x, \varepsilon)\right\}w_{r+1},$$

with

$$\varepsilon^{r+2}R_{r+2}(x, \varepsilon)\sim\sum_{\nu=r+2}^{\infty}\left\{\sum_{\substack{(r+1)i+j+(r+1)k=\nu\\ i\geqq 0,\, j\geqq 0,\, k\geqq 0}}(-1)^{\delta(k)}Q_{r+1}^{i}R_{r+1,j}Q_{r+1}^{k}\right.$$
$$\left.-\sum_{(k+1)(r+1)+h=\nu} Q_{r+1}^{k}Q_{r+1}'\sqrt{p_0}^{-1}\right\}\varepsilon^\nu,$$

where $\delta(k)=0$ if $k=0$ and $=1$ if $k\neq 0$.

Suppose that

$$R_{r+1,r+1}=\begin{bmatrix}g_{11}(x) & g_{12}(x)\\ g_{21}(x) & g_{22}(x)\end{bmatrix},$$

and if we put

$$Q_{r+1}(x)=\begin{bmatrix}0 & -\dfrac{g_{12}(x)}{2}\\ \dfrac{g_{21}(x)}{2} & 0\end{bmatrix},$$

then we have

$$G_{r+1}(x)=\begin{bmatrix}g_{11}(x) & 0\\ 0 & g_{22}(x)\end{bmatrix}.$$

Thus the matrix Q_{r+1} is antidiagonal, G_{r+1} is diagonal, then the elements of $Q_{r+1}(x)$ and $G_{r+1}(x)$ have the form (3.1) with $\nu=r+1<h$. And the above expression of $\varepsilon^{r+2}R_{r+2}(x,\varepsilon)$ shows that this can be written as (3.7) and this prove the Lemma for $m=0$. For $m\geqq 1$, we can prove it by the same method as for $\nu=r+1$ and will not repeat it.

From the above Lemma, we can determine the grouth order when $x\to\infty$, and the order of poles at the turning points for the elements of Q_ν, G_ν and $R_{m+h+1}(x,\varepsilon)$.

LEMMA 3.2. *The grouth order of elements of the matrices Q_ν and G_ν as x tends to infinity is at most*

$$\nu(\alpha+\beta-q)$$

$$\nu\alpha+\beta-q$$

and the order of pole of elements of the matrices Q_ν and G_ν at a turning point a_k is at most ν/ρ_{a_k}, that is, if x approachs to a_k,

$$Q_\nu \quad \textit{and} \quad G_\nu=O[(x-a_k)^{-\nu/\rho_{a_k}}].$$

Proof. At first let us denote the grouth order at infinity of elements of Q_ν or G_ν by $\mathcal{G}(Q_\nu)$ or $\mathcal{G}(G_\nu)$. From (3.8), (3.9) and the assumption (2.1) for $p_\nu(x)$ we have

$$\mathcal{G}(Q_1)=\mathcal{G}(G_1)=\alpha+\beta-q.$$

Assume that the above Lemma is true for $\mathcal{G}(Q_\nu)$ and $\mathcal{G}(G_\nu)$ $(\nu<m)$, then we have from (3.6) and $A_\nu(x)$ in (3.2)

$$\begin{aligned}\mathcal{G}(Q_m)=\mathcal{G}(G_m)\leqq \max_{\{i_1,\cdots,i_{m-1},k\}}&\{i_1(\alpha+\beta-q)+2i_2(\alpha+\beta-q)+\cdots+(m-1)i_{m-1}(\alpha+\beta-q)\\&+\alpha k+(\beta-q)\gamma_k\}=m(\alpha+\beta-q),\ (\beta\geqq q),\end{aligned}$$

$$\begin{aligned}\mathcal{G}(Q_m)=\mathcal{G}(G_m)\leqq \max_{\{i_1,i_2,\cdots,i_{m-1},k\}}&\{i_1(\alpha+\beta-q)+i_2(2\alpha+\beta-q)+\cdots+i_{m-1}((m-1)\alpha+\beta-q)\\&+\alpha k+(\beta-q)\gamma_k\}=\max\{m\alpha+(\beta-q)(i_1+\cdots i_{m-1}+\gamma_k)\}\\&\leqq m\alpha+\beta-q \qquad (\beta<q)\end{aligned}$$

where γ_k is 0 if $k=0$, and 1 if $k\neq 0$.

Next, we consider the order of pole at a turning point a_k of order r. From (2.2) p_ν/p_0 and $p_0'/p_0\sqrt{p_0}$ can be written in the neighborhood of a_k as

$$\frac{p_\nu}{p_0}=\frac{p_{\nu 0}^{(a_k)}+p_{\nu 1}^{(a_k)}(x-a_k)+\cdots+p_{q_\nu}^{(a_k)}(x-x_k)^{q_\nu}}{(x-a_k)^r}\left\{\frac{1}{p_{0r}^{(a_k)}}+O[x-a_k]\right\},$$

$$p_{\nu\mu}^{(a_k)}(x-a_k)^{\mu-r}=p_{\nu\mu}^{(a_k)}(x-a_k)^{-\nu/\rho_{a_k}}(x-a_k)^{\mu-r+\nu/\rho_{a_k}} \qquad (\mu=0,1,\cdots,q_\nu),$$

$$\frac{p_0'}{p_0\sqrt{p_0}}=(x-a_k)^{-r/2-1}\left\{\frac{r}{\sqrt{p_{0r}^{(a_k)}}}+O[x-a_k]\right\},$$

$$(x-a_k)^{-r/2-1}=(x-a_k)^{-h/\rho a_k}(x-a_k)^{h/\rho a_k+r/2-1},$$

but from the definition of characteristic polygon and ρ_{a_k}, we have $\mu-r+\nu/\rho_{a_k}\geqq 0$ and $-r/2-1+h/\rho_{a_k}\geqq 0$. From these facts and the expressions (3.8), (3.9), the order of pole of Q_ν, G_ν is at most $1/\rho_{a_k}$. Suppose that the order of pole of Q_ν, G_ν is at most ν/ρ_{a_k} for $\nu<m$. Then from the expressions (3.6), the order of pole of Q_m and G_m is at most

$$\max\left\{\rho_{a_k}^{-1}(i_1+2i_2+\cdots+(m-1)i_{m-1}+k),\ \rho_{a_k}^{-1}(i_1+2i_2+\cdots+\nu i_\nu+l)+1+\frac{r}{2}\right\}=\frac{m}{\rho_{a_k}},$$

this proves the Lemma.

LEMMA 3.3. *Let δ_1 be sufficiently small positive constant. Then it satisfies*

$$\varepsilon^{m+1}\sqrt{p_0}\ R_{m+h+1}(x,\varepsilon)=\begin{cases}O[|x^{\alpha+\beta-q}\varepsilon|^{m+1}\cdot|x|^{(\alpha+\beta-q)h+q/2}] & (\beta\geqq q),\\ O[|x^{\alpha}\varepsilon|^{m+1}\cdot|x|^{\alpha h+\beta-q/2}] & (\beta<q),\end{cases}$$

as x tends to infinity under the restriction

$$|x^{\alpha_1}\varepsilon|\leqq\delta_1<1,$$

where $\alpha_1=\alpha+\beta-q$ if $\beta\geqq q$ and α if $\beta<q$, and it satisfies also

$$\varepsilon^{m+1}\sqrt{p_0}\ R_{m+h+1}(x,\varepsilon)=O[|(x-a_k)^{-1/\rho a_k}\varepsilon|^{m+1}\cdot|x-a_k|^{r/2-h/\rho a_k}]$$

as x approaches to the turning point a_k in such a way

$$|(x-a_k)^{-1/\rho a_k}\varepsilon|\leqq\delta_1<1.$$

We remark here that if δ_1 is taken small enough, the inequality (3.5) is automatically satisfied. The proof of this Lemma is easily derived from the above two lemmas.

4. Canonical regions and admissible regions.

In this and next sections, we construct an asymptotic expansion of fundamental system of solutions of the differential equation (3.4), and in this section it is defined the canonical region which is fundamental in establishing asymptotic properties of solutions.

The differential equation considered here is

$$\text{(4.1)}\qquad \varepsilon^h\frac{dz_m}{dx}=\sqrt{p_0}\left\{\sum_{\nu=0}^{m+h}G_\nu(x)\varepsilon^\nu+\varepsilon^{m+h+1}R_{m+h+1}(x,\varepsilon)\right\}z_m.$$

For simplification we write

(4.2) $$\frac{\sqrt{p_0}}{\varepsilon}\left\{\sum_{\nu=0}^{h} G_\nu(x)\varepsilon^\nu\right\}=\begin{bmatrix} g(x,\varepsilon) & 0 \\ 0 & -g(x,\varepsilon)\end{bmatrix}+\frac{p_0'}{4p_0}\begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix},$$

where

$$g(x,\varepsilon)=\frac{\sqrt{p_0}}{\varepsilon^h}\left\{1+\frac{p_1}{2p_0}\varepsilon+\left(\frac{p_2}{2p_0}-\frac{1}{8}\left(\frac{p_1}{p_0}\right)^2\right)\varepsilon^2+\cdots+\frac{p_h}{2p_0}\varepsilon^h\right\},$$

and define the functions $\xi(x, x_0)$, $\xi_h(x, x_0, \varepsilon)$ and matrix $\Lambda_h(x, x_0, \varepsilon)$ by

(4.3) $$\xi(x,x_0)=\int_{x_0}^{x}\sqrt{p_0}\,dx, \qquad \xi_h(x,x_0,\varepsilon)=\int_{x_0}^{x}\varepsilon^h g(x,\varepsilon)dx,$$
$$\Lambda_h(x,x_0,\varepsilon)=\varepsilon^{-h}\begin{bmatrix} \xi_h(x,x_0,\varepsilon) & 0 \\ 0 & -\xi_h(x,x_0,\varepsilon)\end{bmatrix}.$$

Moreover we introduce here matrix functions $\hat{z}_m(x,\varepsilon)$, $\hat{w}_m(x,\varepsilon)$, $w_m(x,\varepsilon)$, $\hat{u}_m(x,\varepsilon)$ and $u_m(x,\varepsilon)$ by

$$z_m(x,\varepsilon)=\hat{z}_m(x,\varepsilon)p_0^{-1/4}\exp\Lambda_h(x,x_0,\varepsilon),$$
$$\hat{w}_m(x,\varepsilon)=\exp\int_{x_0}^{x}\sqrt{p_0}\left\{\sum_{\nu=0}^{m} G_{\nu+h}(x)\varepsilon^\nu\right\}dx,$$
(4.4) $$w_m(x,\varepsilon)=\hat{w}_m(x,\varepsilon)p_0^{-1/4}\exp\Lambda_h(x,x_0,\varepsilon)=\exp\int_{x_0}^{x}\varepsilon^{-h}\sqrt{p_0}\left\{\sum_{\nu=0}^{m+h} G_\nu(x)\varepsilon^\nu\right\}dx,$$
$$u_m(x,\varepsilon)=z_m(x,\varepsilon)-w_m(x,\varepsilon),$$
$$u_m(x,\varepsilon)=\hat{u}_m(x,\varepsilon)p_0^{-1/4}\exp\Lambda_h(x,x_0,\varepsilon),$$

then $\hat{u}_m(x,\varepsilon)$ satisfies

(4.5) $$\hat{u}_m(x,\varepsilon)=\hat{z}_m(x,\varepsilon)-\hat{w}_m(x,\varepsilon),$$
$$\frac{d\hat{u}_m}{dx}=\begin{bmatrix} g(x,\varepsilon) & 0 \\ 0 & -g(x,\varepsilon)\end{bmatrix}\hat{u}_m-\hat{u}_m\begin{bmatrix} g(x,\varepsilon) & 0 \\ 0 & -g(x,\varepsilon)\end{bmatrix}$$
$$+\left\{\sum_{\nu=1}^{m}\sqrt{p_0}\,G_{\nu+h}(x)\varepsilon^\nu\right\}\hat{u}_m+\varepsilon^{m+1}\sqrt{p_0}\,R_{m+h+1}(x,\varepsilon)(\hat{u}_m+\hat{w}_m).$$

Let $u_{ij}(x,\varepsilon)$ and $w_{ij}(x,\varepsilon)$ $(i,j=1,2)$ be components of $\hat{u}_m(x,\varepsilon)$ and $\hat{w}_m(x,\varepsilon)$ respectively, then the above equation becomes for each component

(4.1)$_1$ $$\begin{cases} u_{11}'= g_1(x,\varepsilon)u_{11}+\varepsilon^{m+1}\sqrt{p_0}\,R_{m+h+1}[u_{11},u_{21},w_{11},w_{21}]_{11}, \\ u_{21}'=-2g(x,\varepsilon)u_{21}+g_2(x,\varepsilon)u_{21}+\varepsilon^{m+1}\sqrt{p_0}\,R_{m+h+1}[u_{11},u_{21},w_{11},w_{21}]_{21},\end{cases}$$

(4.5)$_2$ $$\begin{cases} u_{12}'= 2g(x,\varepsilon)u_{12}+g_1(x,\varepsilon)u_{12}+\varepsilon^{m+1}\sqrt{p_0}\,R_{m+h+1}[u_{12},u_{22},w_{12},w_{22}]_{12}, \\ u_{22}'= g_2(x,\varepsilon)u_{22}+\varepsilon^{m+1}\sqrt{p_0}\,R_{m+h+1}[u_{12},u_{22},w_{12},w_{22}]_{22},\end{cases}$$

where the functions $g_1(x, \varepsilon)$ and $g_2(x, \varepsilon)$ are the diagonal elements of diagonal matrix $[\sum_{\nu=1}^{m} \sqrt{p_0} G_{h+\nu} \varepsilon^{\nu}]$, and the last term in each equation denotes a linear combination of the components in the bracket whose coefficients are elements of $\varepsilon^{m+1}\sqrt{p_0}R_{m+h+1}$.

Now we shall prove the existence of solutions of the above equation. To do so, it will be convenient to introduce the notion of canonical region with respect to $\xi(x, x_0)$ following to Evgrafov and Fedoryuk [2].

The family of curves

$$\mathrm{Re}\,\xi(x, x_0) = \text{constant}$$

does not depend on initial value x_0, the choice of the path of integration in the x-plane and the determination of the square root of $p_0(x)$, and has branch points at turning points. The curves which pass through turning points are called the Stokes curves, and they divide the x-plane into a finite number of simply connected unbounded regions: Stokes regions. Here we consider the function $\xi(x, x_0)$ as the mapping of the x-plane into the ξ-plane. Since each Stokes curve is mapped onto a straight segment or a ray parallel to the imaginary ξ-axis, the image of Stokes region is a vertical strip or a half plane.

The canonical region with respect to $\xi(x, x_0)$ is a union of an appropriate number of Stokes curves and adjacent Stokes regions bounded by the Stokes curves, con-

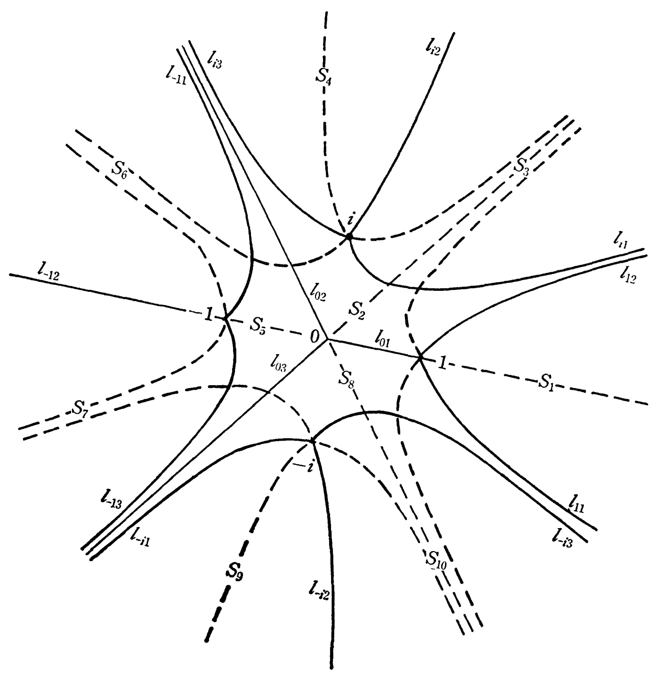

Fig. 2.

tains no turning points in its interior, and is mapped by $\xi(x, x_0)$ onto the whole ξ-plane cut by a finite number of verticals. Each canonical region contains in its interior at least one Stokes curve.

We give here a simple example of canonical regions in the case $p_0(x)=x^5-x$, and then $\xi(x, x_0)=\int_{x_0}^{x} \sqrt{x^5-x}\,dx$.

The Stokes curve configuration and Stokes regions are given at Fig. 2. The real curves and the dotted curves denote the Stokes curves and anti-Stokes curves on which Im $\xi(x, x_0)$=const. respectively. Each Stokes curve and Stokes region are numbered as in the Fig. 2.

For the above example, we give here a set of canonical regions. The union of all canonical regions of this set covers the whole x-plane two times except turning points.

$$D^{(1)}=S_1\cup S_2\cup S_3\cup l_{12}\cup l_{i1}, \qquad D^{(2)}=S_3\cup S_4\cup l_{i2},$$
$$D^{(3)}=S_2\cup S_4\cup S_5\cup S_6\cup l_{02}\cup l_{i3}\cup l_{-11}, \quad D^{(4)}=S_6\cup S_7\cup l_{-12},$$
$$D^{(5)}=S_5\cup S_7\cup S_8\cup S_9\cup l_{-13}\cup l_{03}\cup l_{-11}, \quad D^{(6)}=S_9\cup S_{10}\cup l_{-i2},$$
$$D^{(7)}=S_1\cup S_8\cup S_{10}\cup l_{-i3}\cup l_{11}.$$

The corresponding images $\mathcal{D}^{(i)}$ of the above $D^{(1)}$ and $D^{(2)}$ in the ξ-plane are described in figures, Fig. 3-1 and Fig. 3-2.

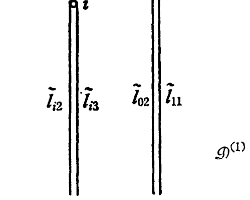

Fig. 3-1.

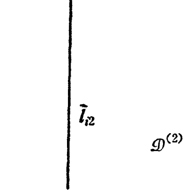

Fig. 3-2.

After introducing the canonical regions, we construct admissible regions by deforming the canonical regions appropriately. Let X and Σ be the complex x and

ξ plane respectively and D be one of the canonical regions. The image of D under the transformation $\xi=\xi(x, x_0)$ of X into Σ will be denoted by $\mathcal{D}$.

Suppose that $a_1, a_2, \cdots, a_n, b_1, \cdots, b_m$ be the set of turning points that are on the boundary of D, a Stokes curve from each a_j is going into the interior of D and b_j is on the Stokes curve issuing from one of a_k. For example, if we take $D^{(1)}$ as a canonical region D, then the turning point $x=0$ is considered as b_1, and $x=1$, i as a_1 and a_2 respectively.

We denote the inverse images of a_k and b_j under the mapping $\xi=\xi(x, x_0)$ by $\tilde{a}_k$ and $\tilde{b}_j$ respectively. Then the region $\mathcal{D}$ in Σ is a whole plane with several cuts issuing from $\tilde{a}_k (k=1, 2, \cdots, n)$, and each $\tilde{b}_j$ is on one of these cuts.

For all sufficiently small ε such that $0<\varepsilon\leqq\varepsilon_0$, a region $\mathcal{D}[\varepsilon]$ is introduced by

$$\mathcal{D}[\varepsilon]=\mathcal{D}\cap\{\xi\in\Sigma\colon |\xi^{2\alpha_2/(q+2)}\varepsilon|\delta_2<1,$$

where δ_2 is sufficiently small positive constant and α_2 is a positive rational number defined by

$$\alpha_2=\begin{cases}(h+1)(\alpha+\beta-q)+q/2+1 & (\beta\geqq q),\\ (h+1)\alpha+\beta+1-q/2 & (q/2-1-\alpha h\leqq\beta<q),\\ \alpha & (q/2-1-\alpha h>\beta),\end{cases}\tag{4.6}$$

and if $\alpha=\beta=0$, we put $\mathcal{D}=\mathcal{D}[\varepsilon]$.

Now we change the region $\mathcal{D}[\varepsilon]$ into $\mathcal{D}[\gamma, \varepsilon]$ for small positive number γ by deleting small neighborhoods of cuts and some portions near the boundary, so that it satisfies following conditions:

Let $\eta^{(+)}$ and $\eta^{(-)}$ be two points on the boundary of $\mathcal{D}[\gamma, \varepsilon]$ such that $\eta^{(\pm)}=\pm(\delta_2/\varepsilon)^{(q+2)/2\alpha_2}$, then for every point ξ in $\mathcal{D}[\gamma, \varepsilon]$ we can describe two piecewise smooth curves $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$ for $0\leqq s\leqq s^{(+)}$ and $\boldsymbol{c}^{(-)}(s, \xi, \eta^{(-)})$ connecting ξ and $\eta^{(\pm)}$ respectively, and they satisfy

(1) $\boldsymbol{c}^{(\pm)}(s, \xi, \eta^{(\pm)})$ are contained in $\mathcal{D}[\gamma, \varepsilon]$, $\boldsymbol{c}^{(\pm)}(0, \xi, \eta^{(\pm)})=\xi$, $\boldsymbol{c}^{(\pm)}(s^{(\pm)}, \xi, \eta^{(\pm)})=\eta^{(\pm)}$, where s denotes the arc length of the curves from ξ.

(2) On these curves, the following inequalities are satisfied

$$\begin{cases}\dfrac{d\,\mathrm{Re}\,\xi_h(x, x_0, \varepsilon)}{ds}\geqq\gamma & \text{on}\quad \boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)}),\\[2ex] \dfrac{d\,\mathrm{Re}\,\xi_h(x, x_0, \varepsilon)}{ds}\leqq-\gamma & \text{on}\quad \boldsymbol{c}^{(-)}(s, \xi, \eta^{(-)}).\end{cases}$$

We take the inverse image of this region $\mathcal{D}[\gamma, \varepsilon]$ as the admissible region.

Now we specify how to construct the region $\mathcal{D}[\gamma, \varepsilon]$ and two curves $\boldsymbol{c}^{(\pm)}(s, \xi, \eta^{(\pm)})$. Let us define the argument φ by $\tan\varphi=2\gamma/\sqrt{1-4\gamma^2}$.

(1) We describe two lines issuing from $\eta^{(+)}$ with arguments $\pi/2+\varphi$ and $-\pi/2-\varphi$, and cut off from $\mathcal{D}[\varepsilon]$ the right hand parts of lines. Analogously, drawing two lines from $\eta^{(-)}$ with arguments $\pi/2\varphi$ and $-\pi/2+\varphi$, we delete the left hand

parts of lines from $\mathscr{D}[\varepsilon]$.

(2) In the neighborhood of $\tilde{a}_k=\xi(a_k, x_0)$ for which we assume that the vertical cut is directed downward and there are no $\tilde{b}_j=\xi(b_j, x_0)$ on this cut, we firstly draw concentric circles C and C' around $\tilde{a}_k$ whose radii are $N\varepsilon^{2\lambda a_k/(r+2)}$ and ρ' $(N\varepsilon^{2\lambda a_k/(r+2)}<\rho')$ respectively. Here r is the order of turning point a_k and λ_{a_k} is the positive number defined in § 2: $\lambda_{a_k}=((h+1)/\rho_{a_k}-r/2-1)^{-1}$. Moreover write two segments L_1 and L_2 starting from $\tilde{a}_k$ with arguments φ and $\pi-\varphi$ respectively. Let P_1, Q_1 be cross points of L_1 with C and C', and P_2, Q_2 be cross points of L_2 with C and C' respectively. From P_1 we draw a segment of argument $-\pi/2+2\varphi$ to cross point R_1 with C', and from R_1 continue a line of argument $-\pi/2+\varphi$ to the boundary of $\mathscr{D}[\varepsilon]$. Let us denote this polygonal segment by l_1. Analogously we describe a point R_2 and a polygonal segment l_2. We delete from $\mathscr{D}[\varepsilon]$ the neighborhood of vertical cut that is a region surrounded by l_1, upper circle P_1P_2, l_2 and the boundary of $\mathscr{D}[\varepsilon]$ (Fig. 4).

(3) When the vertical cut from $\tilde{a}_k$ direct upward, the modification to be made is trivial, and if it brings on it some point $\tilde{b}_j$, we can analogously define neighborhood of cut which is to be deleted from $\mathscr{D}[\varepsilon]$. In what follows, we omit the detailed descriptions about this case for simplicity.

Thus we obtained the region $\mathscr{D}[\gamma, \varepsilon]$ by perfoming the above procedures for all $\tilde{a}_k$ and cuts. Next it is defined the curves $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$ and $\boldsymbol{c}^{(-)}(s, \xi, \eta^{(-)})$ for every ξ in $\mathscr{D}[\gamma, \varepsilon]$. Clearly the cueve $\boldsymbol{c}^{(-)}(s, \xi, \eta^{(-)})$ is drawn by the same method as for $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$, and then we only explain how to construct the curve $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$ (Fig. 4).

(4) Before defining $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$, we divide $\mathscr{D}[\gamma, \varepsilon]$ into several subregions.

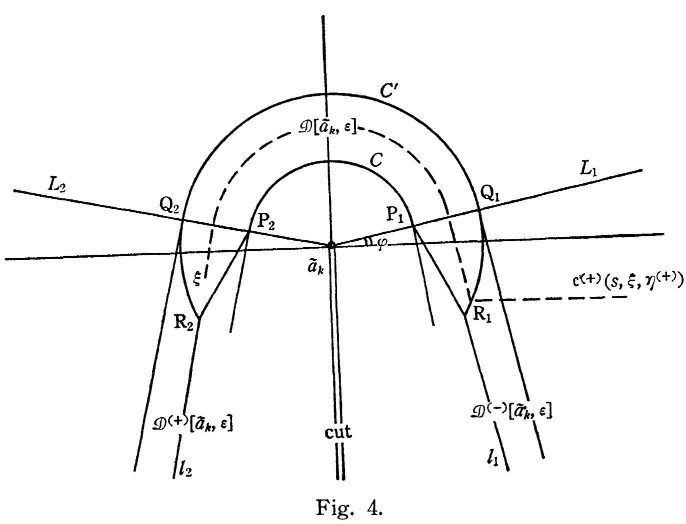

Fig. 4.

Let $\mathscr{D}[\tilde{a}_k, \varepsilon]$ be a region such that

$$\mathscr{D}[\tilde{a}_k, \varepsilon] = \mathscr{D}[\gamma, \varepsilon] \cap \{\xi \colon N' \varepsilon^{2\lambda a_k/(r+2)} \leqq |\xi - \tilde{\tau}_k| \leqq \rho'\}.$$

We draw a line of argument $-\pi/2-\varphi$ from the point Q_2, and denote by $\mathscr{D}^{(+)}[\tilde{a}_k, \varepsilon]$ the region bounded by this line, the arc Q_2R_2 and the boundary of $\mathscr{D}[\gamma, \varepsilon]$. Analogously the region $\mathscr{D}^{(-)}[\tilde{a}_k, \varepsilon]$ is defined which is surrounded by the line of argument $-\pi/2+\varphi$ starting from Q_1, the arc Q_1R_1 and the boundary of $\mathscr{D}[\gamma, \varepsilon]$. Then the region $\mathscr{D}[\gamma, \varepsilon]$ is divided into a finite number of subregions:

$$\mathscr{D}[\gamma, \varepsilon] = \bigcup_{k=1}^{n} \{\mathscr{D}[\tilde{a}_k, \varepsilon] \cup \mathscr{D}^{(+)}[\tilde{a}_k, \varepsilon] \cup \mathscr{D}^{(-)}[\tilde{a}_k, \varepsilon]\} \cup \mathscr{D}'[\gamma, \varepsilon],$$

where

$$\mathscr{D}'[\gamma, \varepsilon] = \mathscr{D}[\gamma, \varepsilon] - \bigcup_{k=1}^{n} \{\mathscr{D}[\tilde{a}_k, \varepsilon] \cup \mathscr{D}^{(+)}[\tilde{a}_k, \varepsilon] \cup \mathscr{D}^{(-)}[\tilde{a}_k, \varepsilon]\}.$$

(5) For ξ in $\mathscr{D}'[\gamma, \varepsilon]$ and $\mathscr{D}^{(-)}[\tilde{a}_k, \varepsilon]$, we can easily draw a curve $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(-)})$ contained in $\mathscr{D}'[\gamma, \varepsilon]$ and $\mathscr{D}^{(-)}[\tilde{a}_k, \varepsilon]$ on which it satisfies except for a finite number of points

$$\frac{d\xi}{ds} > 2\gamma$$

by connecting several segments, owing to their shapes.

(6) Let us divide $\mathscr{D}[\tilde{a}_k, \varepsilon]$ into three parts: $\mathscr{D}_1[\tilde{a}_k, \varepsilon]$, $\mathscr{D}_2[\tilde{a}_k, \varepsilon]$ and $\mathscr{D}_3[\tilde{a}_k, \varepsilon]$ where $\mathscr{D}_1[\tilde{a}_k, \varepsilon]$ is a part of $\mathscr{D}[\tilde{a}_k, \varepsilon]$ below the segments L_1, $\mathscr{D}_2[\tilde{a}_k, \varepsilon]$ between L_1 and L_2 $\mathscr{D}_3[\tilde{a}_k, \varepsilon]$ below L_2.

(7) For ξ in $\mathscr{D}_1[\tilde{a}_k, \varepsilon]$, we draw downward a segment of argument $-\pi/2+\varphi$ from ξ to a point of C' or to a point on the segment P_1R_1. In the former case we connect it with a curve defined in (6), and in the latter case we continue it along P_1R_1 and then combine it with the one described in (6).

(8) For ξ in $\mathscr{D}_2[\tilde{a}_k, \xi]$, $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$ is a curve along the circle of radius $|\xi - a_k|$ from ξ to a point on L_1 and connect it with the curve defined in (7).

(9) For ξ in $\mathscr{D}_3[\tilde{a}_k, \varepsilon]$ and $\mathscr{D}^{(+)}[\tilde{a}_k, \varepsilon]$, $\boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)})$ consists of a segment of argument $\pi/2-\varphi$ from ξ to the point on L_2 and the connected curve described in (8).

From the method of above construction, we can prove the following two lemmas.

LEMMA 4.1. *For sufficiently small* ε_0, δ_2 *and sufficiently large* N' *in the construction of* $\mathscr{D}[\gamma, \varepsilon]$, *we have*

$$\frac{d \operatorname{Re} \xi_h(x, x_0, \varepsilon)}{ds} \geqq \gamma \quad \textit{along} \quad \boldsymbol{c}^{(+)}(s, \xi, \eta^{(+)}),$$

$$\frac{d \operatorname{Re} \xi_h(x, x_0, \varepsilon)}{ds} \leqq -\gamma \quad \textit{along} \quad \boldsymbol{c}^{(-)}(s, \xi, \eta^{(-)}).$$

LEMMA 4.2. *For all ξ in $\mathcal{D}'[\gamma, \varepsilon]$ and $\mathcal{D}^{(+)}[\tilde{a}_k, \varepsilon]$, there exists a constant K independent of ξ and ε such that*

$$(4.7)\qquad \left|\int_{c^{(+)}(s,\xi,\eta^{(+)})} \{\exp(-2\xi_h(x, \tau, \varepsilon))\}\eta^r d\eta\right| \leqq K(|\xi|^r+1) \quad (r\geqq 0),$$

where $\eta=\int_{x_0}^{\tau}\sqrt{p_0}\,dx$, *and for all ξ in $\mathcal{D}[\tilde{a}_k, \varepsilon]$,*

$$(4.8)\qquad \left|\int_{c^{(+)}(s,\xi,\eta^{(+)})} (\eta-\tilde{a}_k)^{-r}d\eta\right| \leqq K|\xi-\tilde{a}_k|^{-r+1} \quad (r\neq 1).$$

Analogous inequalities hold when the integrals are taken along $c^{(-)}(s, \xi, \eta^{(-)})$.

The letter K is used here to denote some positive constant independent of ε and x or ξ, and this K will be used often in later, but in all cases it does not always mean the same constant number.

Proof of Lemma 4.1. It is sufficient if we prove the first inequality of the Lemma. We recall that

$$\xi(x, x_0)=\int_{x_0}^{x}\sqrt{p_0}\,dx,$$

$$\xi_h(x, x_0, \varepsilon)=\int_{x_0}^{x}\sqrt{p_0}\left\{1+\frac{p_1}{2p_0}\varepsilon+\left[\frac{p_1}{2p_0}-\frac{1}{8}\left(\frac{p_1}{p_0}\right)^2\right]\varepsilon^2+\cdots+\frac{p_h}{2p_0}\varepsilon^h\right\}dx,$$

then

$$\frac{d\xi}{ds}=\sqrt{p_0}\frac{dx}{ds},$$

$$\frac{d\xi_h}{ds}=\frac{d\xi}{ds}\left\{1+\frac{p_1}{2p_0}\varepsilon+\left[\frac{p_2}{2p_0}-\frac{1}{8}\left(\frac{p_1}{p_0}\right)^2\right]\varepsilon^2+\cdots+\frac{p_h}{2p_0}\varepsilon^h\right\},$$

$$\frac{d\,\mathrm{Re}\,\xi_h}{ds}=\frac{d\,\mathrm{Re}\,\xi}{ds}+\mathrm{Re}\left[\frac{d\xi}{ds}\left\{\frac{p_1}{2p_0}\varepsilon+\left[\frac{p_1}{2p_0}-\frac{1}{8}\left(\frac{p_1}{p_0}\right)^2\right]\varepsilon^2+\cdots+\frac{p_h}{2p_0}\varepsilon^h\right\}\right].$$

In the last equation, if each term of the expression

$$(4.9)\qquad \left\{\frac{p_1}{2p_0}\varepsilon+\left[\frac{p_2}{2p_0}-\frac{1}{8}\left(\frac{p_1}{p_0}\right)^2\right]\varepsilon^2+\cdots+\frac{p_h}{2p_0}\varepsilon^h\right\}$$

is sufficiently small in the region considered, then we have

$$\frac{d\,\mathrm{Re}\,\xi_h}{ds}\geqq\gamma \quad \text{on} \quad c^{(+)}(s, \xi, \eta^{(+)})$$

owing to the method of construction of $c^{(+)}(s, \xi, \eta^{(+)})$. From the grouth order of $p_\nu(x)$ as $x\to\infty$ and the order of pole at the turning points, the expression (4.9) becomes as small as we want if ε, $|x^{\alpha_1}\varepsilon|$ and $|(x-a_k)^{-1/\rho a_k}\varepsilon|$ are taken small, and these are much smaller if ε, $|x^{\alpha_1}\varepsilon|$ and $|(x-a_k)^{-1/\lambda a_k}\varepsilon|$ are taken sufficiently small since $\alpha_1 \leqq \alpha_2$, $\rho_{a_k} \geqq \lambda_{a_k}$.

Thus in the ξ-plane if the constants ε_0, δ_2 are taken sufficiently small and N' sufficiently large we have the desired inequality.

Proof of the Lemma 4.2. First, the inequality (4.7) is proved. For ξ in $\mathcal{D}'[\gamma, \varepsilon]$ the integral path $c^{(+)}(s, \xi, \eta^{(+)})$ can be written

$$\eta = \xi + t(s) \qquad (0 \leqq s \leqq s^{(+)},\ t(0)=0,\ t(s^{(+)}) = \eta^{(+)} - \xi,$$

where $t(s)$ can be a piecewise linear function and along the path, we have

$$\mathrm{Re}\, \xi_h(x, \tau, \varepsilon) = \mathrm{Re}\, \{\xi_h(x, x_0, \varepsilon) - \xi_h(\tau, x_0, \varepsilon)\} \geqq \gamma s.$$

Then

$$\left| \int_{c^{(+)}(s,\xi,\eta^{(+)})} \{\exp(-2\xi_h(x, \tau, \varepsilon))\} \eta^r d\eta \right| \leqq \int_0^{s^{(+)}} e^{-2\gamma s} |\xi + t(s)|^r ds.$$

Here we consider the two cases: when $|\xi| \leqq M$ and $|\xi| \geqq M$. If $|\xi| \leqq M$, the last expression is clearly bounded for some K, since

$$\int_0^{s^{(+)}} e^{-2\gamma s} |\xi + t(s)|^r ds \leqq \int_0^{\infty} e^{-2\gamma s} (M + |t(s)|)^r ds \leqq K,$$

and if $|\xi| \geqq M$,

$$\int_0^{s^{(+)}} e^{-2\gamma s} |\xi + t(s)|^r ds \leqq |\xi|^r \int_0^{\infty} e^{-2\gamma s} \left(1 + \frac{|t(s)|}{M}\right) ds \leqq K |\xi|^r.$$

Second, we prove the inequality (4.8).

(1) Let ξ be in $\mathcal{D}_1[\tilde{a}_k, \varepsilon]$ and $\xi - \xi(a_k, x_0) = |\xi - \xi(a_k, x_0)| e^{i\varphi}$. The integral path $c^{(+)}(s, \xi, \eta^{(+)})$ consists of possibly three parts, $c_2^{(+)}(s, \xi, \eta^{(+)})$, $c_1^{(+)}(s, \xi, \eta^{(+)})$ and $c_3^{(+)}(s, \xi, \eta^{(+)})$. Here $c_1^{(+)}(s, \xi, \eta^{(+)})$ is in the exterior of $\mathcal{D}[\tilde{a}_k, \varepsilon]$, $c_2^{(+)}(s, \xi, \eta^{(+)})$ is a straight segment of argument $-\pi/2 + \varphi$ from ξ to a point S_ξ which is on the C' or P_1R_1, and if S_ξ is on P_1R_1, the segment $S_\xi R_1$ is the $c_3^{(+)}(s, \xi, \eta^{(+)})$.

Now we estimate the integral of (4.8) for each part. Firstly we have

$$\int_{c_1^{(+)}(s,\xi,\eta^{(+)})} |\eta - \xi(a_k, x_0)|^r |d\eta| \leqq \rho'^{-r} \int_{c_1^{(+)}(s,\xi,\eta^{(+)})} |ds| \leqq K$$

for some constant K.

Next, $c_2^{(+)}(s, \xi,, \eta^{(+)})$ can be written

$$\eta - \xi(a_k, x_0) = \xi - \xi(a_k, x_0) + s e^{-i(\pi/2 - \varphi)},$$

$$s = |\xi - \xi(a_k, x_0)| \{\cos(\varphi - \phi) \tan(\varphi - \phi + \theta) - \sin(\varphi - \phi)\},$$

where

$$-\frac{\pi}{2} + 2\varphi \leqq \phi \leqq \varphi, \qquad 0 \leqq \varphi - \phi \leqq \frac{\pi}{2} - \varphi,$$

$$0 \leqq \theta \leqq \frac{\pi}{2} - \varphi - (\varphi - \phi), \quad 0 \leqq \varphi - \phi + \theta \leqq \frac{\pi}{2} - \varphi.$$

Then we have

$$|\eta - \xi(a_k, x_0)| \geqq |\xi - \xi(a_k, x_0)| \cos(\varphi - \phi)$$

$$\geqq |\xi - \xi(a_k, x_0)| \sin \varphi = 2\gamma |\xi - \xi(a_k, x_0)|,$$

$$ds = |\xi - \xi(a_k, x_0)| \frac{\cos(\varphi - \phi)}{\cos^2(\varphi - \phi + \theta)} d\theta.$$

$$\int_{c_2^{(+)}(s, \xi, \eta^{(+)})} |\eta - \xi(a_k, x_0)|^{-r} |d\eta| \leqq (2\gamma)^{-r} |\xi - \xi(a_k, x_0)^{-r+1} \int \frac{|\cos(\varphi - \theta)|}{\cos^2(\varphi - \phi + \theta)} d\theta$$

$$\leqq K'' |\xi - \xi(a_k, x_0)|^{-r+1}.$$

Lastly we consider the contribution of the part $c_3^{(+)}(s, \xi, \eta^{(+)})$. On this segment

$$ds \leqq \frac{d|\eta - \xi(a_k, x_0)|}{\cos\left(\frac{\pi}{2} - \varphi\right)} = (2\gamma)^{-1} d|\eta - \xi(a_k, x_0)|,$$

and we have

$$|S_\xi - \xi(a_k, x_0)| = |\xi - \xi(a_k, x_0)| \cos(\varphi - \phi) \tan \phi_\xi,$$

where ϕ_ξ is a certain argument satisfying $0 < \phi_\xi < \pi/2 - 2\varphi$. Then we have

$$\int_{c_3^{(+)}(s, \xi, \eta^{(+)})} |\eta - \xi(a_k, x_0)|^{-r} |d\eta| \leqq (2\gamma)^{-1} \int_{|S_\xi - \xi(a_k, x_0)|}^{\rho'} |\eta - \xi(a_k, x_0)|^{-r} d|\eta - \xi(a_k, x_0)|$$

$$\leqq \frac{1}{2\gamma|1-r|} |S_\xi - \xi(a^k, x_0)|^{-r+1} + \frac{\rho'^{-r+1}}{2\gamma|1-r|} \leqq K'' |\xi - \xi(a_k, x_0)|^{-r+1}.$$

Thus by adding the above three estimates, we obtain the desired inequality (4.8) for ξ in $\mathscr{D}_1[\tilde{a}_k, \varepsilon]$.

(2) For ξ in $\mathscr{D}_2[\tilde{a}_k, \varepsilon]$, we denote by $c_1^{(+)}(s, \xi, \eta^{(+)})$ the part of $c^{(+)}(s, \xi, \eta^{(+)})$ in the interior of $\mathscr{D}_2[\tilde{a}_k, \varepsilon]$, and by $c_2^{(+)}(s, \xi, \eta^{(+)})$ the remaining part. From (1),

$$\int_{c_2^{(+)}(s, \xi, \eta^{(+)})} |\eta - \xi(a_k, x_0)|^{-r} |d\eta| \leqq K' |\xi - \xi(a_k, x_0)|^{-r+1}$$

for some constant K'. On the other hand, $c_1^{(+)}(s, \xi, \eta^{(+)})$ can be written as

$$\eta - \xi(a_k, x_0) = |\xi - \xi(a_k, x_0)| e^{i\theta} \quad (\phi \leqq \theta < \varphi),$$

$$ds = |\xi - \xi(a_k, x_0)| d\theta,$$

then we have

$$\int_{c_1^{(+)}(s, \xi, \eta^{(+)})} |\eta - \xi(x_k, x_0)|^{-r} |d\eta| \leqq |\xi - \xi(a_k, x_0)|^{-r+1} \int_\phi^\varphi d\theta \leqq \pi |\xi - \xi(a_k, x_0)|^{-r+1}.$$

Thus by adding the above two inequalities, we obtain the inequality (4.8).

(3) For ξ in $\mathscr{D}_3[\tilde{a}_k, \varepsilon]$ the contribution of the integral from the path in the $\mathscr{D}_3[\tilde{a}_k, \varepsilon]$ is obtained by the same method as (1), and the contribution from other part is obtained from (2), and then inequality (4.8) is proved for this case.

Therefore we can conclude that the inequality (4.8) holds for all ξ in $\mathscr{D}[\tilde{a}_k, \varepsilon]$.

Finally for ξ in $\mathscr{D}^{(\pm)}[\tilde{a}_k, \varepsilon]$, the inequality (4.7) can be proved easily by combining the above procedures and we omit them here.

Thus we have proved the Lemma 4.2.

Now let us denote the inverse images in the x-plane of $\mathscr{D}[\gamma, \varepsilon]$, $\mathscr{D}[\tilde{a}_k, \varepsilon]$ and $\boldsymbol{c}^{(\pm)}(s, \xi, \eta^{(\pm)})$ under the mapping

$$\xi=\int_{x_0}^{x} \sqrt{p_0(x)}\,dx \quad \text{by} \quad D[\gamma, \varepsilon],\ D[a_k, \varepsilon]$$

and $\boldsymbol{c}^{(\pm)}(s, x, x^{(\pm)})$, where

$$\eta^{(\pm)}=\int_{x_0}^{x^{(\pm)}} \sqrt{p_0(x)}\,dx.$$

We call $D[\gamma, \varepsilon]$ the admissible region and also the inverse image of a region $\{\xi: |\xi-\tilde{a}_k| \leqq N'\varepsilon^{2\lambda a_k/(r+2)}\}$ the domain of influence at a_k. It is shown in Fig. 5 below the admissible regions corresponding to $\mathscr{D}^{(1)}$ and $\mathscr{D}^{(2)}$ of Fig. 3.

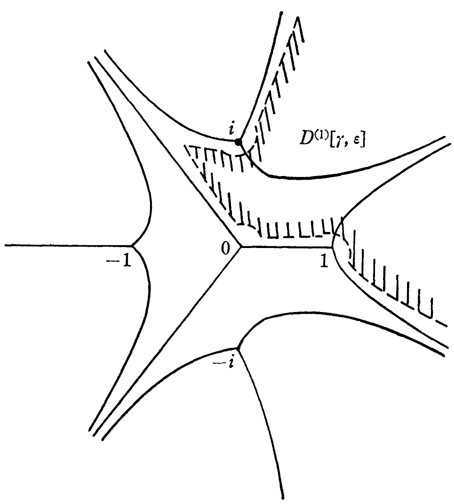

Fig. 5-1.

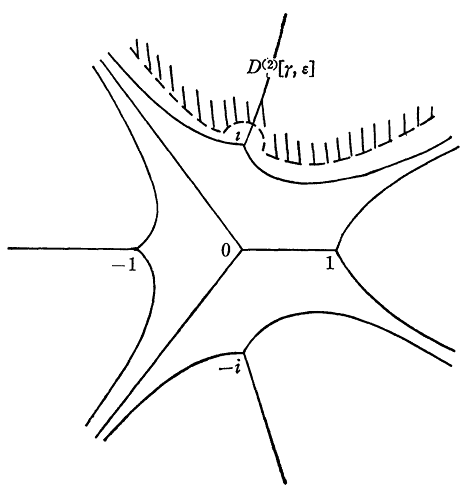

Fig. 5-2.

5. Existence theorem.

We consider in this section the existence and estimates of solutions of differential equations $(4.5)_1$ and $(4.5)_2$. Since the method of analysis is almost parallel for the both equations, it is treated only the equation $(4.5)_1$. As usual the differential equation $(4.5)_1$ change into the following integral equation

$$
(5.1)\qquad
\begin{cases}
u_{11}(x, \varepsilon)=\varepsilon^{m+1}\displaystyle\int_{\gamma_{11}(x)}\left[\exp\int_{\tau}^{x} g_1(s, \varepsilon)ds\right]\sqrt{p_0}\,R_{m+h+1}\{u_{11}, u_{21}, w_{11}, w_{21}\}d\tau,\\
u_{11}(x, \varepsilon)=\varepsilon^{m+1}\displaystyle\int_{\gamma_{21}(x)}\left[\exp\left(-2\xi_h(x, \tau, \varepsilon)+\int_{\tau}^{x} g_2(s, \varepsilon)ds\right)\right]\\
\qquad\qquad\times\sqrt{p_0}\,R_{m+h+1}\{u_{11}, u_{21}, w_{11}, w_{21}\}d\tau,
\end{cases}
$$

where the pathes of integration of $\int_{\tau}^{x} g_i(s, \varepsilon)ds$ $(i=1, 2)$ are appropriate curves connecting x and τ and lying in the interior of $D[\gamma, \varepsilon]$, $\gamma_{11}(x)$ is a curve connecting x and some bounded fixed point x_0 in $D[\gamma, \varepsilon]$, and $\gamma_{21}(x)$ is the curve $c^{(-)}(s, x, x_0^{(+)})$ defined in the previous section.

It is noted at first that the function $w_{11}(\tau, \varepsilon)$, $w_{21}(\tau, \varepsilon)$ and the integral $\int_{\tau}^{x} g_i(s, \varepsilon)ds$ are uniformly bounded for x, τ in the region $D[\gamma, \varepsilon]$, and hence so is for $\exp\int_{\tau}^{x} g_i(s, \varepsilon)ds$ since we have from the Lemma 3.2

$$
|\sqrt{p_0}\,G_{h+\nu}|\leqq
\begin{cases}
K\{1+|x|^{(\alpha+\beta-q)(h+\nu)+q/2}\}, & \beta\geqq q,\ x\in D^{(\infty)}[\gamma, \varepsilon],\\
K\{1+|x|^{\alpha(h+\nu)+\beta-q/2}\}, & \beta<q,\ x\in D^{(\infty)}[\gamma, \varepsilon],\\
K|x-a_k|^{-(h+\nu)/\rho a_k+r/2}, & x\in D[a_k, \varepsilon],
\end{cases}
$$

and then

$$
(5.2)\qquad \left|\int_{x_0}^{x}\sqrt{p_0}\,G_{h+\nu}\varepsilon^{\nu}ds\right|\leqq
\begin{cases}
K'\{1+|x|^{\alpha_2\nu}\}\varepsilon^{\nu}, & x\in D^{(\infty)}[\gamma, \varepsilon],\\
K'\{|x-a_k|^{-\nu/\lambda a_k}\}\varepsilon^{\nu}, & x\in D[a_k, \varepsilon]
\end{cases}
$$

for $\nu=1, 2, \cdots, m$ and for some positive constant K', where $D[a_k, \varepsilon]$ was introduced in the previous section and

$$
D^{(\infty)}[\gamma, \varepsilon]=D[\gamma, \varepsilon]-\bigcup_{k=1}^{n} D[a_k, \varepsilon].
$$

We now prove the following existence theorem by the method of successive approximation.

THEOREM 5.1. *There exists a region* $D[\gamma, \varepsilon]$ *and a system of solutions* $\{u_{11}(x, \varepsilon), u_{21}(x, \varepsilon)\}$ *of the integral equation* (5.1) *in* $D[\gamma, \varepsilon]$ *such that*

$$
(5.3)\qquad |u_{11}(x, \varepsilon)|,\ |u_{21}(x, \varepsilon)|\leqq
\begin{cases}
K(1+|x|^{(m+1)\alpha_2})\varepsilon^{m+1} & \textit{for}\quad x\in D^{(\infty)}[\gamma, \varepsilon],\\
K(|x-a_k|^{-(m+1)/\lambda a_k})\varepsilon^{m+1} & \textit{for}\quad x\in D[a_k, \varepsilon],
\end{cases}
$$

for some positive constant K.

Proof. Let us define the successive approximation by

$$u_{11}^{(0)}(x,\varepsilon)=u_{21}^{(0)}(x,\varepsilon)=0,$$

$$u_{11}^{(k)}(x,\varepsilon)=\varepsilon^{m+1}\int_{\gamma_{11}(x)}\left[\exp\int_{\tau}^{x}g_1(s,\varepsilon)ds\right]\sqrt{p_0}\,R_{m+h-1}\{u_{11}^{(k-1)},u_{21}^{(k-1)},w_{11},w_{21}\}_{11}\,d\tau,$$

$$u_{21}^{(k)}(x,\varepsilon)=\varepsilon^{m+1}\int_{\gamma_{21}(x)}\left[\exp\left(-2\xi_h(x,\tau,\varepsilon)+\int_{\tau}^{x}g_2(s,\varepsilon)\,ds\right)\right]$$
$$\times\sqrt{p_0}\,R_{m+h+1}\{u_{11}^{(k-1)},u_{21}^{(k-1)},w_{11},w_{21}\}_{21}\,d\tau.$$

From the Lemma 3.3 we have

$$\|\varepsilon^{m+1}\sqrt{p_0}\,R_{m+h+1}\{u_{11},u_{21},w_{11},w_{21}\}i_1\|\qquad(i=1,2)$$

$$(5.4)\qquad\leqq\begin{cases}L(1+|x|^{(\alpha+\beta-q)(m+h+1)+q/2})\varepsilon^{m+1}\{|u_{11}|+|u_{21}|+|w_{11}|+|w_{21}|\}, & \beta\geqq q,\\ L(1+|x|^{\alpha(m+h-1)-q/2})\varepsilon^{m+1}\{|u_{11}|+|u_{21}|+|w_{11}|+|w_{21}|\}, & \beta<q,\ x\in D^{(\infty)}[\gamma,\varepsilon],\\ L(|x-a_k|^{-(m+h+1)/\rho a_k+r/2})\varepsilon^{m+1}, & x\in D[a_k,\varepsilon]\end{cases}$$

for some L. Then from the remark stated above and the Lemma 4.2, we have

$$(5.5)\qquad|u_{i1}^{(1)}(x,\varepsilon)|\leqq\begin{cases}K(1+|x|^{(m+1)\alpha_2})\varepsilon^{m+1} & \textit{for}\quad x\in D^{(\infty)}[\gamma,\varepsilon].\\ K(|x-a_k|^{-(m+1)/\lambda a_k})\varepsilon^{m+1} & \textit{for}\quad x\in D[a_k,\varepsilon],\end{cases}\qquad(i=1,2)$$

where K is a positive constant.

Since $R_{m+h+1}[u_{11},u_{21},w_{11},w_{21}]_{i1}$ is a linear function of its variables,

$$(5.6)\qquad\begin{aligned}u_{i1}^{(k+1)}(x,\varepsilon)-u_{i1}^{(k)}(x,\varepsilon)&=\varepsilon^{m+1}\int_{\gamma_{i1}}[\exp\Gamma(x,\tau,\varepsilon)]\sqrt{p_0}\,R_{m+h+1}\\&\times\{u_{11}^{(k)}-u_{11}^{(k-1)},u_{21}^{(k)}-u_{21}^{(k-1)},0,0\}_{i1}d\tau\end{aligned}$$

where

$$\Gamma(x,\tau,\varepsilon)=\begin{cases}\displaystyle\int_{\tau}^{x}g_1(s,\varepsilon)ds & \text{if}\quad i=1,\\ \displaystyle-2\xi_h(x,\tau,\varepsilon)+\int_{\tau}^{x}g_2(s,\varepsilon)ds & \text{if}\quad i=2.\end{cases}$$

From (5.4), (5.5), (5.6) and the Lemma 4.2, we have for some constant L',

$$|u_{i1}^{(2)}(x,\varepsilon)-u_{i1}^{(1)}(x,\varepsilon)|\leqq\begin{cases}KL'(1+|x|^{2(m+1)\alpha_2})\varepsilon^{2(m+1)} & \text{for}\quad x\in D^{(\infty)}[\gamma,\varepsilon],\\ KL'(x-a_k|^{-2(m+1)/\lambda a_k})\varepsilon^{2(m+1)} & \text{for}\quad x\in D[a_k,\varepsilon]\end{cases}$$

and by the induction it is easily proved

$$(5.7)\qquad |u_{i1}^{(k+1)}(x,\varepsilon)-u_{i1}^{(k)}(x,\varepsilon)|\leqq\begin{cases}K(L')^k(1+|x|^{k(m+1)\alpha_2})\varepsilon^{k(m+1)} & \text{for } x\in D^{(\infty)}[\gamma,\varepsilon],\\ K(L')^k(|x-a_k|^{-k(m+1)/\lambda a_k\varepsilon})^{k(m+1)} & \text{for } x\in D[a_k,\varepsilon].\end{cases}$$

Then if the quantities ε, $|x^{\alpha_2}\varepsilon|$ and $|x-a_k|^{-1/\lambda a_k\varepsilon}$ are taken sufficiently small, that is, if we take the constant ε_0, δ_2 and N in the Lemma 4.1 smaller if necessary, the series

$$(5.8)\qquad \sum_{k=0}^{n-1}\{u_{i1}^{(k+1)}(x,\varepsilon)-u_{i1}^{(k)}(x,\varepsilon)\}\qquad (i=1,2)$$

are uniformly and absolutely convergent to a bounded and holomorphic function $u_{i1}(x,\varepsilon)$ in the region $D[\gamma,\varepsilon]$. The functions $u_{i1}(x,\varepsilon)$ constitute a system of solution of the integral equation (5.1). The estimate (5.3) can be proved from (5.5), (5.7) and (5.8), and hence the Theorem 5.1 is proved.

The solution of the integral equation (5.1) is a solution of the differential equation $(4.5)_1$ and by the parallel arguments, the solution of $(4.5)_2$ are obtained. Thus we have a system of solutions of the differential equation (4.5).

From the Lemma and (5.2), the following two lemmas are clearly satisfied.

LEMMA 5.1. *If the matrix function $\hat{w}_m(x,\varepsilon)$ in* (4.4) *is expanded in power series of ε with coefficients $\hat{w}_m^{(i)}(x)$, it satisfies*

$$\|\hat{w}_m(x,\varepsilon)-\{E+\hat{w}_m^{(1)}(x)\varepsilon+\cdots+\hat{w}_m^{(m)}(x)\varepsilon^m\}\|\leqq\begin{cases}K(1+|x|^{(m+1)\alpha_2})\varepsilon^{m+1} & \textit{for } x\in D^{(\infty)}[\gamma,\varepsilon],\\ K(|x-a_k|^{-(m+1)/\lambda a_k})\varepsilon^{m+1} & \textit{for } x\in D[a_k,\varepsilon],\end{cases}$$

for some positive constant K.

LEMMA 5.2. *If the matrix function*

$$F(x,\varepsilon)=\{(E-\varepsilon Q_1)(E-\varepsilon^2Q_2)\cdots(E-\varepsilon^{m+h}Q_{m+h})\}\{E+\hat{w}_m^{(1)}(x)\varepsilon+\cdots+\hat{w}_m^{(m)}(x)\varepsilon^m\}$$

is rearranged in power series of ε with coefficients $y_i(x)$ such that

$$E+y_1(x)+\cdots y_{2m+h}(x)\varepsilon^{2m+h},$$

then we have

$$\|F(x,\varepsilon)-\{E+y_1(x)\varepsilon+\cdots+y_m(x)\varepsilon^m\}\|\leqq\begin{cases}K(1+|x|^{(m+1)\alpha_2})\varepsilon^{m+1} & \textit{for } x\in D^{(\infty)}[\gamma,\varepsilon],\\ K(|x-a_k|^{-(m+1)/\lambda a_k})\varepsilon^{m+1} & \textit{for } x\in D[a_k,\varepsilon].\end{cases}$$

From the above two lemmas and the Theorem 5.1, we have the following main theorem.

THEOREM 5.2. *The differential equation* (3.1) *has a fundamental system of solutions such that*

$$(5.9)\qquad y(x,\varepsilon)=\begin{bmatrix}1 & 1\\ \sqrt{p_0(x)} & -\sqrt{p_0(x)}\end{bmatrix} p_0(x)^{-1/4}\{E+y_1(x)\varepsilon+\cdots+y_m(x)\varepsilon^m + Y_{m+1}(x,\varepsilon)\}\exp \Lambda_h(x,x_0,\varepsilon),$$

where the remainder term $Y_{m+1}(x,\varepsilon)$ *satisfies*

$$\|Y_{m+1}(x,\varepsilon)\|\leqq\begin{cases}K(1+|x|^{(m+1)\alpha_2})\varepsilon^{m+1} & \text{for}\quad x\in D^{(\infty)}[\gamma,\varepsilon],\\ K(|x-a_k|^{-(m+1)/\lambda a_k})\varepsilon^{m+1} & \text{for}\quad x\in D[a_h,\varepsilon]\end{cases}$$

for some constant K. *The constant numbers* α_2 *and* λ_{a_k} *are defined at* (4.6) *and* (2.2) *respectively.*

6. Example.

In this section we consider as one of the example of applications of our theory the central connection problem of the differential equation

$$\varepsilon^2\frac{d^2y}{dx^2}=(x^5-\varepsilon x)y,$$

in the neighborhood of the origin $D=\{x: |x|\leqq d\}$. As usual it is more convenient to consider the above equation in the vector form

$$(6.1)\qquad \varepsilon\frac{dy}{dx}=\begin{bmatrix}0 & 1\\ x^5-\varepsilon x & 0\end{bmatrix}y.$$

In the notation of the Introduction, each constants becomes $q=5$, $m_1=r=1$, $\rho_1=1/4$, $\rho_2=1/3$, $\gamma_1=5/8$ and $\gamma_2=2/3$, and corresponding to the descriptions of the Introduction we consider the following three differential equations in some subregions of D.

(1) $\qquad D_1'$: $\quad M\varepsilon^{1/4}\leqq|x|\leqq d,$

$$(\mathrm{A}_1)\qquad \varepsilon x^{-5/2}\frac{dz_1}{dx}=A_1(x,\varepsilon)z_1,\qquad y=\begin{bmatrix}1 & 0\\ 0 & x^{5/2}\end{bmatrix}z_1,$$
$$A_1(x,\varepsilon)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}-\varepsilon\begin{bmatrix}0 & 0\\ x^{-4} & \frac{5}{2}x^{-7/2}\end{bmatrix}.$$

(2) $\qquad D_2'$: $\quad m\varepsilon^{1/4}\leqq|x|\leqq M\varepsilon^{1/4},$

$$(\mathrm{A}_2)\qquad \varepsilon^{1/8}\frac{dz_2}{ds}=A_2(s,\varepsilon)z_2,\qquad y=\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{5/8}\end{bmatrix}z_2,\qquad x=\varepsilon^{1/4}s,$$
$$A_2(s,\varepsilon)=\begin{bmatrix}0 & 1\\ s^5-s & 0\end{bmatrix}.$$

(3) $$D_4': \ |x|\leqq M\varepsilon^{1/3},$$

(A_4)
$$\frac{dz_4}{dt}=A_4(t,\varepsilon),\qquad y=\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{2/3}\end{bmatrix}z_4,\qquad x=\varepsilon^{1/3}t,$$
$$A_4(t,\varepsilon)=\begin{bmatrix}0 & 1\\ -t & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ t^5\varepsilon^{1/3} & 0\end{bmatrix}.$$

The fundamental systems of solutions of the equations (A_1) and (A_4) are obtained by the usual method of constructing the outer and inner solutions as in [6]. The equation (A_2) is analysed by the method described in this paper.

1. Firstly we construct the fundamental system of solutions of the equation (A_1). In what follow, η^α means $|\eta^\alpha|\exp i\alpha\arg\eta$ for all complex η and rational number α.

If we put

$$y=\begin{bmatrix}1 & 0\\ 0 & x^{5/2}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\frac{\varepsilon}{4}\begin{bmatrix}0 & x^{-4}-\frac{5}{2}x^{-7/2}\\ x^{-4}+\frac{5}{2}x^{-7/2} & 0\end{bmatrix}\right\}z_1^*,$$

the equation (6.1) becomes

$$(\varepsilon x^{-5/2})\frac{dz_1^*}{dx}=\left\{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}-\frac{\varepsilon}{2}\begin{bmatrix}x^{-4}+\frac{5}{2}x^{-7/2} & 0\\ 0 & -x^{-4}+\frac{5}{2}x^{-7/2}\end{bmatrix}+O((\varepsilon x^{-4})^2)\right\}z_1^*.$$

From the method used in my previous papers [6, 7], we can construct an asymptotic expansion of a fundamental system of solutions of the above equation. Let T be a sector in the x-plane such that

$$T:\ -\frac{\pi}{7}<\arg x<\frac{3\pi}{7}.$$

Then there exist a region D_1 of ε, x-plane defined by

$$D_1:\ \arg x\in T,\qquad 0<\varepsilon\leqq\varepsilon_1,\qquad c_1\varepsilon^{1/4}\leqq|x|\leqq c_2,$$

where ε_1, c_1, c_2 are small constants independent of ε, and an actual solution $y_1(x,\varepsilon)$ of (6.1) of the form

(6.2)
$$y_1(x,\varepsilon)=\begin{bmatrix}1 & 0\\ 0 & x^{5/2}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\hat{z}_1(x,\varepsilon)x^{-5/4}$$
$$\times\exp\begin{bmatrix}-\frac{2}{7}\frac{x^{7/2}}{\varepsilon}\left(1+\frac{7}{2}x^{-4}\varepsilon\right) & 0\\ 0 & -\frac{2}{7}\frac{x^{7/2}}{\varepsilon}\left(1+\frac{2}{7}x^{-4}\varepsilon\right)\end{bmatrix}.$$

Here $\hat{z}_1(x, \varepsilon)$ satisfies the following asymptotic property

$$\|\hat{z}_1(x, \varepsilon)-E\| \leqq K(1+|x|^{-4})\varepsilon,$$

for some positive constant K.

2. Next, the equation (A_2) will be analysed. According to the results obtained in the preceding sections, we choose one of admissible regions which is covenient for matching procedures with D_1 in the paragraph 1 and with D_4 in the next paragraph 3. In this case, let $D_2[\gamma, \varepsilon]$ be the admissible region constructed from the canonical region $D^{(1)}$ defind in the example of § 4. The letters q, α, and β used in the theory become $q=5$, $\alpha=\beta=0$, and the order of turning points located at $s=0$, ± 1 and $\pm i$ are all one.

By the transformation

$$z_2=\begin{bmatrix} 1 & 1 \\ \sqrt{p} & -\sqrt{p} \end{bmatrix}\left\{E+\frac{1}{2}r\varepsilon^{1/8}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}z_2^*$$

the equation (A_2) becomes

$$\varepsilon^{1/8}\frac{dz_2^*}{ds}=\sqrt{p}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}+\varepsilon^{1/8}r\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}+O((r\varepsilon^{1/8})^2)\right\}z_2^*,$$

where

$$p(s)=s^5-s, \quad \text{and} \quad r(s)=p'(s)/4p(s)\sqrt{p(s)}.$$

Then applying the results of the Theorem 5. 2, we have

$$y_2(x, \varepsilon)=\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{5/8} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \sqrt{p} & -\sqrt{p} \end{bmatrix}\hat{z}_2(s, \varepsilon)p^{-1/4}$$
$$\times\exp\begin{bmatrix} \int^s \sqrt{p}\,\varepsilon^{-1/8}ds & 0 \\ 0 & -\int^s \sqrt{p}\varepsilon^{-1/8}ds \end{bmatrix}. \tag{6.3}$$

Here the matrix function $\hat{z}_2(s, \varepsilon)$ has the following asymptotic properties

$$\|\hat{z}_2(s, \varepsilon)-E\| \leqq \begin{cases} K\varepsilon^{1/8} & \text{for} \quad s\in D_2^{(\infty)}[\gamma, \varepsilon], \\ K(|s-a_k|^{-3/2})\varepsilon^{1/8} & \text{for} \quad s\in D_2[a_k, \varepsilon], \end{cases}$$

where $a_k=0$, ± 1 or $\pm i$.

3. In this paragraph, we construct the fundamental system of solutions of the equation (A_4). At first, we find out a formal solution of the form

$$z_4(t, \varepsilon)\sim\sum_{\nu=0}^{\infty} v_i(t)\varepsilon^{i/3}.$$

Then each function $v_i(t)$ must satisfy

$$\frac{dv_0}{dt}=C_0(t)v_0,$$

$$\frac{dv_i}{dt}=C_0(t)v_i+C_1(t)v_{i-1} \qquad (i\geqq 1),$$

where

$$C_0(t)=\begin{bmatrix}0 & 0\\ -t & 0\end{bmatrix}, \qquad C_1(t)=\begin{bmatrix}0 & 0\\ t^5 & 0\end{bmatrix}.$$

The differential equation for $v_0(t)$ is the so-called Airy equation. The fundamental system of solutions is given explicitly using the Hankel functions of order $\nu=1/3$, (see [7] section 3)

$$v_0(t)=\begin{bmatrix}1 & 0\\ 0 & t^{1/2}\end{bmatrix}\xi^{\nu}\begin{bmatrix}H_{\nu}^{(1)}(\xi) & H_{\nu}^{(2)}(\xi)\\ H_{\nu-1}^{(1)}(\xi) & H_{\nu-1}^{(2)}(\xi)\end{bmatrix},$$

where $\xi=(2/3)t^{3/2}$, and

$$H_{\nu}^{(1)}(\xi)=\frac{i}{\sin\nu\pi}\{e^{-\nu\pi i}J_{\nu}(\xi)-J_{+\nu}(\xi)\},$$

$$H_{\nu}^{(2)}(\xi)=\frac{-i}{\sin\nu\pi}\{e^{\nu\pi i}J_{\nu}(\xi)-J_{-\nu}(\xi)\}.$$

Here $J_{\nu}(\xi)$, $J_{-\nu}(\xi)$ are the Bessel functions of order $\nu=1/3$, and the formulas of convergent power series expression of $J_{\nu}(\xi)$ for $|\xi|<\infty$ and the asymptotic expansion of $H_{\nu}^{(i)}(\xi)$ $(i=1,2)$ for $|\xi|>1$, $|\arg\xi|<\pi$ are well known:

$$J_{\nu}(\xi)=\left(\frac{\xi}{2}\right)^{\nu}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!\Gamma(\nu+l+1)}\left(\frac{\xi}{2}\right)^{2l} \qquad (|\xi|<\infty),$$

$$H_{\nu}^{(1)}(\xi)=\sqrt{\frac{2}{\pi\xi}}\left[\exp\left\{i\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\}\right]\left\{\sum_{m=0}^{M-1}\frac{(\nu,m)}{(-2i\xi)^m}+O(|\xi|^{-M})\right\},$$

$$(|\xi|>1,\ |\arg\xi|<\pi),$$

$$H_{\nu}^{(2)}(\xi)=\sqrt{\frac{2}{\pi\xi}}\left[\exp\left\{-i\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right\}\right]\left\{\sum_{m=0}^{M-1}\frac{(\nu,m)}{(2i\xi)^m}+O(|\xi|^{-M})\right\}$$

where $(\nu,m)=\Gamma(\nu+m+1/2)/m!\Gamma(\nu-m+1/2)$.

The nonhomogeneous equation for $v_j(t)$ has the solution which is holomorphic in the neighborhood of the origin, and can be expanded asymptotically at infinity such that

$$v_j(t)=\begin{bmatrix}1 & 0\\ 0 & t^{1/2}\end{bmatrix}\xi^{\nu}V_j(t)\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}\begin{bmatrix}g_1(\xi) & 0\\ 0 & g_2(\xi)\end{bmatrix},$$

where

$$g_1(\xi)=\sqrt{\frac{2}{\pi\xi}}\exp\left\{i\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\},$$

$$g_2(\xi)=\sqrt{\frac{2}{\pi\xi}}\exp\left\{-i\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\},$$

$$V_j(t)\cong\xi^{11j/3}\{V_{j0}+V_{j1}\xi^{-1}+\cdots\} \qquad (j\geqq 0).$$

Thus we obtain formal solution $z_4(t,\varepsilon)$, and from this we can construct a fundamental system of solutions $y_4(x,\varepsilon)$ of (6.1). Let $x=\varepsilon^{1/3}t$, $t=(3\xi/2)^{2/3}$ and a region D_4 be a neighborhood of the origin such that

$$D_4\colon\ |t^{33/2}\varepsilon|\leqq c_4 \quad or \quad |x^{11/3}\varepsilon^{-1}|\leqq c_4',\ 0<\varepsilon\leqq\varepsilon_1$$

for sufficiently small c_4, c_4'.

Then the equation (6.1) has a fundamental system of solutions $y_4(x,\varepsilon)$ such that

$$y_4(x,\varepsilon)=\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{2/3}\end{bmatrix}z_4(t,\varepsilon),$$

(6.4)
$$\begin{cases} \|z_4(t,\varepsilon)-v_0(t)\|\leqq K\varepsilon^{1/3} & ((t,\varepsilon)\in D_4,\ |t|\leqq t_0),\\ \left\|\begin{bmatrix}1 & 0\\ 0 & t^{-1/2}\end{bmatrix}\{z_4(t,\varepsilon)-v_0(t)\}\xi^{-\nu}\begin{bmatrix}g_1^{-1}(\xi) & 0\\ 0 & g_2^{-1}(\xi)\end{bmatrix}\right\|\leqq K|t^{11/2}\varepsilon^{1/3}| & \begin{pmatrix}(t,\varepsilon)\in D_4,\ |t|>t_0\\ |\arg t|<\frac{2}{3}\pi\end{pmatrix}, \end{cases}$$

for some positive constants K and t_0.

4. Now we are on the position of calculating the relation between the solutions $y_1(x,\varepsilon)$ and $y_4(x,\varepsilon)$ by the matching procedure. We remark at first that connection matrices between $y_1(x,\varepsilon)$ and $y_2(x,\varepsilon)$, and $y_4(x,\varepsilon)$ are asymptotically diagonal (see for example [7], [12]). Let us define the connection matrices L_1 and L_2 by

(6.5)
$$y_1(x,\varepsilon)=y_2(x,\varepsilon)L_1, \qquad y_2(x,\varepsilon)=y_4(x,\varepsilon)L_2.$$

The matrix L_1 is obtained by comparing the expressions (6.2) and (6.3), where we need the asymptotic expansion of (6.3) when s tends to infinity. The indefinite integral in the bracket of exponential matrix function is to be defined by

$$\varepsilon^{-1/8}\int^s\sqrt{p}\,ds=\varepsilon^{-1/8}\left\{\int_0^s s^{5/2}ds-\frac{1}{2}\int_\infty^s s^{-3/2}ds+\int_\infty^s\left(\sqrt{p}-s^{5/2}+\frac{1}{2}s^{-3/2}\right)ds\right.$$

$$=\frac{2}{7}x^{7/2}\varepsilon^{-1}+x^{-1/2}+\varepsilon^{-1/8}\int_\infty^s\left(\sqrt{p}-s^{5/2}+\frac{1}{2}s^{-3/2}\right)ds.$$

When s tends to infinity, the last term of the above expression is the order of $O(\varepsilon)$.

Thus we have after a short calculation that

(6.6)
$$L_1\cong\varepsilon^{-1/16}\{E+O(\varepsilon^{1/8})\}.$$

The matrix L_2 is asymptotically determined from the expressions of $y_2(x,\varepsilon)$ in the neighborhood of $x=s=0$ $(a_k=0)$ and $y_4(x,\varepsilon)$ when t tends to infinity. From (6.3) (6.4) and (6.5) we have

$$\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{5/8}\end{bmatrix}\begin{bmatrix}1 & 1\\ \sqrt{p} & -\sqrt{p}\end{bmatrix}\hat{z}_2(s,\varepsilon)p^{-1/4}\exp\begin{bmatrix}\int^s \sqrt{p}\,\varepsilon^{-1/8}ds & 0\\ 0 & -\int^s \sqrt{p}\,\varepsilon^{-1/8}ds\end{bmatrix}$$

(6.7)

$$=\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{2/3}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & t^{1/2}\end{bmatrix}\xi^{\nu}\hat{z}_4(t,\varepsilon)\begin{bmatrix}g_1(\xi) & 0\\ 0 & g_2(\xi)\end{bmatrix}L_2,$$

where $\hat{z}(t,\varepsilon)=\sum_{j=0}^{\infty}V_j(t)\varepsilon^{i/3}$.

Here we put $s=s_\eta=\eta\varepsilon^{1/12}$ and $t=\eta$ with $|\eta\varepsilon^{2/33}|$ sufficiently small and $|\eta|$ large. Then we have

$$\sqrt{p(s)}=\sqrt{s^5-s}=\sqrt{\eta^5\varepsilon^{5/12}-\eta\varepsilon^{1/12}}=i\eta^{1/2}\varepsilon^{1/24}(1+O(\eta\varepsilon^{1/12})^4),$$

$$p(s)^{-1/4}=(s^5-s)^{-1/4}=e^{-\pi i/4}\eta^{-1/4}\varepsilon^{-1/48}(1+O(\eta\varepsilon^{1/12})^4).$$

By putting the above quantities into (6.7), we have

$$e^{-\pi i/4}\sqrt{\frac{\pi}{2}}\left(\frac{2}{3}\right)^{1/6}\varepsilon^{-1/24}\left\{\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}+O[(\eta\varepsilon^{1/12})^4]\right\}\hat{z}_2(s_\eta,\varepsilon)\hat{z}_4(\eta,\varepsilon)^{-1}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}^{-1}$$

$$=\exp\begin{bmatrix}i\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) & 0\\ 0 & -i\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\end{bmatrix}L_2\exp$$

$$\times\begin{bmatrix}-\int^{s_\eta}\varepsilon^{-1/8}\sqrt{p}\,ds & 0\\ 0 & \int^{s_\eta}\varepsilon^{-1/8}\sqrt{p}\,ds\end{bmatrix}$$

From the asymptotic nature of $\hat{z}_2(s,\varepsilon)$ and $\hat{z}_4(t,\varepsilon)$.

$$\hat{z}_2(s_\eta,\varepsilon)=E+O(|\eta|^{-3/2}),$$

$$\hat{z}_4(\eta,\varepsilon)=E+O(|\eta|^{-3/2}+|\eta^{11/2}\varepsilon^{1/3}|),$$

and definition of integral $\int^s \sqrt{p(s)}\,ds$ gives

$$\int^{s_\eta}\varepsilon^{-1/8}\sqrt{p}\,ds=\varepsilon^{-1/8}\int_0^{s_\eta}\sqrt{p}\,ds-\varepsilon^{-1/8}\int_0^{\infty}(\sqrt{p}-s^{5/2})ds$$

$$=-c\varepsilon^{-1/8}+i\frac{2}{3}\eta^{3/2}+O(\eta^{11/2}\varepsilon^{1/3}),$$

where we put $c=\int_0^\infty(\sqrt{p(s)}-s^{5/2})ds$.

Since the connection matrix is asymptotically diagonal and does not depend on η, we can conclude that

$$(6.8)\qquad L_2\cong\sqrt{\frac{\pi}{2}}\left(\frac{2}{3}\right)^{1/6}\varepsilon^{-1/24}\exp\begin{bmatrix}-c\varepsilon^{-1/8}+\frac{1}{6}\pi i & 0\\ 0 & c\varepsilon^{-1/8}-\frac{2}{3}\pi i\end{bmatrix}\{E+O(\varepsilon^{1/3})\}.$$

Therefore the central connection problem can be solved by using (6.5), (6.6) and (6.8), that is, the outer solution of the equation (6.1) expressed asymptotically by (6.2) in the region D_1 has the asymptotic expansion at the origin of the form

$$y_1(0,\varepsilon)\cong\sqrt{\frac{\pi}{2}}\left(\frac{2}{3}\right)^{1/6}\varepsilon^{-5/48}\begin{bmatrix}1 & 0\\ 0 & \varepsilon^{2/3}\end{bmatrix}z^4(0,\varepsilon)$$

$$\times\exp\begin{bmatrix}-c\varepsilon^{-1/8}+\frac{1}{6}\pi i & 0\\ 0 & c\varepsilon^{-1/8}-\frac{2}{3}\pi i\end{bmatrix}\{E+O(\varepsilon^{1/8})\},$$

where

$$z_4(0,\varepsilon)=\begin{bmatrix}\dfrac{i2^\nu}{\Gamma(1-\nu)\sin\nu\pi} & 0\\ 0 & \left(\dfrac{3}{2}\right)^{1/3}\dfrac{i2^{1-\nu}}{\Gamma(\nu)\sin\nu\pi}\end{bmatrix}\begin{bmatrix}-1 & 1\\ e^{-\nu\pi i} & -e^{\nu\pi i}\end{bmatrix}\{E+O(\varepsilon^{1/3})\}$$

$$\left(\nu=\frac{1}{3}\right).$$

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