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ON PSEUDO-PRIME MEROMORPHIC FUNCTIONS

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A transcendental meromorphic function $F(z)$ is called *pseudo-prime* if $F(z) = f\circ g(z)$ implies that either $f(z)$ or $g(z)$ is a rational function.

The notion of asymptotic spots of meromorphic functions was introduced by Heins [7], [8]. In this paper we shall give several sufficient conditions for meromorphic functions $F(z)$ to be pseudo-prime involving restrictions on the asymptotic spots of $F(z)$.

At first we shall show the following.

THEOREM 1. *Let $F(z)$ be a transcendental meromorphic function of finite order ρ_F which takes a value b at most a finite number of times and has a finite number of asymptotic spots σ_i $(i=1, \cdots, k)$ over a $(a \neq b)$ such that, for any simply-connected Jordan region Ω containing a, $\cup_{i=1}^{k} \sigma_i(\Omega)$ contains infinitely many roots of $F(z)=a$. Further assume that there exist at mot a finite number of roots of $F(z)=a$ outside $\cup_{i=1}^{k} \sigma_i(\Omega)$. Then $F(z)$ is pseudo-prime.*

Proof. We may assume that $a=0$ and $b=\infty$. Suppose that f and g are both transcendental and $F(z)$ has a factorization of the form $F(z)=f\circ g(z)$. Then we have $\rho_f=0$ by a result of Edrei-Fuchs [2], in view of $\rho_F < +\infty$. Since $F(z)$ has only a finite number of poles, $f(\zeta)$ has also a finite number of poles. If $f(\zeta)$ has a finite number of zeros, then $\rho_f \geqq 1$. This is a contradiction. Hence $f(\zeta)$ has infinitely many unbounded zeros $\{\zeta_i\}_{i=1}^{\infty}$. By the assumption, $\cup_{i=1}^{k} g(\sigma_i(\Omega))$ contains $\{\zeta_i\}_{i=1}^{\infty}$ except for at most one ζ_i. Therefore at least one of $g(\sigma_i(\Omega))$ which we denote $g(\sigma(\Omega))$ contains infinitely many unbounded $\{\zeta_i\}$. Hence $g(\sigma(\Omega))$ is unbounded and $\Omega(\supset f\circ g(\sigma(\Omega)))$ is unbounded by an extension of Wiman's theorem to meromorphic functions [6] (p. 119). This is a contradiction. Therefore $F(z)$ is pseudo-prime.

An application. Theorem 1 can apply to the function $F(z)=R(z)\cdot\sin z$ where $R(z)$ is a rational function satisfying $R(z)\to 0$ as $z\to\infty$.

In [10], Ozawa gave several sufficient conditions for entire functions to be pseudo-prime. We shall give two theorems (Theorem 2 and Theorem 3), as sufficient conditions for meromorphic functions to be pseudo-prime, which are analogous to his theorems (Theorem 6 and Theorem 7 in [10], respectively).

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In order to prove our theorems we shall need the following lemma which refers to the existence of asymptotic spots for composed meromorphic functions.

LEMMA. *Let $F(z)=f\circ g(z)$ be a meromorphic function of finite order where $f(\zeta)$ and $g(z)$ are both transcendental and let σ be an asymptotic spot of $F(z)$ over w_0. Further assume that $\delta(\infty, f)>0$. Then there exists an asymptotic spot $\sum$ of $g(z)$ over a root α of $f(\zeta)=w_0$ such that $\sum(\omega)=\sigma(\Omega)$ where $\omega(\ni\alpha)$ is a component of $f^{-1}(\Omega)$.*

Proof. Let Ω_0 be a simply-connected region containing w_0 in w-plane. Assume that $g(\sigma(\Omega_0))$ is unbounded. Since $\delta(\infty, f)>0$, $\Omega_0(\subset f\circ g(\sigma(\Omega_0)))$ is unbounded by an extended Wiman's theorem. This is a contradiction. Hence $g(\sigma(\Omega_0))$ is bounded. Suppose that $g(\sigma(\Omega_0))$ is contained in the disk $|\zeta|<R$. We may assume that $f(\zeta)$ has no w_0-points on $|\zeta|=R$. Denote by ζ_i $(i=1, \cdots, k)$ the w_0-points of $f(\zeta)$ in $|\zeta|<R$. Consider disks K_i $(i=1, \cdots, k)$ centered at ζ_i such that $K_i\cap K_j=\phi$ $(i\neq j)$ and $f(K_i)\subset\Omega_0$. Let Ω be a simply-connected region contained in $\cap_{i=1}^{k} f(K_i)$. Then $g(\sigma(\Omega))$ is contained in only one disk centered at α of K_i. Denote by ω a component of $f^{-1}(\Omega)$ containing α. Then we can define an asymptotic spot Σ of $g(z)$ over α, putting $\Sigma(\omega)=\sigma(\Omega)$.

THEOREM 2. *Let $F(z)$ be a transcendental meromorphic function of finite order ρ_F which takes a value b at most a finite number of times, and let H be the grand total of harmonic indices of all the asymptotic spots of $F(z)$. Further assume that the order of $N(r; a, F)$ for a value $a(\neq b)$ is less than $H/2$. Then $F(z)$ is pseudo-prime.*

Proof. We may assume that $a=0$ and $b=\infty$. Suppose that $F(z)$ has a factorization $F(z)=f\circ g(z)$ where f and g are both transcendental.

By the same reasoning as in Theorem 1, $f(\zeta)$ has infinitely many zeros. Take two zeros ζ_1 and ζ_2. Then we have

$$\begin{aligned} N(r; 0, F) &= N(r; 0, f\circ g) \\ &\geqq N(r; \zeta_1, g)+N(r; \zeta_2, g) \\ &\geqq m(r, g)-O(\log (r\cdot m(r, g))) \end{aligned}$$

by the second fundamental theorem for g. Hence

$$\rho_g \leqq \rho_{N(r;\, 0,\, F)} < \frac{H}{2}.$$

Let σ be an asymptotic spot of $F(z)$ over w_0 with harmonic index $h(\sigma)$. Then we show that there exists an asymptotic spot Σ of $g(z)$ over α (a root of $f(\zeta)=w_0$) with harmonic index not less than $h(\sigma)$.

Since $F(z)$ has only a finite number of poles, $f(\zeta)$ has also only a finite number of poles. Hence $\delta(\infty, f)=1$. Therefore, by our Lemma we can find an asymptotic spot Σ of $g(z)$ over α.

Since $f(\zeta)$ has the expansion in ω (a component of $f^{-1}(\Omega)$ containing α)

$$f(\zeta)-w_0=c(\zeta-\alpha)^n\{1+o(1)\}$$

with a non-zero constant c, we have

$$f\circ g(z)-w_0=c(g(z)-\alpha)^n\{1+o(1)\}$$

in $\sigma(\Omega)$. Hence we have

$$\log\frac{\varepsilon}{|f\circ g(z)-w_0|}\leqq\log^+\frac{\varepsilon}{|c|}+n\cdot\log^+\frac{1}{|g(z)-\alpha|}+\log 2$$

in $\sigma(\Omega)$ where $\Omega=\{w;\ |w-w_0|<\varepsilon\}$.

On the other hand, we have

$$\log^+\frac{1}{|\zeta-\alpha|}\leqq\mathfrak{G}_\omega(\zeta,\alpha)+M$$

in ω where $\mathfrak{G}_\omega(\zeta,\alpha)$ is the Green's function of ω with the pole at α and M is a positive constant, and hence

$$\log^+\frac{1}{|g(z)-\alpha|}\leqq\mathfrak{G}_\omega(g(z),\alpha)+M$$

in $\sigma(\Omega)$. Therefore we have

$$\log\frac{\varepsilon}{|f\circ g(z)-w_0|}\leqq n\cdot\mathfrak{G}_\omega(g(z),\alpha)+\log^+\frac{\varepsilon}{|c|}+\log 2+M\cdot n$$

in $\sigma(\Omega)$. If we put

$$u_{\sigma(\Omega)}(z)=\text{G.H.M.}\log\frac{\varepsilon}{|f\circ g(z)-w_0|},$$

then we have

$$u_{\sigma(\Omega)}(z)\leqq n\cdot\mathfrak{G}_\omega(g(z),\alpha)+M'$$

in $\sigma(\Omega)$ with a positive constant M'. Since the harmonic index of σ is $h(\sigma)$, $u_{\sigma(\Omega)}(z)$ dominates $h(\sigma)$ positive minimal harmonic functions $u_i(z)$ $(i=1,\cdots,h(\sigma))$ in (Ω);

$$u_i(z)\leqq n\cdot\mathfrak{G}_\omega(g(z),\alpha)+M'.$$

Thus it follows that

$$u_i(z)-n\cdot\mathfrak{G}_\omega(g(z),\alpha)\leqq 0$$

in $\Sigma(\omega)=\sigma(\Omega)$, by the maximum principle of subharmonic functions. Therefore the harmonic index of the asymptotic spot Σ of $g(z)$ over α is not less than $h(\sigma)$.

Now applying the Heins' main theorem [8], we have

$$H\leqq 2\rho_g.$$

This contradicts $\rho_g < H/2$. Thus we have the desired result.

THEOREM 3. *Let $F(z)$ be a transcendental meromorphic function of finite order ρ_F which has at most a finite number of poles, and let H be the grand total of harmonic indices of all the asymptotic spots of $F(z)$. Further assume that the order of $N(r; 0, F')$ is less than $H/2$. Then $F(z)$ is pseudo-prime.*

Proof. Suppose that $F(z)$ has a factorization $F(z)=f\circ g(z)$ where f and g are both transcendental. Then, at first we shall prove that $f'(\zeta)$ has only a finite number of poles.

Let z_0 be a pole of $f'\circ g(z)$. If we put

$$g(z)=\zeta_0+(z-z_0)^q g_1(z); \qquad g_1(z_0)\neq 0, \qquad \zeta_0=g(z_0)$$

and

$$f(\zeta)=\frac{f_1(\zeta)}{(\zeta-\zeta_0)^p}; \qquad f_1(\zeta_0)\neq 0,\ \infty,$$

then from the right hand of the derived equation;

$$F'(z)=f'\circ g(z)\cdot g'(z),$$

we have

$$F'(z)=\frac{F_1(z)}{(z-z_0)^{pq+1}}; \qquad F_1(z_0)\neq 0,\ \infty,$$

and since $pq+1\geqq 2$, z_0 is a pole of $F'(z)$. This means that $f'(\zeta)$ has only a finite number of poles.

Hence, if $f'(\zeta)$ has only a finite number of zeros, then $\rho_f=\rho_{f'}\geqq 1$. But since we have $\rho_f=0$ by a result of Edrei-Fuchs [2], this is a contradiction. Therefore $f'(\zeta)$ has infinitely many zeros. Take two zeros ζ_1 and ζ_2. Then we have

$$\begin{aligned} N(r; 0, F') &\geqq N(r; 0, f'\circ g) \\ &\geqq N(r; \zeta_1, g)+N(r; \zeta_2, g), \end{aligned}$$

and by the second fundamental theorem,

$$N(r; 0. F')\geqq m(r, g)-O(\log (r\cdot m(r, g))).$$

Hence we have

$$2\rho_g < H.$$

The remaining reasoning is the same as in Theorem 2. Hence we have the desired result.

Now, Goldstein gave a sufficient condition for meromorphic functions to be pseudo-prime involving restrictions on the asymptotic values. We shall give a modification of his result (Theorem 1 in [5]), by using asymptotic spots instead of asymptotic values.

THEOREM 4. *Let $F(z)$ be a taanscendental meromorphic function of finite order ρ_F which takes a value b at most a finite number of times and has an asymptotic spot σ over a $(a \neq b)$, and let $\Omega_m = \{w;\ |w-a| < 1/m\}$ and $J(r) = \{re^{i\theta};\ 0 \leqq \theta \leqq 2\pi,\ 1/|F(re^{i\theta}) - a| > \exp(K \cdot T(r, F))\}$ with a positive constant K. Further assume that there exists a sequence $\{r_m\}_{m=1}^{\infty}$ such that the angular measure of $J(r_m) \cap \sigma(\Omega_m)$ is not less than a positive number A. Then $F(z)$ is pseudo-prime.*

Proof. Suppose that $F(z)$ has a factorization $F(z) = f \circ g(z)$ where f and g are both transcendental. We may assume that $b = \infty$. Since $F(z)$ has only a finite number of poles, $f(\zeta)$ has also only a finite number of poles. Hence $\delta(\infty, f) = 1$. Therefore by our Lemma, there exists an asymptotic spot Σ of $g(z)$ over α (a root of $f(\zeta) = a$).

Let s be the order of this a-point α of $f(\zeta)$. Then there exists a constant m_0 such that for every $m \geqq m_0$

$$|f(\zeta) - a| > B \cdot |\zeta - \alpha|^s$$

in ω_m, where B is a positive constant, ω_m is a component of $f^{-1}(\Omega_m)$ containing α and $\Sigma(\omega_m) = \sigma(\Omega_m)$. Hence we have

$$|F(z) - a| = |f \circ g(z) - a| > B \cdot |g(z) - \alpha|^s$$

in $\sigma(\Omega_m)$.

On the other hand we have

$$\frac{1}{|F(z) - a|} > e^{K \cdot T(r_m, F)}$$

in $J(r_m)$, and hence

$$\log^+ \frac{1}{|g(z) - a|} \geqq \frac{K}{s} T(r_m, F) - \frac{1}{s} \log \frac{1}{B}$$

in $J(r_m) \cap \sigma(\Omega_m)$. Integrating both sides in the above inequality, it follows that for every $m \geqq m_0$

$$T(r_m, g) \geqq C \cdot T(r_m, F)$$

with a positive constant C. But since we have

$$\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} = +\infty,$$

by a result of Clunie [1], this is a contradiction. Therefore $F(z)$ is pseudo-prime.

Goldstein also proved the following [4], [5].

THEOREM A. *Let $F(z)$ be a transcendental meromorphic function of finite order which takes a value b at most a finite number of times and is such that*

$$\sum_{a \neq b} \delta(a, F) = 1.$$

Then $F(z)$ is pseudo-prime.

We shall prove Theorem 5 concerning the special class of meromorphic functions such that $\sum \delta(a)=2$, by using the following theorem of Nevanlinna [9].

THEOREM B. *Let $F(z)$ be a meromorphic function of finite order ρ_F without multiple values. Then the total sum of the deficiencies of $F(z)$ is 2 and $\rho_F \geqq 1$.*

The simple proof of this theorem was given by Fuchs [3].

THEOREM 5. *Let $F(z)$ be a meromorphic function of finite order ρ_F without multiple values, then $F(z)$ is pseudo-prime.*

Proof. Suppose that $F(z)$ has a factorization $F(z)=f\circ g(z)$ where f and g are both transcendental. Since $\rho_F<+\infty$, $\rho_f=0$ by a result of Edrei-Fuchs [2]. On the other hand, $f(\zeta)$ is a meromorphic function of finite order without multiple values. In fact, if $f(\zeta)$ has a multiple value, then $F(z)$ has also a multiple value. Thus we have $\rho_f \geqq 1$ by Theorem B. This is a contradiction. Therefore $F(z)$ is pseudo-prime.

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