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ON THE EXISTENCE OF ANALYTIC MAPPINGS

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1. Let R be an ultrahyperelliptic surface defined by $y^2=G(x)$ where $G(x)$ is an entire function having only an infinite number of simple zeros. Let S be a similar surface defined by $y^2=g(x)$ with a similar $g(x)$ as $G(x)$. We already discussed the existence problem of non-trivial analytic mappings of R into S [4]. The following theorem is central theorem in the existence problem of non-trival analytic mappings.

THEOREM A. *Suppose that there is a non-trivial analytic mapping of R into S. Then there are entire $h(z)$ and meromorphic $f(z)$ satisfying $g\circ h(z)=f^2(z)G(z)$ and vice versa.*

Hiromi and Mutō [2] proved the following

THEOREM B. *Suppose that the order of $N(r, 0, G)$ is finite and the one of $N(r, 0, G)$ is finite positive and that there is a non-trivial analytic mapping of R into S. Then the corresponding $h(z)$ is a polynomial of degree* $\mathrm{ord}\, N(r, 0, G)$ $/\mathrm{ord}\, N(r, 0, g)$.

In this paper we shall prove the following

THEOREM 1. *Suppose that the assumptions of Theorem* B *are satisfied and further that G is an entire periodic function of finite order. Then the existence of a non-trivial analytic mapping of R into S gives the following relation*

$$\mathrm{ord}\, G=\nu\, \mathrm{ord}\ N(r, 0, g) \qquad \nu=\begin{cases} 1, 2 & \text{if } \mathrm{ord}\, G\neq 2, \\ 1, 2, 3, 4, 6 & \text{if } \mathrm{ord}\, G=2. \end{cases}$$

There are examples which show the occurence of all the possible cases. In order to prove the above theorem we need several lemmas on number theory.

We shall give an application of Theorem 1.

THEOREM 2. *Besides the assumptions of Theorem* 1 *assume that g is also periodic. Then every non-trivial analytic mapping of R into S reduces to a conformal mapping of R onto S.*

Our results do not depend on any representations of R and S. We can for-

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mulate them in an intrinsic manner. There are lots of related results. If $G(x)$ is $(e^{H(x)}-\gamma)(e^{H(x)}-\delta)$ with a polynomial $H(x)$ and $\gamma\delta(\gamma-\delta)\neq 0$, then $0<\operatorname{ord} g<\infty$ and the existence of non-trivial analytic mappings of R into S imply $\operatorname{ord} G=\nu \operatorname{ord} g$, $\nu=1$ or 2 [5]. Rényi [6] proved that $g\circ h(z)$ with periodic $G(z)$ and a polynomial $h(z)$ implies $\deg h=1$ or 2. In this tendency we can extend the Rényi's result.

THEOREM 3. *Let $G(z)$ be an entire periodic function, $g(z)$ a non-constant entire function and $p(z)$, $Q(z)$ polynomials. If $g\circ P(z)=Q(z)G(z)$, then* $\deg P\leqq 2$.

THEORE 4. *Let $G(z)$ be an entire periodic function of finite order and with only simple zeros. Let $g(z)$ be an entire function of non-zero finite order and with only simple zeros. Let $f(z)$ be an entire function of order less than G. If $g\circ P(z)$ $=f(z)^2G(z)$ with entire $P(z)$, then $P(z)$ is a polynomial of degree less than* 3.

2. Lemmas on number theory

LEMMA 1. *The cyclotomic equation is irreducible over the rational number field.*

LEMMA 2. *The degree of the cyclotomic equation, which corresponds to the primitive n^{th} roots of unity, is larger than 2, unless $n=1, 2, 3, 4, 6$.*

Lemma 2 is very easy to prove by using the so-called Euler's function $\phi(n)$. See [3], [7] for Lemmas 1 and 2.

LEMMA 3. *Let η be $\exp(2\pi i/n)$. If $n\neq 1, 2, 3, 4, 6$, then there are infinitely many triples of integers p, q, r such that*

$$0<|p+q\eta+r\eta^2|<\varepsilon$$

for any given $\varepsilon>0$.

Proof of Lemma 3. Assume that

$$p+q\eta+r\eta^2=0.$$

Then

$$\cos\frac{2\pi}{n}=-\frac{q}{2r},\qquad p=r.$$

Since η and $\bar{\eta}$ are n^{th} primitive roots of unity,

$$\begin{aligned} x^2-(\eta+\bar{\eta})x+\eta\bar{\eta} &= x^2-2x\cos\frac{2\pi}{n}+1 \\ &= x^2+\frac{q}{r}x+1 \end{aligned}$$

should be a factor of the cyclotomic equation. By Lemmas 1 and 2 n should be $1, 2, 3, 4, 6$ in this case. Hence for $n\neq 1, 2, 3, 4, 6$

$$p+q\eta+r\eta^2\neq 0$$

for any non-zero triple (p, q, r). Since $\cos(2\pi/n)$ is irrational for $n\neq 1, 2, 3, 4, 6$, we have

$$0<\left|q\sin\frac{2\pi}{n}+2r\cos\frac{2\pi}{n}\sin\frac{2\pi}{n}\right|<\frac{\varepsilon}{2}$$

for an infinite number of pairs (q, r). Take $p=r$. Then

$$0<\left|p+q\cos\frac{2\pi}{n}+r\cos\frac{4\pi}{n}\right|<\frac{\varepsilon}{2}.$$

Thus for an infinite number of triples (p, q, r), $p=r$

$$0<|p+q\eta+r\eta^2|<\varepsilon.$$

3. Proof of Theorem 1. We may start from

$$g\circ h(z)=f^2(z)G(z).$$

Here we may take g as the canonical product of its zeros. By Hiromi-Muto's theorem B $h(z)$ is a polynomial of degree n such that

$$\operatorname{ord} N(r, 0, G)=n \operatorname{ord} N(r, 0, g).$$

Further since G and g have only simple zeros $f(z)$ is entire and has a finite number of zeros. We may assume that

$$h(z)=z^n+a_pz^p+\cdots+a_0, \qquad a_p\neq 0,\ 1\leqq p\leqq n-2.$$

Let τ be the period of G. The assumption $a_p\neq 0$ leads us to a contradiction. This is our first aim. Let z_j be the j^{th} roots of $h(z)=w$. If w is sufficiently large, then $z_j\neq z_k$ for $j\neq k$. Further

$$z_j=\eta^{j-1}w^{1/n}-\frac{a_p}{n}\frac{1}{\eta^{-(p+1)(j-1)}w^{(n-p-1)/n}}-\frac{a_{p-1}}{n}\frac{1}{\eta^{-p(j-1)}w^{(n-p)/n}}+O\left(\frac{1}{w^{(n-p+1)/n}}\right).$$

Let $\phi(z_1)$ be z_2 and $\phi^{-1}(z_2)=z_1$. Consider

$$\phi^{-1}(\phi(\phi^{-1}(\phi(z_1+\tau)+\tau)-\tau)-\tau)\equiv L(z_1).$$

Then with $\eta=\exp(2\pi i/n)$

$$\begin{aligned} z_2&=\phi(z_1)\\ &=\eta z_1+\frac{a_p}{n}\frac{\eta-\eta^{p+1}}{z_1{}^{n-p-1}}+\frac{a_{p-1}}{n}\frac{\eta-\eta^p}{z_1{}^{n-p}}+O\left(\frac{1}{z_1{}^{n-p+1}}\right),\\ z_1&=\phi^{-1}(z_2)\\ &=\frac{z_2}{\eta}-\frac{a_p}{n}\frac{\eta-\eta^{p+1}}{\eta^{p+2}}\frac{1}{z_2{}^{n-p-1}}-\frac{a_{p-1}}{n}\frac{\eta-\eta^p}{\eta^{p+1}}\frac{1}{z_2{}^{n-p}}+O\left(\frac{1}{z_2{}^{n-p+1}}\right)\end{aligned}$$

and

$$L(z_1)=z_1+\frac{a_p}{n}\,\frac{\eta-\eta^{p+1}}{\eta}\tau\,\frac{\sum_{j=0}^{n-p-2}(z_1+\tau+\tau/\eta)^j(z_1+r)^{n-p-2-j}}{(z_1+\tau)^{n-p-1}(z_1+\tau+\tau/\eta)^{n-p-1}}$$

$$-\frac{a_p}{n}\,\frac{\eta-\eta^{p+1}}{\eta}\tau\eta^{n-1}\frac{\sum_{j=0}^{n-p-2}(z_1+\tau/\eta)^j z_1{}^{n-p-2-j}}{z_1{}^{n-p-1}(z_1+\tau/\eta)^{n-p-1}}+O\left(\frac{1}{z_1{}^{n-p+1}}\right).$$

Assume $z_1=z_0+k\tau$ is a zero of $G(z)$. Then $L(z_1)$ is also a zero of $G(z)$ satisfying

$$|L(z_1)-z_1|=\frac{|a_p|}{n}|1-\eta^p||1-\eta^{n-1}|\frac{n-p-1}{|\tau|^{n-p-1}}\,\frac{1}{k^{n-p}}+O\left(\frac{1}{k^{n-p+1}}\right).$$

Hence $L(z_1)$ is a zero of $G(z)$ being very close to z_1. Hence around z_0 there is an infinite number of zeros of G. In order to prove this we make $L(z_0+k^p\tau)$, $p=1,2,\cdots$ and reduce them around z_0 by subtracting $k^p\tau$. Then they are different and are close to z_0. This is a contradiction. Hence a_p must be zero and hence

$$h(z)=z^n+a_0.$$

We next prove $n=1,2,3,4,6$. Let z be a zero of G. Then $\{z+k\tau\}$, $k=0,\ \pm 1,$ $\cdots$ is a set of zeros of $G(z)$. Further $\phi(\phi^{-1}(z)+\ell\tau)=z+\ell\eta\tau$ and $\phi^2(\phi^{-2}(z)+m\tau)=z+\ell\eta^2\tau$ are also sets of zeros of $G(z)$. Therefore there are three vectors τ, $\eta\tau$, $\eta^2\tau$ such that $z+p\tau+q\eta\tau+r\eta^2\tau$ indicates a set of zeros of G for any triple of integers p, q, r. If $n\neq 1,2,3,4,6$, then Lemma 3 gives a cluster point of zeros of G, which is a contradiction. Thus we have $n=1,2,3,4,6$.

Next we prove that if $n=3$, 4 or 6 then

$$\operatorname{ord} N(r, 0, G)=2.$$

In order to do it we pick up a zero z of G. Then we have zeros of G by $z+k_0\tau$ with integral coefficient k_0. Further $\phi(z)=\eta z$ is a zero of G and hence $\{\eta z+k_1\tau\}$ is a set of zeros of G. Then returning back to z by $\phi^{-1}(z)$ we have a set $\{z+\eta^{-1}k_1\tau\}$ of zeros of G. Repeat this process for $\eta^p, p=1, 2, \cdots, n-1$. Then we have a set of zeros of G, whose form is $\{z+\eta^{-p}k_p\tau\}$. This set with $p=0, 1, \cdots, n-1$; $k_p=0,$ $\pm 1, \pm 2, \cdots$ is called the zero point lattice $\mathcal{L}(z)$ attached to z. We construct lattices for all zeros of G. $\mathcal{L}(z_m)$ may coincide with $\mathcal{L}(z_\ell)$ for $m\neq\ell$. This occurs if and only if $z_\ell\in\mathcal{L}(z_m)$. If there are infinitely many different lattices, then in any fundamental parallelogram P with vertices a, $a+\eta^{-1}\tau$, $a+\eta^{-1}\tau+\tau$, $a+\tau$ there appear infinitely many zeros of G. Of course we consider P as a torus. This is a contradiction. Each $\mathcal{L}(z_m)$ has $A_m r^2$ points in $|z|\leqq r$ with a positive constant A_m if $r\geqq r_0$. Thus $N(r,0, G)\sim Ar^2$ with a positive constant A. Therefore $\operatorname{ord} N(r, 0, G)=2$.

We prove $\operatorname{ord} N(r, 0, G)\equiv\lambda$. $\operatorname{ord} G\geqq\lambda$ is trivial. Assume that $\operatorname{ord} G>\lambda$. Then by $\operatorname{ord} G<\infty$ $G(z)$ has the form

$$Bz^\nu e^{H(z)}\prod E(z, a_m),$$

where $E(z, a_m)$ is the Weierstrass prime factor

$$\left(1-\frac{z}{a_m}\right)\exp\left\{\frac{z}{a_m}+\frac{1}{2}\left(\frac{z}{a_m}\right)^2+\cdots+\frac{1}{q}\left(\frac{z}{a_m}\right)^q\right\}, \qquad q=[\lambda]$$

and Π is taken over all zeros of G excepting $a_m=0$, $\nu=0$ or 1. Since $\operatorname{ord}\Pi=\operatorname{ord} N(r, 0, G)=\lambda$,

$$\deg H\geqq[\lambda]+1>\lambda.$$

We put $H(z)=a_\mu z^\mu+a_{\mu-1}z^{\mu-1}+\cdots+a_1 z$, $a_\mu\neq 0$, $\mu\geqq[\lambda]+1$. Then the asymptotic behavior of $e^{H(z)}$ around $z=\infty$ is well known. It has

$$e^{\Re a_\mu r^\mu e^{i2\mu\pi}(1+O(1/r))}.$$

Thus on putting $a_\mu=|a_\mu|e^{i\theta_\mu}$

$$\cos(\theta_\mu+2\mu\pi)$$

controlls the asymptotic behavior of $e^{H(z)}$. If $\cos(\theta_\mu+2\mu\pi)>0$, then $e^{H(z)}\to\infty$ as

$$|e^{H(z)}|\sim e^{|a|r\cos(\theta_\mu+2\mu\pi)}, \qquad r\to\infty.$$

If it is negative, then $e^{H(z)}\to 0$ as

$$|e^{H(z)}|\sim e^{|a_\mu|r^\mu\cos(\theta_\mu+2\mu\pi)}, \qquad r\to\infty.$$

G is uniformly bounded for the τ-direction, that is, G is bounded along any curve $z=z_0+t\tau$, $t>0$, $t\to\infty$ and $|z_0|\leqq M$. On the other hand $|\Pi|\leqq e^{r^{\lambda+\varepsilon}}$, $\varepsilon>0$ for $r\geqq r_0$. Further we can use the the following estimation [1, p. 73]: For every $\zeta>0$, $\xi>0$ there is a $K(\rho, \zeta, \xi)$ such that

$$\log|\Pi(re^{i\theta})|>-K(\rho, \zeta, \xi)A\,V(r)$$

for $\zeta R\leqq r\leqq R$, except perhaps in circles the sum of whose radii is at most ξR, provided that $R>R_0(\zeta, \xi)$. Here A is a constant and $V(r)=r^{\rho(r)}$ with the Lindelöf proximate order $\rho(r)$ of Π. Of course $\rho(r)<\lambda+\varepsilon$ for every $\varepsilon>0$ if $r\geqq r_0$. Hence in any sector in which $e^H\to\infty$ G is not bounded. Further in any sector in which $e^H\to 0$ G tends to zero as $z\to\infty$. But G does not tend to zero for the τ-direction. Thus the τ-direction should be a direction defined by $\cos(\theta_\mu+2\mu\pi)=0$. Since $\mu\geqq[\lambda]+1$, there are $\eta^{\nu/4}\tau$, $\eta^{\sigma/4}\tau$-directions such that $\eta^{\nu/4}\neq\pm 1$, $\eta^{\sigma/4}\neq\pm 1$ and $G\to 0$ uniformly as $z\to\infty$ along these two directions. These directions along which $G\to 0$ as $z\to\infty$ are equally spaced around $z=0$. Hence we may take them so that $\eta^{\nu/4}\tau$, $\eta^{\sigma/4}\tau$ lie in the opposite side with respect to the τ-direction. Hence $G(z)$ is bounded in any half period strip by the periodicity of G and so in the plane. This is a contradiction. Thus we have

$$\operatorname{ord} G=\operatorname{ord} N(r, 0, G).$$

This completes the proof of Theorem 1.

4. We shall show that theorem 1 is best possible. For $\nu=1$, it is trivial. For

$\nu=2$, take $g=\cos\sqrt{w}$, $h=z^2$, $f=1$ and $G=\cos z$. For $\nu=3$ set

$$\psi(z)=\prod_{-\infty}^{\infty}\left(1-\frac{\cos 2\pi(z+n\omega)}{\cos 4\pi n\omega}\right), \qquad \omega=e^{2\pi i/3}.$$

It is easy to verify the convergence of the infinite product in the right hand side. Let $M(R)$ be $\max_{|z|=R}|\psi(z)|$. Since $\psi(z)$ has period 1, $M(R)$ is taken by ψ at a point on the arc $|z|=R$, $|\Re z|\leqq 1/2$. Further ψ is even. So we evaluate $|\psi(z)|$ for $z=x+iy$, $|x|\leqq 1/2$, $y>0$. We have

$$|\psi(z)|\leqq|1-\cos 2\pi z|\prod_{-\infty}^{\infty}{}'\left(1+\frac{e^{n\pi\sqrt{3}+y}+e^{-(n\pi\sqrt{3}+y)}}{e^{2\pi|n|\sqrt{3}}-e^{-2\pi|n|\sqrt{3}}}\right)$$

and

$$|1-\cos 2\pi z|\leqq 1+\frac{e^{2\pi y}+e^{-2\pi y}}{2}\leqq e^{2\pi y},$$

where Π' indicates the product omitting $n=0$. Hence for large y

$$|\psi(z)|\leqq e^{2\pi y}\prod_{n=1}^{\infty}\left(1+\frac{e^{\pi(n\sqrt{3}+2y)}+e^{-\pi(n\sqrt{3}+2y)}}{e^{2\pi n\sqrt{3}}-e^{-2\pi n\sqrt{3}}}\right)^2$$

and

$$\log|\psi(z)|\leqq 2\pi y+2\left(\sum_{y>2\pi n\sqrt{3}}+\sum_{y\leqq 2\pi n\sqrt{3}}\right)\log\left(1+\frac{e^{\pi(n\sqrt{3}+2y)}+e^{-\pi(n\sqrt{3}+2y)}}{e^{2\pi n\sqrt{3}}-e^{-2\pi n\sqrt{3}}}\right)$$

$$\leqq\frac{y}{2\pi\sqrt{3}}(2\pi y+\log 2)+Ky.$$

Since $y\sim R$ for large y, we have $\operatorname{ord}\psi\leqq 2$. On the other hand ψ has double zeros at $z=3n\omega$ and simple zeros at $z=(3n+1)\omega$, $(3n+2)\omega$. Considering the periodicity of ψ, we see that the convergence exponent of its zeros is equal to 2. Hence $\operatorname{ord}\psi=2$. We define G as

$$G=(\psi(z)\psi(z+\omega)\psi(z+2\omega))^{1/4}.$$

Let α_ν be the images of the points $n+m\omega$ under $w=z^3$. Set

$$g(w)=w\prod_{\nu=1}^{\infty}\left(1-\frac{w}{\alpha_\nu}\right)$$

and

$$f(z)^2=z^3\prod_{\nu=1}^{\infty}\left(1-\frac{z^3}{\alpha_\nu}\right)\Big/G(z)$$

$$=z^2e^{P(z)}.$$

Thus a solution is constructed.

For $\nu=6$ we put

$$G(z)=\frac{(\psi(2z)\psi(2z+\omega)\psi(2z+2\omega))^{1/4}}{(\psi(z)\psi(z+\omega)\psi(z+2\omega))^{1/4}}$$

and replace the α_ν by the images of zeros of G under $w=z^6$. Set

$$g(w)=w\prod_{\nu=1}^{\infty}\left(1-\frac{w}{\alpha_\nu}\right)$$

and

$$f(z)^2=z^6\prod_{\nu=1}^{\infty}\left(1-\frac{z^6}{\alpha_\nu}\right)\Big/G(z)$$
$$=e^{P(z)}.$$

Thus a solution is constructed.

For $\nu=4$ set

$$\psi(z)=\prod_{-\infty}^{+\infty}\left(1+\frac{\cos 2\pi(z+i(2n-1)/2)}{\cosh 4\pi n}\right)$$

and

$$G(z)=(\psi(z)\psi(z+i)\psi(z+2i))^{1/4}.$$

By taking α_ν in the expression of g as the images of zeros of G under $w=z^4$ and $h=z^4$, we get a solution.

5. Proof of Theorem 2. By Theorem 1

$$\operatorname{ord} G=\nu \operatorname{ord} N(r, 0, g), \qquad \nu=\begin{cases} 1, 2 & \text{if } \operatorname{ord} G\neq 2, \\ 1, 2, 3, 4, 6 & \text{if } \operatorname{ord} G=2. \end{cases}$$

Every periodic function is of order $\geqq 1$. Hence $\operatorname{ord} g\geqq 1$. Further $\operatorname{ord} g=\operatorname{ord} N(r, 0, g)$ by the proof of Theorem 1, since g has a zero and hence $\infty>\operatorname{ord} N(r, 0, g)\geqq 1$ by its periodicity. If $\operatorname{ord} G=2$, then ν should be 1 or 2. Hence in any case ν should be 1 or 2. If $\nu=1$, then we have the desired result. Assume $\nu=2$, that is the degree of $h(z)$ is equal to 2. Then we put

$$w\equiv h(z)=A(z-\alpha)^2+B, \qquad A\neq 0.$$

We have

$$z=\alpha\pm\sqrt{\frac{w-B}{A}}.$$

Since $g(w)$ is periodic with period s and $f(z)$ has only a finite number of zeros,

$$z_p^{\pm}=\alpha\pm\sqrt{\frac{w_0+p_s-B}{A}}, \qquad p\geqq p_0$$

are two zeros of $G(z)$ if $g(w_0)=0$. Hence

$$z_{p+1}^{+}-z_{p}^{+}=\sqrt{\frac{s}{A}}\frac{1}{2\sqrt{p}}+O\left(\frac{1}{p\sqrt{p}}\right),$$

$$z_{p+1}^{-}-z_{p}^{-}=-\sqrt{\frac{s}{A}}\frac{1}{2\sqrt{p}}+O\left(\frac{1}{p\sqrt{p}}\right).$$

By the periodicity of $G(z)$ we have a cluster point of zeros of G, which is a contradiction. Thus we have the desired result.

6. Proof of Theorem 3. In order to prove Theorem 3 we again need the following fact: Let τ be the period of G. The $\pm\tau$ directions divide the plane into two sides. If $G(z)$ is bounded in a half period strip, then this is true in a half plane lying in one side which contains the half period strip.

Let ν be the degree of $P(z)$. We may assume that

$$P(z)=z^{\nu}+p_{\nu-2}z^{\nu-2}+\cdots+p_0.$$

Consider the equation $P(z)=P(z_0+k\tau)$, k: integer >0. If k is sufficiently large, then there are ν solutions $z_{\ell,k}$, $\ell=0, 1, \cdots, \nu-1$, $z_{0,k}=z_0+k\tau$. Then

$$z_{\ell,k}=(z_0+k\tau)e^{2\pi i\ell/\nu}\left(1+O\left(\frac{1}{k^2}\right)\right).$$

Then

$$G(z_{\ell,k})=\frac{Q(z_{0,k})}{Q(z_{\ell,k})}G(z_{0,k})$$

$$=\frac{1+O(1/k^2)}{e^{(2\pi i\ell/\nu)q}}G(z_0),\qquad q=\deg Q.$$

This means that $G(z)$ is bounded uniformly for a direction

$$\tau e^{2\pi i\ell/\nu}.$$

If $\nu\geqq 3$, then $\ell=1$ and $\ell=\nu-1$ give two directions which are not parallel to $\pm\tau$ and lie in the opposite side with respect to the τ-direction. Hence $G(z)$ is bounded in both of half period strips. This is a contradiction.

7. Proof of Theorem 4. Evidently $f(z)$ has a finite number of zeros. If $f(z)$ is a polynomial, then theorem 2 gives the result. Hence $f^2(z)=Q(z)e^{H(z)}$, $\deg H<\operatorname{ord} G$. Q is a polynomial. Let $H(z)$ be

$$h_N z^N+\cdots+h_1 z,$$

$q=\deg Q$. Then

$$G(z_{\ell,k})=\frac{1+O(1/k)}{e^{(2\pi i\ell)q/\nu}}\frac{e^{h_N z_{0,k}^N(1+O(1/k))}}{e^{h_N z_{0,k}^N e^{2\pi i\ell N/\nu}}(1+O(1/k))}G(z_0).$$

If $\nu\geqq 3$, then

$$\log G(z_{\ell,k})=O\left(h_N z_{0,k}^N\left(1+O\left(\frac{1}{k}\right)\right)\right).$$

Thus by the periodicity of $G(z)$ in any direction

$$\log G(z)=O(z^N) \qquad \text{uniformly.}$$

Hence

$$m(z, G)=O(r^N).$$

Since $N<\operatorname{ord} G$, this implies a contradiction.

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