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A NOTE ON NONCOMPACT RIEMANNIAN MANIFOLDS

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Let M be a complete connected Riemannian manifold. Every geodesic is always parameterized with respect to arclength. A geodesic $c: [0, \infty)\rightarrow M$ ($c: (-\infty, \infty)\rightarrow M$) is called a ray (a line, respectively), if any segment of c is minimal. d denotes the metric distence in M. A subset A of M is called totally convex, if for any $p, q\in A$, any geodesic segment joining p and q is contained in A. A point p of M is called a simple point, if $\{p\}$ is a totally convex set. S_M denotes the set of all simple points of M. A point p of M is called a pole, if the map $\exp_p$: $T_p(M)\rightarrow M$ has maximal rank everywhere. P_M denotes the set of all poles of M. If M is simply connected and $p\in P_M$, then all geodesics $c: [0, \infty)\rightarrow M$ starting from p are rays. For a point p of M, let $C(p)$ and $Q(p)$ be the cut locus and the first conjugate locus of M, respectively. Then, the function $d_M: M\rightarrow R\cup\{\infty\}$ defined by

$$d_M(p):=\begin{cases}\inf_{q\in C(p)} d(p, q) & (\text{when } C(p)\neq\phi),\\ \infty & (\text{when } C(p)=\phi)\end{cases}$$

is continuous. Hence, let $\tilde{M}$ be the universal covering manifold of M and π be its projection, then by the fact

$$P_M=\pi(P_{\tilde{M}})=\pi(d_{\tilde{M}}^{-1}(\infty))$$

P_M is a closed subset of M.

Now, assuming furthermore that M is noncompact and of positive sectional curvature, it it is proved in [2] that there exists a point $p\in M$ such that $C(p)\cap Q(p)\neq\phi$. This result extends to some manifolds of nonnegative sectional curvature. If M is noncompact and of nonnegative sectional curvature, it is proved in [1] that M is homeomorphic to an n-dimentional Euclidean space E^n if and only if $S_M\neq\phi$.

THEOREM. *Let M be a complete connected noncompact Riemannian manifold with nonnegative sectional curvature and not flat. If $S_M\neq\phi$, then there exists a point $q\in M$ such that $C(q)\cap Q(q)\neq\phi$.*

Proof. We distinguish two cases

1) $P_M=M$. Let $c: (-\infty, \infty)\rightarrow M$ be a geodesic and $\{s_i\}$ be a sequence such that $s_i\rightarrow-\infty$ as $i\rightarrow\infty$. For each i, $c|[s_i, \infty)$ is a ray, hence letting $i\rightarrow\infty$, we see that c: $(-\infty, \infty)\rightarrow M$ is a line. Hence, by Toponogov's splitting theorem, M splits as $M=E^1\times\bar{M}$ (see Theorem 4.3, [1]). Hence $P_M=E^1\times P_{\bar{M}}$. By induction, we obtain $M=E^n$.

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2) $P_M \subsetneqq M$. Since M is simply connected, $P_M \subset S_M$. We show that $P_M \subsetneqq S_M$. If $P_M = S_M$, there exists a sequence of points $\{p_i\}$ such that $p_i \in M - P_M$ and $p_i \to p \in P_M$ as $i \to \infty$, because P_M is a closed subset of M and $P_M \neq M$. For this point $p \in M$, we apply the basic construction in the argument in [1]. That is, there exists a family of compact totally convex set $C_t, t \geqq 0$ such that

$$t_2 \geqq t_1 \quad \text{implies} \quad C_{t_2} \supset C_{t_1}$$

and

$$C_{t_1} = \{q \in C_{t_2}: d(q, \partial C_{t_2}) \geqq t_2 - t_1\}.$$

More precisely, C_t is given by

$$C_t = \bigcap_c (M - B_{c_t})$$

where $B_{c_t} := \cup_{s>0} B_s(c(t+s))$, $B_s(q) := \{q^1 \in M: d(q, q^1) < s\}$ and the intersection is taken over all rays $c: [0, \infty) \to M$ starting from p. Since all geodesics starting from p are rays, we easily see that $C_t = \overline{B_t(p)}$. We choose $t_0 > 0$ such that $B_{t_0}(p)$ is a convex neighborhood of p. Choose $t_1 > 0$ such that $t_1 < t_0$. Then we can find i_0 such that $p_{i_0} \in B_{t_1}(p)$. Since $p_{i_0} \in M - S_M$, there exists a geodesic loop γ starting from p_{i_0}. But $B_{t_0}(p)$ is totally convex, it must be $\gamma \subset B_{t_0}(p)$. This is a contradiction.

From the above argument, there exists a point $q \in S_M - P_M$. $C(q) \neq \phi$, because $q \notin P_M$. If $C(q) \cap Q(q) = \phi$, then as is well known, there exists a geodesic loop of length $2d_M(q)$ starting from q. This contradicts $q \in S_M$. Q.E.D.

COROLLARY. *Let M be a manifold homeomorphic to E^n and with nonnegative sectional curvature. Then M is isometric to E^n if and only if, for all $q \in M$, $C(q) \cap Q(q) = \phi$. Moreover if $n = 2, 3$, the assumption that M is homeomorphic to E^n is replaced by that M is simply connected.*

Proof. By the classification theorem in [1], when $n=2$, M is homeomorphic to E^2 and when $n=3$, M is homeomorphic to E^3 or $E^1 \times S^2$. For a manifold $\bar{M}$ which is homeomorphic to S^2, all points $q \in \bar{M}$ satisfy $C(q) \cap Q(q) \neq \phi$; see Theorem 5.1 of [3]. Q.E.D.

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