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ON NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR ELLIPTIC DIFFERENTIAL OPERATORS

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§ 0. Introduction.

This paper is inspired by Seeley's work [10]. In [10], he has shown that one can reduce the study of boundary value problems for elliptic differential operators to the study of operators defined on the boundary by means of appropriate surface potentials and volume potentials (cf. [3], [4], [5], [7], [11]).

In this paper, we show by the same approach that the question of the validity of a priori estimates, the question of solvability and the question of regularity for the non-homogeneous boundary value problems formulated in a slightly more general framework can be reduced to the corresponding questions for the operators defined on the boundary.

The paper is organized as follows. § 1 establishes the notation and the definitions and summarizes some of the results in Seeley [10], [11]. § 2 formulates the non-homogeneous boundary value problems and states main theorem (Theorem 2.2). § 3, § 4 and § 5 are devoted to the proof of this theorem.

§ 1. Preliminaries.

1. 1. Spaces. Let M_1^+ be an n-dimensional compact C^∞ manifold with boundary. Then we may assume that M_1^+ is the closure of a relatively compact open subset M^+ of an n-dimensional compact C^∞ manifold M without boundary in which M^+ has a C^∞ boundary X (see Palais [9], p. 170). We also assume a C^∞ volume element on M. Let E be a complex C^∞ vector bundle over M with e-dimensional fiber, let $C^\infty(E)$ denote the space of C^∞ sections of E and let $C_0^\infty(E_{M^+})$ denote the subspace of $C^\infty(E)$ with compact support in M^+. We assume a C^∞ Hermitian inner product in E. (A complex C^∞ vector bundle with such a structure will be called an Hermitian vector bundle.) Thus we get the inner product on $C^\infty(E)$ which we denote by ((,)). For each real s, we denote by $H^s(E)$ the Sobolev space of E and by $\|\ \|_{s,M}$ its norm (for the definition, we refer to Palais [9], pp. 147–155). Then $H^s(E)$ and $H^{-s}(E)$ are antidual with respect to an extension of the inner product $((u, v))$ defined for $u, v \in C^\infty(E)$ and the duality will be denoted by

$$((\underset{(s,M)}{u}, \underset{(-s,M)}{v})); \ u \in H^s(E),\ v \in H^{-s}(E).$$

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Let $E_{M_1^+}$ be the restriction of E to $M_1^+=M^+\cup X$ and let $C^\infty(E_{M_1^+})$ denote the space of C^∞ sections of $E_{M_1^+}$. For each real s, we define the following spaces:

$H^s(E_{M^+})$=the space of restrictions to M^+ of elements in $H^s(E)$; the norm of $u\in H^s(E_{M^+})$ is defined by

$$\|u\|_s=\inf \|U\|_{s,M},$$

the infimum being taken over all $U\in H^s(E)$ such that $U=u$ in M^+;

$H_0^s(E_{M^+})$=the subspace of $H^s(E)$ with support in $M_1^+=M^+\cup X$. Then $C^\infty(E_{M_1^+})$ is dense in each $H^s(E_{M^+})$ and $C_0^\infty(E_{M^+})$ is dense in each $H_0^s(E_{M^+})$. Moreover, $H^s(E_{M^+})$ and $H_0^{-s}(E_{M^+})$ are antidual with respect to an extension of the inner product $((u, v))$ defined for $u\in C^\infty(E_{M_1^+})$, $v\in C_0^\infty(E_{M^+})$ (cf. Hörmander [6], p. 51) and the duality will be denoted by

$$\underset{(s,M^+)(-s,M^+)}{((u,\ v))};\ u\in H^s(E_{M^+}),\ v\in H_0^{-s}(E_{M^+}).$$

Let E_X be the restriction of E to X and let $C^\infty(E_X)$ denote the space of C^∞ sections of E_X. Then, using the induced volume element on X, we get the inner product on $C^\infty(E_X)$ which we denote by $(\ ,\ )$. For each real s, we denote by $H^s(E_X)$ the Sobolev space of E_X and by $|\ |_s$ its norm. Then $H^s(E_X)$ and $H^{-s}(E_X)$ are antidual with respect to an extension of the inner product (g, h) defined for $g, h\in C^\infty(E_X)$. Here we define the following spaces for each real s: $B^s(X)=\oplus_{j=0}^{\omega-1}H^{s-j-1/2}(E_X)$; the norm of $g=(g_0, \cdots, g_{\omega-1})\in B^s(X)$ is defined by

$$|g|_s=\left(\sum_{j=0}^{\omega-1}|g_j|^2_{s-j-1/2}\right)^{1/2};$$

$(B^s(X))^*=\oplus_{j=0}^{\omega-1}H^{-s+j+1/2}(E_X)$; the norm of $h=(h_0, \cdots, h_{\omega-1})\in(B^s(X))^*$ is defined by

$$|h|_{s^*}=\left(\sum_{j=0}^{\omega-1}|h_j|^2_{-s+j+1/2}\right)^{1/2}.$$

Then $B^s(X)$ and $(B^s(X))^*$ are antidual with respect to an extension of

$$\sum_{j=0}^{\omega-1}(g_j, h_j);\ g_j, h_j\in C^\infty(E_X),$$

and the duality will be denoted by

$$\underset{(s,X)\ (s^*,X)}{(g,\ h)};\ g\in B^s(X),\ h\in(B^s(X))^*.$$

Let G be an Hermitian vector bundle over X with g-dimensional fiber and let $C^\infty(G)$ denote space of C^∞ sections of G. Then, using the induced volume element on X, we get the inner product on $C^\infty(G)$ which we denote by $[\ ,\ ]$. For each real s, we denote by $H^s(G)$ the Sobolev space of G and by $\langle\ \rangle_s$ its norm. Then $H^s(G)$ and $H^{-s}(G)$ are antidual with respect to an extension of the inner product $[\varphi, \psi]$ defined for φ, $\psi\in C^\infty(G)$ and the duality will be denoted by

$$\underset{(s,G)(-s,G)}{[\varphi,\ \psi]};\ \varphi\in H^s(G),\ \psi\in H^{-s}(G).$$

1.2. Pseudo-differential operators on vector bundles. Let $A: C^\infty(E)\to C^\infty(E)$ be a pseudo-differential operator of order ω and let $\sigma_\omega(A)$ denote the principal symbol of A (for the definitions, we refer to Seeley [10], pp. 230–237). Then $\sigma_\omega(A)$ is the map of $T'(M)$ (the cotangent bundle minus the zero section) into Hom (E, E). A is called elliptic if and only if $\sigma_\omega(A)(\xi_x)$ is an isomorphism E_x onto E_x for all $(x, \xi_x)\in T'(M)$. Here E_x denotes the fiber of E at $x\in M$.

A has the formal adjoint $A^*: C^\infty(E)\to C^\infty(E)$ such that

$$((Au, v))=((u, A^*v)); \quad u, v\in C^\infty(E). \tag{1}$$

A^* is also a pseudo-differential operator of order ω and the principal symbol of A^* is the adjoint of $\sigma_\omega(A)$ with respect to the Hermitian inner product in E. Thus A is elliptic if and only if A^* is elliptic. Further, in view of (1), we see that if A is an elliptic differential operator of order ω, then A^* is also an elliptic differential operator of order ω.

1.3. Restriction maps γ. We assume that near the boundary X a normal coordinate t has been chosen so that the points of M are represented as (x, t), $x\in X$, $-1<t<1$ and that the bundle E is represented as $E_X\times(-1,1)$: $t>0$ in M^+ and $t<0$ in M^- (the complement of $M_1^+=M^+\cup X$) and $t=0$ only on X; and $u\in C^\infty(E)$ is represented as $u(x, t)\in E_X$ with $x\in X$, $-1<t<1$.

Now we define γ_0: $C^\infty(E)\to C^\infty(E_X)$ by

$$(\gamma_0 u)(x)=\lim_{t\to 0+} u(x, t), \ u\in C^\infty(E)$$

and γ_j: $C^\infty(E)\to C^\infty(E_X)$ $(j=1, 2, \cdots)$ by

$$(\gamma_j u)(x)=\lim_{t\to 0+} D_t^j u(x, t), \ D_t=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}.$$

Then we define the restriction map γ: $C^\infty(E)\to\bigoplus_{j=0}^{\omega-1}C^\infty(E_X)$ by

$$\gamma u=(\gamma_0 u, \cdots, \gamma_{\omega-1}u), \ u\in C^\infty(E)$$

(Cauchy data of order $<\omega$ on X). It is well known that γ extends to a continuous linear map $\gamma: H^\sigma(E)\to B^\sigma(X)=\bigoplus_{j=0}^{\omega-1}H^{\sigma-j-1/2}(E_X)$ for $\sigma>\omega-1/2$ and also to a continuous linear map $\gamma: H^\sigma(H_{M^+})\to B^\sigma(X)$ for $\sigma>\omega-1/2$ (cf. Lions and Magenes [8], 47; Palais [9], p. 171).

We define $H^{-\infty}(E)=\bigcup_{s\in R}H^s(E)$ (resp. $H^{-\infty}(E_X)=\bigcup_{s\in R}H^s(E_X)$) and topologize $H^{-\infty}(E)$ (resp. $H^{-\infty}(E_X)$) as the inductive limit. Then $H^{-\infty}(E)$ is the antidual of $C^\infty(E)=\bigcap_{s\in R}H^s(E)$ (cf. Palais [9], p. 126) and the duality will be denoted by

$$\{\{u, v\}\}; \ u\in C^\infty(E), v\in H^{-\infty}(E).$$

Also $H^{-\infty}(E_X)$ is the antidual of $C^\infty(E_X)=\bigcap_{s\in R}H^s(E_X)$ and the duality will be denoted by

$$\{g, h\}; \ g\in C^\infty(E_X), h\in H^{-\infty}(E_X).$$

We define the adjoint $\gamma^*\colon \bigoplus_{j=0}^{\omega-1} H^{-\infty}(E_X)\to H^{-\infty}(E)$ by

$$\{\{u, \gamma^* h\}\}=\sum_{j=0}^{\omega-1} \{\gamma_j u, h_j\},\ u\in C^\infty(E)$$

for $h=(h_0, \cdots, h_{\omega-1})\in\bigoplus_{j=0}^{\omega-1} H^{-\infty}(E_X)$. In particular, since γ is continuous from $H^\sigma(E)$ into $B^\sigma(X)$ for $\sigma>\omega-1/2$, γ^* is continuous from $(B^\sigma(X))^*$ into $H^{-\sigma}(E)$ and satisfies

$$((u, \gamma^* h))_{(\sigma,M)\,(-\sigma,M)}=(\gamma u, h)_{(\sigma,X)\,(\sigma^*,X)};\ u\in H^\sigma(E),\ h\in(B^\sigma(X))^*. \tag{2}$$

1.4. Extension maps E_k. Let k be a positive integer. For real τ with $|\tau|\leqq k$, we define the extension map $E_k\colon H^\tau(E_{M^+})\to H^\tau(E)$ by

$$(E_k u)(x, t)=\begin{cases} u(x, t) & \text{if}\quad t\geqq 0,\\ \varphi_k(t)\sum_{j=1}^{2k} a_j u(x, -jt) & \text{if}\quad t<0,\end{cases} \tag{3}$$

where $\sum_{j=1}^{2k}(-1)^l a_j=1$ for $-k\leqq l\leqq k-1$, and φ_k is C^∞, $\varphi_k(t)=1$ for $t\geqq 0$, $\varphi_k(t)=0$ for $t\leqq -1/2k$.

Since $E_k\colon H^\tau(E_{M^+})\to H^\tau(E)$ is continuous (cf. Lions and Magenes [8], p. 83), there exists the adjoint $E_k^*\colon H^{-\tau}(E)\to H_0^{-\tau}(E_{M^+})$ such that

$$((u, E_k^* v))_{(\tau,M+)\,(-\tau,M+)}=((E_k u, v))_{(\tau,M)\,(-\tau,M)};\ u\in H^\tau(E_{M^+}),\ v\in H^{-\tau}(E). \tag{4}$$

Further, assuming that near the boundary X the volume element on M is the product of the induced volume element on X and Lebesgue measure on $(-1, 1)$, we can easily prove from the definition (3) of E_k that if v is C^∞ up to X in M^+ and also in M^-, then

$$(E_k^* v)(x, t)=v(x, t)+\sum_{j=1}^{2k}\frac{a_j}{j}\, v\left(x, -\frac{t}{j}\right)\varphi_k\left(-\frac{t}{j}\right),\ t>0, \tag{5}$$

so $E_k^* v$ is C^∞ up to X in M^+.

1.5. Summary of known results. From now on, let $A\colon C^\infty(E)\to C^\infty(E)$ be an elliptic differential operator of order ω and let K denote a generic positive constant.

Now we summarize some of the results in Seeley [10], [11] (cf. Hörmander [5], pp. 187–194).

We define for each real s

$$N(A, s)=\{u\in H^s(E_{M^+})\colon Au=0\ \text{in}\ M^+\}.$$

Then we have

THEOREM 1.1. (Seeley [11], p. 803).

(i) *For any $u\in N(A, s)$, γu exists in $B^s(X)$.*

(ii) *$\gamma\colon N(A, s)\to B^s(X)$ is continuous:*

$$|\gamma u|_s\leqq K\|u\|_s,\ u\in N(A, s). \tag{6}$$

By Theorem 1.1, we can define for each real s

(7) $$R_0(A, s)=\{g\in B^s(X)\colon g=\gamma u \text{ for some } u\in N(A, s)\}.$$

Further, we define

$$N_0(A)=\{u\in C^\infty(E)\colon \mathrm{supp}(u)\subset M_1^+,\ Au=0 \text{ in } M^+\}.$$

Then we have for the surface potentials

THEOREM 1.2. (Seeley [10], pp. 274–275).

(i) $\dim N_0(A)<\infty$.

(ii) *There is a linear map* $P\colon \bigoplus_{j=0}^{\omega-1}C^\infty(E_X)\to C^\infty(E_{M_1^+})$ *which, for each real s, extends to a continuous linear map* $P\colon B^s(X)\to N(A, s)$:

(8) $$||Pg||_s\leqq K|g|_s,\ g\in B^s(X).$$

(iii) $u\in N(A, s)$ *can be decomposed into* $u=u_0+u_1$ *(direct sum), where* $u_0\in N_0(A)$ *and* $u_1=P\gamma u_1$.

(iv) *For any* $g\in B^s(X)$, $\gamma(Pg)$ *exists in* $B^s(X)$; *call it* P^+g.

(v) P^+ *is a pseudo-differential operator from* $B^s(X)$ *into* $B^s(X)$:

(9) $$|P^+g|_s\leqq K|g|_s,\ g\in B^s(X).$$

(vi) P^+ *is a projection onto* $R_0(A, s)$: $P^+(P^+g)=P^+g\in R_0(A, s)$, $g\in B^s(X)$.

REMARK 1.3. Since the formal adjoint $A^*\colon C^\infty(E)\to C^\infty(E)$ is also an elliptic differential operator of order ω (see 1.2), Theorem 1.2 remains valid for A^*. In particular, $\dim N_0(A^*)<\infty$ and $N_0(A^*)\subset H_0^{-\tau}(E_{M^+})$ (the dual of $H^\tau(E_{M^+})$) for all real τ (see 1.1).

For the volume potentials, we have

THEOREM 1.4. (Seeley [10], pp. 276–277). *There is a pseudo-differential operator C of order* $-\omega$ *with the following properties*:

(i) *The map* $g\to C^*\gamma^*g$ *of* $\bigoplus_{j=0}^{\omega-1}C^\infty(E_X)$ *into* $\bigcap_{\varepsilon>0}H^{1/2-\varepsilon}(E)$ *extends to a continuous linear map* $C^*\gamma^*\colon (B^\sigma(X))^*=\bigoplus_{j=0}^{\omega-1}H^{-\sigma+j+1/2}(E_X)\to H^{\omega-\sigma}(E)$ *for* $\sigma>\omega-1/2$, *where* C^* *is the formal adjoint of C.*

(ii) *The map* $g\to C^*\gamma^*g|_{M^+}$ *(the restriction of* $C^*\gamma^*g$ *to* M^+*) of* $\bigoplus_{j=0}^{\omega-1}C^\infty(E_X)$ *into* $C^\infty(E_{M_1^+})$ *extends to a continuous linear map* $C^*\gamma^*|_{M^+}\colon (B^s(X))^*\to H^{\omega-s}(E_{M^+})$ *for each real s, and the same holds with* M^+ *replaced by* M^-.

(iii) *If f is in* $H^\tau(E_{M^+})$ *and orthogonal to* $N_0(A^*)$, *i.e.*,

$$\underset{(\tau,M^+)(-\tau,M^+)}{((f,\ v))}=0$$

for all $v\in N_0(A^*)$ *(see Remark* 1.3*), then* $ACE_kf=f$ *in* M^+, *where k is some positive integer such that* $|\tau|\leqq k$.

REMARK 1.5. P of Theorem 1.2 and $C^*\gamma^*|_{M^+}$ of Theorem 1.4 are pseudo-Poisson kernels in the sense of Boutet de Monvel (see [3], p. 278).

§ 2. Statement of main theorem.

2.1. Spaces $H_A^{\sigma,\tau}$. For real σ, τ, we define the following space: $H_A^{\sigma,\tau}=\{u\in H^\sigma(E_{M^+}): Au\in H^\tau(E_{M^+})\}$; the norm of $u\in H_A^{\sigma,\tau}$ is defined by

$$||u||_{\sigma,\tau}=(||u||_\sigma^2+||Au||_\tau^2)^{1/2}.$$

The basic properties of the spaces $H_A^{\sigma,\tau}$ are presented in the following proposition.

PROPOSITION 2.1. *Let $\sigma\leqq\tau+\omega$, $\tau>-1/2$ and let k be the minimum positive integer such that $|\tau|\leqq k$. Then*

(i) *$u\in H_A^{\sigma,\tau}$ can be decomposed into $u=z+w$, where $z=CE_kAu|_{M^+}\in H^{\tau+\omega}(E_{M^+})$ and $w=u-z\in N(A,\sigma)$.*

(ii) *The decomposition is continuous: More precisely,*

$$||z||_{\tau+\omega}\leqq K||Au||_\tau; \tag{10}$$

$$||w||_\sigma\leqq K||u||_{\sigma,\tau}. \tag{11}$$

(iii) *The restriction map γ: $H^{\tau+\omega}(E_{M^+})\to B^{\tau+\omega}(X)$ extends to a linear map γ: $H_A^{\sigma,\tau}\to B^\sigma(X)$.*

(iv) *γ: $H_A^{\sigma,\tau}\to B^\sigma(X)$ is continuous:*

$$|\gamma u|_\sigma\leqq K||u||_{\sigma,\tau},\ u\in H_A^{\sigma,\tau}.$$

Proposition 2.1 will be proved in the appendix.

2.2. Formulation of non-homogeneous boundary value problems. Now suppose that we are given a linear map $B=B_0\oplus\cdots\oplus B_{\omega-1}$: $B^\sigma(X)\to H^{\sigma-\omega+1/2+\lambda}(G)$, where λ is some real constant. Let $\sigma\leqq\tau+\omega$, $\tau>-1/2$. Then by Proposition 2.1 (iii) the boundary condition $B\gamma$ can be defined for elements in $H_A^{\sigma,\tau}$. So our non-homogeneous boundary value problem for A is formulated as follows: For given $f\in H^\tau(E_{M^+})$ and $\varphi\in H^\rho(G)$ ($\rho=\sigma-\omega+1/2+\lambda$) with $\sigma\leqq\tau+\omega$, $\tau>-1/2$, find $u\in H_A^{\sigma,\tau}$ such that

$$\begin{cases}Au=f & \text{in } M^+,\\ B\gamma u=\varphi & \text{on } X.\end{cases} \tag{*}$$

With (*), we can associate the following operators, spaces and integers:

$(A\oplus B\gamma;\sigma,\tau)=$ the operator $A\oplus B\gamma$ from $H_A^{\sigma,\tau}$ into $H^\tau(E_{M^+})\oplus H^\rho(G)$;
$N(A\oplus B\gamma;\sigma,\tau)=\{u\in H_A^{\sigma,\tau}: Au=0, B\gamma u=0\}$;
$R(A\oplus B\gamma;\sigma,\tau)=\{(Au, B\gamma u): u\in H_A^{\sigma,\tau}\}$;
$\text{index}\,(A\oplus B\gamma;\sigma,\tau)=\dim N(A\oplus B\gamma;\sigma,\tau)-\text{codim}\,R(A\oplus B\gamma;\sigma,\tau)$;
$(B^+,\sigma)=$ the restriction of B to $R_0(A,\sigma)$;
$N(B^+,\sigma)=\{g\in R_0(A,\sigma): Bg=0\}$;

(12) $R(B^+, \sigma)=\{Bg: g\in R_0(A, \sigma)\}$;

$\text{index}\,(B^+, \sigma)=\dim N(B^+, \sigma)-\text{codim}\, R(B^+, \sigma)$.

2.3. Reduction to the boundary. Our main theorem will be:

THEOREM 2.2. *Let $t<\sigma\leqq\tau+\omega$, $\tau>-1/2$, let λ be some real constant and let B be a continuous linear map of $B^s(X)$ into $H^{s-\omega+1/2+\lambda}(G)$ for each real s. Then the operator $(A\oplus B\gamma; \sigma, \tau)$: $H_A^{\sigma,\tau}\rightarrow H^\tau(E_{M^+})\oplus H^\rho(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ is continuous, and in addition*

I. *Estimates. The estimate*

$$||u||_\sigma\leqq K(||Au||_\tau+\langle B\gamma u\rangle_\rho+||u||_t)$$

is valid for all $u\in H_A^{\sigma,\tau}$ if and only if the estimate

$$|g|_\sigma\leqq K(\langle Bg\rangle_\rho+|g|_t)$$

is valid for all $g\in R_0(A, \sigma)$.

II. *Solvability.*

(1a) $\dim N(A\oplus B\gamma; \sigma, \tau)<\infty$ *if and only if* $\dim N(B^+, \sigma)<\infty$.

(2a) $\dim N(A\oplus B\gamma; \sigma, \tau)=\dim N_0(A)+\dim N(B^+, \sigma)$.

(1b) *$R(A\oplus B\gamma; \sigma, \tau)$ is closed in $H^\tau(E_{M^+})\oplus H^\rho(G)$ if and only if $R(B^+, \sigma)$ is closed in $H^\rho(G)$.*

(2b) $\text{codim}\, R(A\oplus B\gamma; \sigma, \tau)<\infty$ *if and only if* $\text{codim}\, R(B^+, \sigma)<\infty$.

(3b) $\text{codim}\, R(A\oplus B\gamma; \sigma, \tau)=\dim N_0(A^*)+\text{codim}\, R(B^+, \sigma)$.

(1c) $\text{index}\,(A\oplus B\gamma; \sigma, \tau)<\infty$ *if and only if* $\text{index}\,(B^+, \sigma)<\infty$.

(2c) $\text{index}\,(A\oplus B\gamma; \sigma, \tau)=\dim N_0(A)-\dim N_0(A^*)+\text{index}\,(B^+, \sigma)$.

(3c) $\text{index}(A\oplus B_1\gamma; \sigma, \tau)-\text{index}(A\oplus B_2\gamma; \sigma, \tau)=\text{index}(B_1{}^+, \sigma)-\text{index}(B_2{}^+, \sigma)$, *where B_k $(k=1, 2)$ is a linear map having the same property as B.*

III. *Regularity.*

(a) *For every $u\in H^t(E_{M^+})$ such that $Au\in H^\tau(E_{M^+})$ and $B\gamma u\in H^\rho(G)$ we have $u\in H^\sigma(E_{M^+})$ if and only if for every $g\in R_0(A, t)$ such that $B\,g\in H^\rho(G)$, we have $g\in B^\sigma(X)$.*

(b) *$N(A\oplus B\gamma; \sigma, \tau)$ consists of C^∞ sections of $E_{M_1^+}$ if and only if $N(B^+, \sigma)$ consists of C^∞ sections of E_X.*

(c) *$R(A\oplus B\gamma; \sigma, \tau)$ is the orthogonal complement of finitely many elements in $C^\infty(E_{M_1^+})\oplus C^\infty(G)$ if and only if $R(B^+, \sigma)$ is the orthogonal complement of finitely many elements in $C^\infty(G)$.*

§ 3. Estimates.

THEOREM 3.1. *Let $t<\sigma$ and let B: $B^\sigma(X)\rightarrow H^\rho(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ be a continuous linear map. Then the following three statements are equivalent:*

(i) *The estimate*

$$|g|_\sigma\leqq K(\langle Bg\rangle_\rho+|g|_t \tag{13}$$

is valid for all $g\in R_0(A, \sigma)$.

(i)′ *The estimate*

$$|g|_\sigma \leqq K(|(1-P^+)g|_\sigma + \langle Bg\rangle_\rho + |g|_t)$$

is valid for all $g \in B^\sigma(X)$.

(ii) *The estimate*

$$\|u\|_\sigma \leqq K(\langle B\gamma u\rangle_\rho + \|u\|_t) \tag{14}$$

is valid for all $u \in N(A, \sigma)$.

Furthermore, if $\sigma \leqq \tau+\omega, \tau > -1/2$, *then the above three statements are equivalent to*

(ii)′ *The estimate*

$$\|u\|_\sigma \leqq K(\|Au\|_\tau + \langle B\gamma u\rangle_\rho + \|u\|_t) \tag{15}$$

is valid for all $u \in H_A^{\sigma,\tau}$.

REMARK 3.2. The estimate (15) with $\sigma=\omega+\tau, \lambda=0$ is elliptic (see Seeley [10], p. 286) and the one with $\sigma=\omega+\tau-1/2, \lambda=1/2$ is sub-elliptic (cf. Hörmander [5], p. 208). Further, the estimate (15) with $\sigma=\omega-1/2$, $\lambda=0$, $\tau=0$, $t=0$ is important for non-local boundary value problems (see Beals [2], p. 329).

Proof of theorem 3.1. That (i)′⇒(i) and that (ii)′⇒(ii) are obvious.

(i)⇒(i)′: Let $g \in B^\sigma(X)$. Since by Theorem 1.2 (vi) $P^+g \in R_0(A, \sigma)$, we can apply (13) to P^+g and obtain

$$|P^+g|_\sigma \leqq K(\langle BP^+g\rangle_\rho + |P^+g|_t).$$

Hence, using (9) of Theorem 1.2 for $g \in B^\sigma(X) \subset B^t(X)$ $(t<\sigma)$, we get

$$|P^+g|_\sigma \leqq K(\langle BP^+g\rangle_\rho + |g|_t). \tag{16}$$

On the other hand, it follows from the continuity of B: $B^\sigma(X) \to H^\rho(G)$ that

$$\begin{aligned}\langle BP^+g\rangle_\rho &\leqq \langle Bg\rangle_\rho + \langle B(1-P^+)g\rangle_\rho \\ &\leqq \langle Bg\rangle_\rho + K|(1-P^+)g|_\sigma.\end{aligned}$$

Thus, carrying this into (16), we obtain

$$|P^+g|_\sigma \leqq K(|(1-P^+)g|_\sigma + \langle Bg\rangle_\rho + |g|_t).$$

Hence

$$\begin{aligned}|g|_\sigma &\leqq |P^+g|_\sigma + |(1-P^+)g|_\sigma \\ &\leqq K(|(1-P^+)g|_\sigma + \langle Bg\rangle_\rho + |g|_t).\end{aligned}$$

(i)⇒(ii): Let $u \in N(A, \sigma)$. Then by Theorem 1.2 (iii) u can be decomposed into $u=u_0+u_1$, where $u_0 \in N_0(A)$ and $u_1=P\gamma u_1$. Since by Theorem 1.2 (i) $N_0(A) (\subset C^\infty(E))$ is finite dimensional, it follows that the projection $u \to u_0$ is a pseudo-differential operator of order $-\infty$. Hence we have for $t<\sigma$

(17) $$||u_0||_\sigma \leqq K||u||_t.$$

Further, applying (8) of Theorem 1.2 to γu_1, we obtain

(18) $$\begin{aligned}||u_1||_\sigma = ||P\gamma u_1||_\sigma &\leqq K|\gamma u_1|_\sigma \\ &= K|\gamma u|_\sigma,\end{aligned}$$

since $\gamma u_0 = 0$. Thus we obtain from (17) and (18)

$$\begin{aligned}||u||_\sigma &\leqq ||u_0||_\sigma + ||u_1||_\sigma \\ &\leqq K(||u||_t + |\gamma u|_\sigma).\end{aligned}$$

Therefore, applying (13) to $\gamma u \in R_0(A, \sigma)$ and (6) of Theorem 1.1 to $u \in N(A, \sigma) \subset N(A, t)$, we finally obtain

$$\begin{aligned}||u||_\sigma &\leqq K(||u||_t + |\gamma u|_\sigma) \\ &\leqq K(||u||_t + \langle B\gamma u\rangle_\rho + |\gamma u|_t) \\ &\leqq K(\langle B\gamma u\rangle_\rho + ||u||_t).\end{aligned}$$

(ii)⇒(i): Let $g \in R_0(A, \sigma)$. Then by the definition (7) of $R_0(A, \sigma)$ there exists $u \in N(A, \sigma)$ such that $\gamma u = g$. Further, by Theorem 1.2 (iii) u can be decomposed into $u = u_0 + u_1$, where $u_0 \in N_0(A)$ and $u_1 = P\gamma u_1$. Thus, since $\gamma u_0 = 0$, we have $\gamma u_1 = \gamma u = g$ and $u_1 = P\gamma u_1 = Pg$. Hence, applying (6) of Theorem 1.1 and (14) to $u_1 \in N(A, \sigma)$ and (8) of Theorem 1.2 to $g \in R_0(A, \sigma) \subset B^t(X)(t < \sigma)$, we obtain

$$\begin{aligned}|g|_\sigma = |\gamma u_1|_\sigma &\leqq K||u_1||_\sigma \\ &\leqq K(\langle B\gamma u_1\rangle_\rho + ||u_1||_t) \\ &= K(\langle Bg\rangle_\rho + ||Pg||_t) \\ &\leqq K(\langle Bg\rangle_\rho + |g|_t).\end{aligned}$$

(ii)⇒(ii)′: Let $u \in H_A^{\sigma,\tau}$. Then, by Proposition 2.1 (i) u can be decomposed into $u = z + w$, where $z \in H^{\tau+\omega}(E_{M^+})$ and $w \in N(A, \sigma)$. Further, since $t < \sigma \leqq \tau + \omega$, we then have (see (10))

$$\begin{aligned}||z||_t &\leqq ||z||_\sigma \\ &\leqq ||z||_{\tau+\omega} \\ &\leqq K||Au||_\tau.\end{aligned}$$

Hence it follows that

(19) $$\begin{aligned}||u||_\sigma &\leqq ||z||_\sigma + ||w||_\sigma \\ &\leqq K||Au||_\tau + ||w||_\sigma,\end{aligned}$$

that

$$\begin{aligned}\|w\|_t &\leqq \|u\|_t + \|z\|_t \\ &\leqq \|u\|_t + K\|Au\|_\tau \end{aligned} \tag{20}$$

and that

$$\begin{aligned}\langle B\gamma z\rangle_\rho &\leqq K|\gamma z|_\sigma \\ &\leqq K|\gamma z|_{\tau+\omega} \\ &\leqq K\|z\|_{\tau+\omega} \\ &\leqq K\|Au\|_\tau, \end{aligned} \tag{21}$$

since $B: B^\sigma(X)\to H^\rho(G)$ is continuous, and γ is continuous from $H^{\tau+\omega}(E_{M^+})$ into $B^{\tau+\omega}(X)$ for $\tau+\omega>\omega-1/2$ (see 1.3). Then, applying (14) to $w\in N(A,\sigma)$, we obtain from (20) and (21)

$$\begin{aligned}\|w\|_\sigma &\leqq K(\langle B\gamma w\rangle_\rho + \|w\|_t) \\ &\leqq K(\langle B\gamma u\rangle_\rho + \langle B\gamma z\rangle_\rho + \|Au\|_\tau + \|u\|_t) \\ &\leqq K(\langle B\gamma u\rangle_\rho + \|Au\|_\tau + \|u\|_t). \end{aligned}$$

Thus, carrying this into (19), we finally get

$$\|u\|_\sigma \leqq K(\|Au\|_\tau + \langle B\gamma u\rangle_\rho + \|u\|_t).$$

The proof is complete.

§ 4. Solvability.

4.1. Kernels.

THEOREM 4.1. *Let σ, τ be real and let $B: B^\sigma(X)\to H^\rho(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ be a linear map. Then the following three statements are equivalent*:

(i) $\dim N(B^+,\sigma)<\infty$.

(i)′ $\dim N(BP^+,\sigma)/N(P^+,\sigma)<\infty$.

(ii) $\dim N(A\oplus B\gamma;\sigma,\tau)<\infty$.

Here $N(BP^+,\sigma)=\{g\in B^\sigma(X): BP^+g=0\}$ and $N(P^+,\sigma)=\{g\in B^\sigma(X): P^+g=0\}$. Furthermore, we have

$$\dim N(B^+,\sigma)=\dim N(BP^+,\sigma)/N(P^+,\sigma);$$

$$\dim N(A\oplus B\gamma;\sigma,\tau)=\dim N_0(A)+\dim N(B^+,\sigma).$$

Note that by Theorem 1.1 (i) γu exists in $B^\sigma(X)$ for all $u\in N(A\oplus B\gamma;\sigma,\tau)\subset N(A,\sigma)$.

Proof. (i)$\Longleftrightarrow$(i)′: It follows from the isomorphism: $N(BP^+,\sigma)/N(P^+,\sigma)\to N(B^+,\sigma)$ that (i)$\Longleftrightarrow$(i)′ and that $\dim N(B^+,\sigma)=\dim N(BP^+,\sigma)/N(P^+,\sigma)$.

(i)$\Longleftrightarrow$(ii): By Theorem 1.2 (iii), $u \in N(A, \sigma)$ can be decomposed into $u=u_0+u_1$ (direct sum), where $u_0 \in N_0(A)$ and $u_1=P\gamma u_1$. Hence we have $N(A\oplus B\gamma; \sigma, \tau)=N_0(A)+\{Pg: g\in B^\sigma(X), P^+g=g, Bg=0\}$ (direct sum). Thus, since by Theorem 1.2 (vi) γ maps $\{Pg: g\in B^\sigma(X), P^+g=g, Bg=0\}$ isomorphically onto $N(B^+, \sigma)=\{g\in R_0(A, \sigma): Bg=0\}$, we find that $\dim N(A\oplus B\gamma; \sigma, \tau)=\dim N_0(A)+\dim N(B^+, \sigma)$. Hence, since by Theorem 1.2 (i) $\dim N_0(A)<\infty$, it follows that (i) and (ii) are equivalent. The proof is complete.

4.2. Ranges.

LEMMA 4.2. *Let σ be real and let B: $B^\sigma(X)\to H^\rho(G)(\rho=\sigma-\omega+1/2+\lambda)$ be a continuous linear map. Then the following two statements are equivalent*:

(i) *$R(B^+, \sigma)$ is closed in $H^\rho(G)$.*

(i)′ *$R(B\oplus(1-P^+), \sigma)$ is closed in $H^\rho(G)\oplus B^\sigma(X)$.*

Here $R(B\oplus(1-P^+), \sigma)=\{(Bg, (1-P^+)g): g\in B^\sigma(X)\}$.

Proof. (i)$\Rightarrow$(i)′: Let $\{g_n\}\subset B^\sigma(X)$ such that $Bg_n\to\varphi$ in $H^\rho(G)$ and $(1-P^+)g_n\to g_0$ in $B^\sigma(X)$. Then, in view of the continuity of B: $B^\sigma(X)\to H^\rho(G)$, it follows that $BP^+g_n=Bg_n-B(1-P^+)g_n\to\varphi-Bg_0$ in $H^\rho(G)$. Further, since by Theorem 1.2 (vi) $\{P^+g_n\}\subset R_0(A, \sigma)$, it follows from the definition (12) of $R(B^+, \sigma)$ that $\{BP^+g_n\}\subset R(B^+, \sigma)$. Thus we have by (i) $\varphi-Bg_0\in R(B^+, \sigma)$, which implies that there exists $g_1\in R_0(A, \sigma)$ such that $Bg_1=\varphi-Bg_0$. Hence, setting $g=g_0+g_1\in B^\sigma(X)$, we obtain $Bg=\varphi$ and $(1-P^+)g=(1-P^+)g_0=g_0$, since by Theorem 1.2 (vi) $P^+g_1=g_1$ and $P^+g_0=\lim_{n\to\infty}P^+(1-P^+)g_n=0$. So $(\varphi, g_0)\in R(B\oplus(1-P^+), \sigma)$.

(i)′$\Rightarrow$(i): Let $\{g_n\}\subset R_0(A, \sigma)$ such that $Bg_n\to\varphi$ in $H^\rho(G)$. Then, since by Theorem 1.2 (vi) $(1-P^+)g_n=0$, it follows that $(Bg_n, (1-P^+)g_n)=(Bg_n, 0)\to(\varphi, 0)$ in $H^\rho(G)\oplus B^\sigma(X)$. Thus we have by (i)′ $(\varphi, 0)\in R(B\oplus(1-P^+), \sigma)$, which implies that there exists $g\in B^\sigma(X)$ such that $Bg=\varphi$ and $(1-P^+)g=0$. So $\varphi=Bg\in R(B^+, \sigma)$, since $g=P^+g\in R_0(A, \sigma)$. The proof is complete.

LEMMA 4.3. *Let $\sigma\leqq\tau+\omega$, $\tau>-1/2$ and let B: $B^\sigma(X)\to H^\rho(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ be a continuous linear map. Then the operator $(A\oplus B\gamma; \sigma, \tau)$: $H_A^{\sigma,\tau}\to H^\tau(E_{M^+})\oplus H^\rho(G)$ is continuous and in addition the following two statements are equivalent*:

(i) *$R(B^+, \sigma)$ is closed in $H^\rho(G)$.*

(ii) *$R(A\oplus B\gamma; \sigma, \tau)$ is closed in $H^\tau(E_{M^+})\oplus H^\rho(G)$.*

Proof. The continuity of $(A\oplus B\gamma; \sigma, \tau)$ follows immediately from Proposition 2.1 (iv) and the continuity of B.

(ii)$\Rightarrow$(i): Let $\{g_n\}\subset R_0(A, \sigma)$ such that $Bg_n\to\varphi$ in $H^\rho(G)$. Since by the definition (7) of $R_0(A, \sigma)$ there exists $\{w_n\}\subset N(A, \sigma)$ such that $\gamma w_n=g_n$, it then follows that $(Aw_n, B\gamma w_n)=(0, Bg_n)\to(0, \varphi)$ in $H^\tau(E_{M^+})\oplus H^\rho(G)$. Hence we have by (ii) $(0, \varphi)\in R(A\oplus B\gamma; \sigma, \tau)$, which implies that there exists $w\in H_A^{\sigma,\tau}$ such that $Aw=0$ and $B\gamma w=\varphi$. So $\varphi=B\gamma w\in R(B^+, \sigma)$, since $\gamma w\in R_0(A, \sigma)$.

(i)$\Rightarrow$(ii): Let $\{u_n\}\subset H_A^{\sigma,\tau}$ such that $Au_n\to f$ in $H^\tau(E_{M^+})$ and $B\gamma u_n\to\varphi$ in $H^\rho(G)$. Then by Proposition 2.1 (i) u_n can be decomposed into $u_n=z_n+w_n$, where z_n

$=CE_k Au_n|_{M^+}\in H^{\tau+\omega}(E_{M^+})$ and $w_n=u_n-z_n\in N(A,\sigma)$. Since $Au_n\to f$ in $H^\tau(E_{M^+})$ and since $\{Au_n\}\subset H^\tau(E_{M^+})\subset H^{\sigma-\omega}(E_{M^+})(\sigma\leqq\tau+\omega)$, it follows that for all $v\in N_0(A^*)$

$$\underset{(\tau,M^+)(-\tau,M^+)}{((f,\ v))}=\lim_{n\to\infty}\underset{(\tau,M^+)\ (-\tau,M^+)}{((Au_n,\ v))}$$

$$=\lim_{n\to\infty}\underset{(\sigma-\omega,M^+)\ (-\sigma+\omega,M^+)}{((Au_n,\ v))}$$

$$=\lim_{n\to\infty}\underset{(\sigma,M^+)\ (-\sigma,M^+)}{((u_n,\ A^*v))}$$

$$=0$$

(see Remark 1.3). Hence by Theorem 1.4 (iii) $Az=f$, where $z=CE_kf|_{M^+}\in H^{\tau+\omega}(E_{M^+})$. Further, since E_k: $H^\tau(E_{M^+})\to H^\tau(E)$ and C: $H^\tau(E)\to H^{\tau+\omega}(E)$ are continuous, it follows that $z_n\to z$ in $H^{\tau+\omega}(E_{M^+})$ and hence from the continuity of γ: $H^{\tau+\omega}(E_{M^+})\to B^{\tau+\omega}(X)$ for $\tau+\omega>\omega-1/2$ (see 1.3) that $\gamma z_n\to\gamma z$ in $B^{\tau+\omega}(X)$. So we have $\gamma z_n\to\gamma z$ in $B^\sigma(X)$ and $B\gamma z_n\to B\gamma z$ in $H^\rho(G)$, since B: $B^\sigma(X)\to H^\rho(G)$ is continuous. Hence $B\gamma w_n=B\gamma u_n-B\gamma z_n\to\varphi-B\gamma z$ in $H^\rho(G)$. Then, since $\{\gamma w_n\}\subset R_0(A,\sigma)$ and hence $\{B\gamma w_n\}\subset R(B^+,\sigma)$, we have by (i) $\varphi-B\gamma z\in R(B^+,\sigma)$, which implies that there exists $w\in N(A,\sigma)$ such that $B\gamma w=\varphi-B\gamma z$. Thus, setting $u=z+w\in H^\sigma(E_{M^+})$, we obtain $Au=Az+Aw=Az=f\in H^\tau(E_{M^+})$ and $B\gamma u=B\gamma z+B\gamma w=\varphi$. So $(f,\varphi)\in R(A\oplus B\gamma;\sigma,\tau)$. The proof is complete.

Combining Lemma 4.2 and Lemma 4.3, we have proved

THEOREM 4.4. *Let σ be real and let B: $B^\sigma(X)\to H^\rho(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ be a continuous linear map. Then the following two statements are equivalent*:

(i) *$R(B^+,\sigma)$ is closed in $H^\rho(G)$*;

(ii)′ *$R(B\oplus(1-P^+),\sigma)$ is closed in $H^\rho(G)\oplus B^\sigma(X)$.*

Furthermore, if $\sigma\leqq\tau+\omega$, $\tau>-1/2$, then the operator $(A\oplus B\gamma;\sigma,\tau)$: $H_A^{\sigma,\tau}\to H^\tau(E_{M^+})\oplus H^\rho(G)$ is continuous and the above two statements are equivalent to

(ii) *$R(A\oplus B\gamma;\sigma,\tau)$ is closed in $H^\tau(E_{M^+})\oplus H^\rho(G)$.*

4.3. Cokernels.

THEOREM 4.5. *Let $\sigma\leqq\tau+\omega$, $\tau>-1/2$, let k be the minimum positive integer such that $|\tau|\leqq k$ and let B: $B^\sigma(X)\to H^\rho(G)(\rho=\sigma-\omega+1/2+\lambda)$ be a continuous linear map. Suppose that* $\dim N_0(A^*)=m$ *and that the family $\{v_i\}_{i=1}^m$ is a basis of $N_0(A^*)$ (see Remark* 1.3). *Let $\{\psi_j\}_{j=1}^l\subset H^{-\rho}(G)$ and supppose that the family $\{\psi_j\}_{j=1}^l$ satisfies the following two assumptions* (Ai), (Aii):

(Ai) *$\varphi\in H^\rho(G)$ belongs to $R(B^+,\sigma)$ if and only if φ is orthogonal to $\{\psi_j\}_{j=1}^l$, i.e., if and only if*

$$\underset{(\rho,G)\ (-\rho,G)}{[\varphi,\ \psi_j]}=0,\qquad j=1,\cdots,l.$$

(Aii) *The family $\{\psi_j\}_{j=1}^l$ is linearly independent.*

Then the following two conclusions (Ci), (Cii) *hold*:

(Ci) *$(f,\varphi)\in H^\tau(E_{M^+})\oplus H^\rho(G)$ belongs to $R(A\oplus B\gamma;\sigma,\tau)$ if and only if (f,φ) is orthogonal to $\{(v_i,0)\}_{i=1}^m$ and $\{(\tilde v_j,\psi_j)\}_{j=1}^l$ $(\tilde v_j=-E_k^*C^*\gamma^*B^*\psi_j\in H_0^{-\tau}(E_{M^+}))$, i.e., if and* *only if*

$$(22)\qquad \begin{aligned} &\underset{(\tau,M^+)\ (-\tau,M^+)}{((f,\ v_i))}=0, \qquad i=1,\cdots,m;\\ &\underset{(\tau,M^+)\ (-\tau,M^+)}{((f,\ \tilde v_j))}+\underset{(\rho,G)(-\rho,G)}{[\varphi,\ \psi_j]}=0, \qquad j=1,\cdots,l, \end{aligned}$$

where B^*: $H^{-\rho}(G)\to(B^\sigma(X))^*$ *is the adjoint of* B.

(Cii) *The family* $\{(v_i, 0)\}_{i=1}^m$, $\{(\tilde v_j, \psi_j)\}_{j=1}^l$ *is linearly independent.*

REMARK 4.6. Here we introduce the following spaces:

$$\begin{aligned} N((B^+,\sigma)^*)&=\{\psi\in H^{-\rho}(G): \underset{(\rho,G)\quad(-\rho,G)}{[Bg,\ \psi]}=0 \quad\text{for all}\quad g\in R_0(A,\sigma)\}\\ &=\{\psi\in H^{-\rho}(G): \underset{(\rho,G)\quad(-\rho,G)}{[B\gamma u,\ \psi]}=0 \quad\text{for all}\quad u\in N(A,\sigma)\} \text{ (see (7))};\\ N((A\oplus B\gamma;\sigma,\tau)^*)&=\{(v,\psi)\in H_0^{-\tau}(E_{M^+})\oplus H^{-\rho}(G): \underset{(\tau,M^+)\ (-\tau,M^+)}{((Au,\ v))}+\underset{(\rho,G)\quad(-\rho,G)}{[B\gamma u,\ \psi]}\\ &=0 \quad\text{for all}\quad u\in H_A^{\sigma,\tau}\}. \end{aligned}$$

Then it follows from the closed range theorem that the assumptions (Ai), (Aii) imply that the family $\{\psi_j\}_{j=1}^l$ is a basis of $N((B^+,\sigma)^*)$ and that the conclusions (Ci), (Cii) imply that the family $\{(v_i, 0)\}_{i=1}^m$ $\{(\tilde v_j, \psi_j)\}_{j=1}^l$ is a basis of $N((A\oplus B\gamma;\sigma,\tau)^*)$ (see Palais [9], p. 111).

Proof of theorem 4.5. 1) Let $f\in H^\tau(E_{M^+})$ be orthogonal to $\{v_i\}_{i=1}^m$ and let $\varphi-B\gamma CE_k f$ be orthogonal to $\{\psi_j\}_{j=1}^l$, i.e.,

$$(23)\qquad \begin{aligned} &\underset{(\tau,M^+)(-\tau,M^+)}{((f,\ v_i))}=0, \qquad i=1,\cdots,m;\\ &\underset{(\rho,G)\qquad\qquad(-\rho,G)}{[\varphi-B\gamma CE_k f,\ \psi_j]}=0, \qquad j=1,\cdots,l. \end{aligned}$$

Then it follows from Theorem 1.4 (iii) that $Az=f$, where $z=CE_kf|_{M^+}\in H^{\tau+\omega}(E_{M^+})$. Further, since $\varphi-B\gamma z=\varphi-B\gamma CE_kf$ is orthogonal to $\{\psi_j\}_{j=1}^l$, it follows from (Ai) that $\varphi-B\gamma z\in R(B^+,\sigma)$. Hence by the definition (12) of $R(B^+,\sigma)$ there exists $g\in R_0(A,\sigma)$ such that $Bg=\varphi-B\gamma z$. Thus, since by Theorem 1.2 (ii), (vi) $Pg\in N(A,\sigma)$ and $P^+g=g$, setting $u=z+Pg\in H^\sigma(E_{M^+})$, we obtain $Au=Az=f\in H^\tau(E_{M^+})$ and $B\gamma u=B\gamma z+BP^+g$ $=B\gamma z+Bg=\varphi$, which proves that $(f,\varphi)\in R(A\oplus B\gamma;\sigma,\tau)$.

Conversely, let $(f,\varphi)\in R(A\oplus B\gamma;\sigma,\tau)$. Then there exists $u\in H_A^{\sigma,\tau}$ such that $Au=f$ and $B\gamma u=\varphi$. Further, by Proposition 2.1 (i) u can be decomposed into $u=z+w$, where $z=CE_kf|_{M^+}\in H^{\tau+\omega}(E_{M^+})$ and $w=u-z\in N(A,\sigma)$. Thus, since by the definition (7) of $R_0(A,\sigma)$ $\gamma w\in R_0(A,\sigma)$ and hence $B\gamma w\in R(B^+,\sigma)$, it follows from (Ai) that for each $\psi_j\in H^{-\rho}(G)$ $(1\leqq j\leqq l)$

$$\begin{aligned} \underset{(\rho,G)\qquad\qquad(-\rho,G)}{[\varphi-B\gamma CE_kf,\ \psi_j]}&=\underset{(\rho,G)\qquad(-\rho,G)}{[B\gamma u-B\gamma z,\ \psi_j]}\\ &=\underset{(\rho,G)\quad(-\rho,G)}{[B\gamma w,\ \psi_j]}\\ &=0. \end{aligned}$$

Moreover, since $Au\in H^{\tau}(E_{M^+})\subset H^{\sigma-\omega}(E_{M^+})(\sigma\leqq\tau+\omega)$, it follows that for each $v_i\in N_0(A^*)$ $(1\leqq i\leqq m)$

$$\begin{aligned}((f, v_i))_{(\tau,M+)\,(-\tau,M+)} &= ((Au, v_i))_{(\tau,M+)\,(-\tau,M+)}\\ &= ((Au, v_i))_{(\sigma-\omega,M+)\,(-\sigma+\omega,M+)}\\ &= ((u, A^*v_i))_{(\sigma,M+)\,(-\sigma,M+)}\\ &=0.\end{aligned}$$

Therefore we have proved that $(f, \varphi)\in H^{\tau}(E_{M^+})\oplus H^{\rho}(G)$ belongs to $R(A\oplus B\gamma; \sigma, \tau)$ if and only if (23) holds.

2) Since $\gamma CE_k f\in B^{\tau+\omega}(X)\subset B^{\sigma}(X)(\sigma\leqq\tau+\omega)$ and $B^*\psi_j\in(B^{\sigma}(X))^*\subset(B^{\tau+\omega}(X))^*$, it follows that

$$[B\gamma CE_k f, \psi_j]_{(\rho,G)\,(-\rho,G)} = (\gamma CE_k f, B^*\psi_j)_{(\sigma,X)\,(\sigma^*,X)} \tag{24}$$

$$= (\gamma CE_k f, B^*\psi_j)_{(\tau+\omega,X)\,((\tau+\omega)^*,X)}$$

(cf. Palais [9], p. 126). Further, since $\tau+\omega>\omega-1/2$ and $|\tau|\leqq k$, applying (2) to $CE_k f\in H^{\tau+\omega}(E)$ and $B^*\psi_j\in(B^{\tau+\omega}(X))^*$ and (4) to $f\in H^{\tau}(E_{M^+})$ and $C^*\gamma^*B^*\psi_j\in H^{-\tau}(E)$, we obtain

$$\begin{aligned}(\gamma CE_k f, B^*\psi_j)_{(\tau+\omega,X)\,((\tau+\omega)^*,X)} &= ((CE_k f, \gamma^*B^*\psi_j))_{(\tau+\omega,M)\,(-\tau-\omega,M)}\\ &= ((E_k f, C^*\gamma^*B^*\psi_j))_{(\tau,M)\,(-\tau,M)}\\ &= ((f, E_k^*C^*\gamma^*B^*\psi_j))_{(\tau,M+)\,(-\tau,M+)}\end{aligned}$$

Hence, combining this with (24) and setting $\tilde{v}_j=-E_k^*C^*\gamma^*B^*\psi_j\in H_0^{-\tau}(E_{M^+})$ $(1\leqq j\leqq l)$, we get

$$\begin{aligned}[B\gamma CE_k f, \psi_j]_{(\rho,G)\,(-\rho,G)} &= ((f, E_k^*C^*\gamma^*B^*\psi_j))_{(\tau,M+)\,(-\tau,M+)}\\ &= -((f, \tilde{v}_j))_{(\tau,M+)(-\tau,M+)}.\end{aligned}$$

Thus we have

$$[\varphi-B\gamma CE_k f, \psi_j]_{(\rho,G)\,(-\rho,G)} = [\varphi, \psi_j]_{(\rho,G)\,(-\rho,G)} + ((f, \tilde{v}_j))_{(\tau,M+)\,(-\tau,M+)}.$$

3) Now we conclude from parts 1), 2) that $(f, \varphi)\in H^{\tau}(E_{M^+})\oplus H^{\rho}(G)$ belongs to $R(A\oplus B\gamma; \sigma, \tau)$ if and only if (22) holds. Moreover, since the family $\{v_i\}_{i=1}^m$ is a basis of $N_0(A^*)$, we conclude from (Aii) that the family $\{(v_i, 0)\}_{i=1}^m$, $\{(\tilde{v}_j, \psi_j)\}_{j=1}^l$ is linearly independent. The proof is complete.

COROLLARY 4.7. *Let* $\sigma\leqq\tau+\omega$, $\tau>-1/2$ *and let* B: $B^{\sigma}(X)\rightarrow H^{\rho}(G)(\rho=\sigma-\omega+1/2+\lambda)$ *be a continuous linear map. If* codim $R(B^{+},\sigma)<\infty$, *then* codim $R(A\oplus B\gamma;\sigma,\tau)<\infty$. *Furthermore, we have*

$$\operatorname{codim} R(A\oplus B\gamma;\sigma,\tau)=\dim N_{0}(A^{*})+\operatorname{codim} R(B^{+},\sigma).$$

Proof. Suppose that codim $R(B^{+},\sigma)=l$. Then it follows that $R(B^{+},\sigma)$ is closed in $H^{\rho}(G)$ and hence from the closed range theorem that $\dim N((B^{+},\sigma)^{*})=\operatorname{codim} R(B^{+},\sigma)=l$ (see Palais [9], p. 119). So let $\{\psi_{j}\}_{j=1}^{l}$ be a basis of $N((B^{+},\sigma)^{*})$. Then by Theorem 4.5 the family $\{(v_{i},0)\}_{i=1}^{m}$, $\{(\tilde{v}_{j},\psi_{j})\}_{j=1}^{l}$ is a basis of $N((A\oplus B\gamma;\sigma,\tau)^{*})$ (see Remark 4.6). Hence we have

$$\begin{aligned}\operatorname{codim} R(A\oplus B\gamma;\sigma,\tau)&=\dim N((A\oplus B\gamma;\sigma,\tau)^{*})\\&=m+l\\&=\dim N_{0}(A^{*})+\operatorname{codim} R(B^{+},\sigma).\end{aligned}$$

This completes the proof.

Conversely, we have

THEOREM 4.8. *Let* $\sigma\leqq\tau+\omega$, $\tau>-1/2$ *and let* B: $B^{\sigma}(X)\rightarrow H^{\rho}(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ *be a continuous linear map. If* codim $R(A\oplus B\gamma;\sigma,\tau)<\infty$, *then* codim $R(B^{+},\sigma)<\infty$. *Furthermore, we have*

$$\operatorname{codim} R(B^{+},\sigma)=\operatorname{codim} R(A\oplus B\gamma;\sigma,\tau)-\dim N_{0}(A^{*}).$$

Proof. Suppose that codim $R(A\oplus B\gamma;\sigma,\tau)=q$. Then, as in the proof of Corollary 4.7, it follows that $R(A\oplus B\gamma;\sigma,\tau)$ is closed in $H^{\tau}(E_{M^{+}})\oplus H^{\rho}(G)$ and that $\dim N((A\oplus B\gamma;\sigma,\tau)^{*})=\operatorname{codim} R(A\oplus B\gamma;\sigma,\tau)=q$. So let $\{(v_{j},\psi_{j})\}_{j=1}^{q}$ be a basis of $N((A\oplus B\gamma;\sigma,\tau)^{*})$. Then, since by the definitions (7), (12) of $R_{0}(A,\sigma)$ and $R(B^{+},\sigma)$ $\varphi\in H^{\rho}(G)$ belongs to $R(B^{+},\sigma)$ if and only if $(0,\varphi)\in R(A\oplus B\gamma;\sigma,\tau)$, it follows that $\varphi\in H^{\rho}(G)$ belongs to $R(B^{+},\sigma)$ if and only if

$$[\underset{(\rho,G)}{\varphi},\underset{(-\rho,G)}{\psi_{j}}]=0,\qquad j=1,\cdots,q,$$

which implies that the family $\{\varphi_{j}\}_{j=1}^{q}$ generates $N((B^{+},\sigma)^{*})$. Hence codim $R(B^{+},\sigma)=\dim N((B^{+},\sigma)^{*})\leqq q<\infty$. Now we define the linear map R: $N((A\oplus B\gamma;\tau)^{*})\rightarrow N((B^{+},\sigma)^{*})$ by

$$R(v,\psi)=\psi\quad\text{for}\quad(v,\psi)\in N((A\oplus B\gamma;\sigma,\tau)^{*})$$

(see Remark 4.6). Since the family $\{(v_{j},\psi_{j})\}_{j=1}^{q}$ is a basis of $N((A\oplus B\gamma;\sigma,\tau)^{*})$ and since the family $\{\psi_{j}\}_{j=1}^{q}$ generates $N((B^{+},\sigma)^{*})$, it then follows that R is onto. On the other hand, we can easily prove from Theorem 1.4 (iii) that for $\sigma\leqq\tau+\omega$

$$\begin{aligned}R(A;\sigma,\tau)&=\{Au\colon u\in H_{A}^{\sigma,\tau}\}\\&=\{f\in H^{\tau}(E_{M^{+}})\colon \underset{(\tau,M^{+})(-\tau,M^{+})}{((f,\ v))}=0\quad\text{for all}\quad v\in N_{0}(A^{*})\}\\&=\{Au\colon u\in H^{\tau+\omega}(E_{M^{+}})\}\end{aligned}$$

(the first equality is a definition of $R(A;\sigma,\tau)$). This yields

$$\text{the kernel of } R=\{(v,0)\in H_0^{-\tau}(E_{M^+})\oplus\{0\}: \underset{(\tau,M+)\ (-\tau,M+)}{((Au,\ v))}=0 \quad \text{for all} \quad u\in H_A^{\sigma,\tau}\}$$
$$=N_0(A^*)\oplus\{0\}.$$

Hence we have

$$\begin{aligned}\operatorname{codim} R(B^+,\sigma)&=\dim N((B^+,\sigma)^*)\\&=\dim N((A\oplus B\gamma;\sigma,\tau)^*)-\dim \operatorname{Ker} R\\&=\operatorname{codim}(R(A\oplus B\gamma;\sigma,\tau)-\dim N_0(A^*).\end{aligned}$$

Here we have used the notation Ker R=the kernel of R. This completes the proof.

4.4. Indices.

THEOREM 4.9. *Let* $\sigma\leqq\tau+\omega$, $\tau>-1/2$ *and let* B: $B^\sigma(X)\to H^\rho(G)$ $(\rho=\sigma-\omega+1/2+\lambda)$ *be a continuous linear map. Then* index $(A\oplus B\gamma;\sigma,\tau)<\infty$ *if and only if* index $(B^+,\sigma)<\infty$. *Furthermore, we have*

$$\operatorname{index}(A\oplus B\gamma;\sigma,\tau)=\dim N_0(A)-\dim N_0(A^*)+\operatorname{index}(B^+,\sigma).$$

Proof. This is an immediate consequence of Theorem 4.1, Corollary 4.7 and Theorem 4.8.

REMARK 4.10. If $\dim N(BP^+,\sigma)/N(P^+,\sigma)<\infty$ and $\operatorname{codim} R(B^+,\sigma)<\infty$, then we say that B is well-posed for A (see Seeley [11], p. 783). Since by Theorem 4.1 $\dim N(BP^+,\sigma)/N(P^+,\sigma)=\dim N(B^+,\sigma)$, it follows from Theorem 4.9 that index $(A\oplus B\gamma;\sigma,\tau)<\infty$ if any only if B is well-posed for A.

COROLLARY 4.11 (cf. Agranovič [1], p. 105). *Let* B_k $(k=1,2)$ *be well-posed for* A. *Then we have*

$$\operatorname{index}(A\oplus B_1\gamma;\sigma,\tau)-\operatorname{index}(A\oplus B_2\gamma;\sigma,\tau)=\operatorname{index}(B_1^+,\sigma)-\operatorname{index}(B_2^+,\sigma).$$

§ 5. Regularity.

5.1. Kernels. First, we prove the following result (cf. Hörmander [5], p. 197).

THEOREM 5.1. *Let* $t<\sigma$, *let* $\rho=\sigma-\omega+1/2+\lambda$ *and let* B *be a linear map of* $B^s(X)$ *into* $H^{s-\omega+1/2+\lambda}(G)$ *for each real s. Then the following three statements are equivalent*:

(i) *For every* $g\in R_0(A,t)$ *such that* $Bg\in H^\rho(G)$, *we have* $g\in B^\sigma(X)$.

(i)′ *For every* $g\in B^t(X)$ *such that* $(1-P^+)g\in B^\sigma(X)$ *and* $Bg\in H^\rho(G)$, *we have* $g\in B^\sigma(X)$.

(ii) *For every* $u\in N(A,t)$ *such that* $B\gamma u\in H^\rho(G)$, *we have* $u\in H^\sigma(E_{M^+})$.

Furthermore, if $\sigma\leqq\tau+\omega$, $\tau>-1/2$, *then the above three statements are equivalent to*

(ii)′ *For every* $u\in H^t(E_{M^+})$ *such that* $Au\in H^\tau(E_{M^+})$ *and* $B\gamma u\in H^\rho(G)$, *we have*

$u \in H^{\sigma}(E_{M^+})$.

Proof. That (i)′⇒(i) and that (ii)′⇒(ii) are obvious.

(ii)⇒(i): Let $g \in R_0(A, t)$ such that $Bg \in H^{\rho}(G)$. Then by the definition (7) of $R_0(A, t)$ there exists $u \in N(A, t)$ such that $\gamma u = g$. Since $u \in N(A, t)$ and $B\gamma u = Bg \in H^{\rho}(G)$, we have by (ii) $u \in H^{\sigma}(E_{M^+})$ and hence $u \in N(A, \sigma)$. Thus we obtain from Theorem 1.1 (i) $g = \gamma u \in B^{\sigma}(X)$.

(i)⇒(ii): Let $u \in N(A, t)$ such that $B\gamma u \in H^{\rho}(G)$. Then by Theorem 1.2 (iii) u can be decomposed into $u = u_0 + u_1$, where $u_0 \in N_0(A) \subset C^{\infty}(E)$ and $u_1 = P\gamma u_1$. Then, since $\gamma u_0 = 0$, it follows that $\gamma u_1 = \gamma u \in R_0(A, t)$ and hence that $B\gamma u_1 = B\gamma u \in H^{\rho}(G)$. Thus we have by (i) $\gamma u_1 \in B^{\sigma}(X)$ and hence by Theorem 1.2 (iii) $u_1 = P\gamma u_1 \in N(A, \sigma)$, which proves that $u = u_0 + u_1 \in H^{\sigma}(E_{M^+})$.

(i)⇒(i)′: Let $g \in B^{t}(X)$ such that $(1-P^+)g \in B^{\sigma}(X)$ and $Bg \in H^{\rho}(G)$. Then since $(1-P^+)g \in B^{\sigma}(X)$ and $g = P^+g + (1-P^+)g$, we have only to show that $P^+g \in B^{\sigma}(X)$. Now, since $g \in B^{t}(X)$, it follows from Theorem 1.2 (vi) that $P^+g \in R_0(A, t)$. Further, since $Bg \in H^{\rho}(G)$ and $(1-P^+)g \in B^{\sigma}(X)$, it follows from the property of B that $BP^+g = Bg - B(1-P^+)g \in H^{\rho}(G)$. Thus we have by (i) $P^+g \in B^{\sigma}(X)$.

(ii)⇒(ii)′: Let $u \in H^{t}(E_{M^+})$ such that $Au \in H^{\tau}(E_{M^+})$ and $B\gamma u \in H^{\rho}(G)$. Then by Proposition 2.1 (i) u can be decomposed into $u = z + w$, where $z \in H^{\tau+\omega}(E_{M^+})$ and $w \in N(A, t)$. Since $t < \sigma \leqq \tau + \omega$, we have only to show that $w \in H^{\sigma}(E_{M^+})$. Now, since $z \in H^{\tau+\omega}(E_{M^+})$ with $\tau + \omega > \omega - 1/2$, it follows that $\gamma z \in B^{\tau+\omega}(E)$ (see 1.3) and hence that $B\gamma z \in H^{\tau+1/2+\lambda}(G)$. Further, since $\sigma \leqq \tau + \omega$, $\rho = \sigma - \omega + 1/2 + \lambda$ and since $B\gamma u \in H^{\rho}(G)$, it follows that $B\gamma w = B\gamma u - B\gamma z \in H^{\rho}(G)$. Thus we have by (ii) $w \in H^{\sigma}(E_{M^+})$. The proof is complete.

From the proof of Theorem 5.1 and Sobolev's Lemma, we obtain immediately

COROLLARY 5.2. *The following two statements are equivalent*:

(i) $N(B^+, \sigma)$ *consists of* C^{∞} *sections of* E_X, *i.e.*, $N(B^+, \sigma) \subset C^{\infty}(E_X)$;

(ii) $N(A \oplus B\gamma; \sigma, \tau)$ *consists of* C^{∞} *sections of* $E_{M_1^+}$, *i.e.*, $N(A \oplus B\gamma; \sigma, \tau) \subset C^{\infty}(E_{M_1^+})$.

5.2. Cokernels.

THEOREM 5.3. *Let* $\sigma \leqq \tau + \omega$, $\tau > -1/2$ *and let* B *be a continuous linear map of* $B^{s}(X)$ *into* $H^{s-\omega+1/2+\lambda}(G)$ *for each real* s. *Then the following two statement are equivalent*:

(i) $R(B^+, \sigma)$ *is the orthogonal complement of finitely many elements in* $C^{\infty}(G)$, *i.e.*, $\dim N((B^+, \sigma)^*) < \infty$ *and* $N((B^+, \sigma)^*) \subset C^{\infty}(G)$.

(ii) $R(A \oplus B\gamma; \sigma, \tau)$ *is the orthogonal complement of finitely many elements in* $C^{\infty}(E_{M_1^+}) \oplus C^{\infty}(G)$, *i.e.*, $\dim N((A \oplus B\gamma; \sigma, \tau)^*) < \infty$ *and* $N((A \oplus B\gamma; \sigma, \tau)^*) \subset C^{\infty}(E_{M_1^+}) \oplus C^{\infty}(G)$.

For the definitions of $N((B^+, \sigma)^*)$ and $N((A \oplus B\gamma; \sigma, \tau)^*)$, see Remark 4.6.

Proof. (ii)⇒(i): Suppose that the family $\{(v_j, \psi_j)\}_{j=1}^{q} \subset C^{\infty}(E_{M_1^+}) \oplus C^{\infty}(G)$ is a basis of $N((A \oplus B\gamma; \sigma, \tau)^*)$. Then, as in the proof of Theorem 4.8, it follows that the family $\{\psi_j\}_{j=1}^{q} \subset C^{\infty}(G)$ generates $N((B^+, \sigma)^*)$, which implies that $\dim N((B^+, \sigma)^*) \leqq q < \infty$ and that $N((B^+, \sigma)^*) \subset C^{\infty}(G)$.

(i)⇒(ii): Suppose that the family $\{\psi_j\}_{j=1}^{l}\subset C^\infty(G)$ is a basis of $N((B^+,\sigma)^*)$. Then it follows from Theorem 4.5 that the family $\{(v_i, 0)\}_{i=1}^{m}$, $\{(\tilde{v}_j, \psi_j)\}_{j=1}^{l}$ ($\tilde{v}_j = -E_k^* C^* \gamma^* B^* \psi_j$) is a basis of $N((A\oplus B\gamma; \sigma, \tau)^*)$. Since $\{v_i\}_{i=1}^{m}\subset N_0(A^*)\subset C^\infty(E_{M_1^+})$ and $\{\psi_j\}_{j=1}^{l}\subset C^\infty(G)$, we have only to show that $\{\tilde{v}_j\}_{j=1}^{l}\subset C^\infty(E_{M_1^+})$. Now, since B^* is continuous from $H^{-s+\omega-1/2-\lambda}(G)$ into $(B^s(X))^*$ for each real s, it follows from Sobolev's Lemma that $\{B^*\psi_j\}_{j=1}^{l}\subset\bigoplus_{j=0}^{\omega-1}C^\infty(E_X)$. Thus we derive from Theorem 1.4 (i), (ii) that $\{C^*\gamma^*B^*\psi_j\}_{j=1}^{l}\subset\bigcap_{\varepsilon>0}H^{1/2-\varepsilon}(E)$ and that the restrictions of $\{C^*\gamma^*B^*\psi_j\}_{j=1}^{l}$ to M^+ (resp. M^-) belong to $C^\infty(E_{M_1^+})$ (resp. $C^\infty(E_{M_1^-})$), i.e., that $C^*\gamma^*B^*\psi_j (1\leqq j\leqq l)$ is C^∞ up to the boundary X in M^+ and also in M^-. On the other hand, we have known that if v is C^∞ up to X in M^+ and also in M^-, then $E_k^* v$ is C^∞ up to X in M^+ (see (5)). Hence we conclude that $\tilde{v}_j = -E_k^* C^*\gamma^* B^*\psi_j \in C^\infty(E_{M_1^+})$ $(1\leqq j\leqq l)$. The proof is complete.

Appendix.

Proof of Proposition 2.1. (i) Let $u\in H_A^{\sigma,\tau}$. Then, since $E_k: H^\tau(E_{M^+})\to H^\tau(E)$ is continuous (see 1.4) and since by Theorem 1.4 $C: H^\tau(E)\to H^{\tau+\omega}(E)$ is continuous, it follows that $CE_kAu\in H^{\tau+\omega}(E)$ and hence that $z=CE_kAu|_{M^+}\in H^{\tau+\omega}(E_{M^+})$. On the other hand, since $Au\in H^\tau(E_{M^+})\subset H^{\sigma-\omega}(E_{M^+})$ $(\sigma\leqq\tau+\omega)$, it follows that for all $v\in N_0(A^*)$

$$\underset{(\tau,M+)\ (-\tau,M+)}{((Au,\ v))} = \underset{(\sigma-\omega,M+)(-\sigma+\omega,M+)}{((Au,\ v))}$$

$$= \underset{(\sigma,M+)\ \ (-\sigma,M+)}{((u,\ A^*v))}$$

$$=0$$

(see Remark 1.3). Hence we obtain from Theorem 1.4 (iii) $Az=Au$ and thus $w=u-z\in N(A,\sigma)$, since $u\in H_A^{\sigma,\tau}\subset H^\sigma(E_{M^+})$ and $z\in H^{\tau+\omega}(E_{M^+})$. Therefore $u\in H_A^{\sigma,\tau}$ can be decomposed into $u=z+w$, where $z\in H^{\tau+\omega}(E_{M^+})$ and $w\in N(A,\sigma)$.

(ii) Since $E_k: H^\tau(E_{M^+})\to H^\tau(E)$ and $C: H^\tau(E)\to H^{\tau+\omega}(E)$ are continuous, it follows that

$$||z||_{\tau+\omega}\leqq ||CE_kAu||_{\tau+\omega,M}$$

$$\leqq K||E_kAu||_{\tau,M}$$

$$\leqq K||Au||_\tau$$

and hence that

$$||w||_\sigma\leqq ||u||_\sigma+||z||_\sigma$$

$$\leqq ||u||_\sigma+||z||_{\tau+\omega}$$

$$\leqq K||u||_{\sigma,\tau},$$

since $\sigma\leqq\tau+\omega$. These are the desired inequalities (10), (11).

(iii) By part (i), $u\in H_A^{\sigma,\tau}$ can be decomposed into $u=z+w$, where $z\in H^{\tau+\omega}(E_{M^+})$

and $w\in N(A,\sigma)$. Since $\tau+\omega>\omega-1/2$, γz exists in $B^{\tau+\omega}(X)$ (see 1.3) and also in $(B^{\sigma}(X)$ $(\sigma\leqq\tau+\omega)$, and since $w\in N(A,\sigma)$, γw exists in $B^{\sigma}(X)$ (by Theorem 1.1 (i)). Further, if $u\in H^{\tau+\omega}(E_{M^+})\subset H_A{}^{\sigma,\tau}$, it follows that $\gamma z+\gamma w=\gamma z+\gamma(u-z)=\gamma z+\gamma u-\gamma z=\gamma u$. Thus we can extend $\gamma: H^{\tau+\omega}(E_{M^+})\rightarrow B^{\tau+\omega}(X)$ to a map $\tilde{\gamma}: H_A{}^{\sigma,\tau}\rightarrow B^{\sigma}(X)$ by defining

$$\tilde{\gamma}u=\gamma z+\gamma w. \tag{25}$$

Since $\tilde{\gamma}$ agrees with the original γ on $H^{\tau+\omega}(E_{M^+})$, we shall simply drop the tilde and continue to denote it by γ.

(iv) Since $\sigma\leqq\tau+\omega$ and γ is continuous from $H^{\tau+\omega}(E_{M^+})$ into $B^{\tau+\omega}(X)$ for $\tau+\omega>\omega-1/2$ (see 1.3), it follows that

$$\begin{aligned}|\gamma z|_{\sigma}&\leqq|\gamma z|_{\tau+\omega}\\ &\leqq K||z||_{\tau+\omega}.\end{aligned}$$

Hence, combining this with (10), we obtain

$$|\gamma z|_{\sigma}\leqq K||Au||_{\tau}. \tag{26}$$

On the other hand, applying (6) of Theorem 1.1 to $w\in N(A,\sigma)$ and using (11), we obtain

$$\begin{aligned}|\gamma w|_{\sigma}&\leqq K||w||_{\sigma}\\ &\leqq K||u||_{\sigma,\tau}.\end{aligned}$$

Therefore, combining this and (26), we finally obtain (see (25))

$$\begin{aligned}|\gamma u|_{\sigma}&=|\gamma z+\gamma w|_{\sigma}\\ &\leqq|\gamma z|_{\sigma}+|\gamma w|_{\sigma}\\ &\leqq K||u||_{\sigma,\tau},\end{aligned}$$

which proves that $\gamma: H_A{}^{\sigma,\tau}\rightarrow B^{\sigma}(X)$ is continuous. The proof is complete.

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