CHEN B.-Y. AND K. YANO
KODAI MATH. SEM. REP.
25 (1873), 289—296

SUBMANIFOLDS UMBILICAL WITH RESPECT TO A NON-PARALLEL NORMAL SUBBUNDLE

BY BANG-YEN CHEN AND KENTARO YANO

Dedicated to Professor S. Ishihara on his fiftieth birthday

Let V_n be an n-dimensional submanifold of an m-dimensional Riemannian manifold V_m and C be a unit normal vector field of V_n in V_m. If the second fundamental tensor in the normal direction C is proportional to the first fundamental tensor of the submanifold V_n, then V_n is said to be umbilical with respect to the normal direction C. Let N be a subbundle of the normal bundle of V_n in V_m. If the submanifold V_n is umbilical with respect to every normal direction in N, then V_n is said to be umbilical with respect to N. If the covariant derivative of every unit normal direction in N has no component in the complementary normal subbundle $N^\perp$ orthogonal to N, then the subbundle N is said to be parallel. If there exists, in N, a unit normal direction C such that the covariant derivative of C has nonzero component in the subbundle $N^\perp$ everywhere, the subbundle is said to be non-parallel.

In this paper, we shall study submanifolds of a space form which are umbilical with respect to a non-parallel normal subbundle.

§ 1. Preliminaries.

Let V_m be an m-dimensional Riemannian manifold of constant curvature c with the metric $ds^2=g_{\mu\lambda}d\xi^\mu d\xi^\lambda$, $\kappa, \lambda, \mu, \cdots =1, 2, \cdots, m$, where $\{\xi^\kappa\}$ is a local coordinate system in V_m. We denote by $\{{}^{\kappa}_{\mu\lambda}\}$ the Christoffel symbols formed with $g_{\mu\lambda}$ and by $K_{\nu\mu\lambda}{}^\kappa$ the Riemann-Christoffel curvature tensor of V_m:

$$K_{\nu\mu\lambda}{}^\kappa=c(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda}). \tag{1}$$

Let V_n be an n-dimensional submanifold of V_m and the parametric equations of V_n be

$$\xi^\kappa=\xi^\kappa(\eta^h),$$

where $\{\eta^h\}$ is a local coordinate system in V_n and, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

We put

Received June 26, 1972.

(2) $$B_i^\kappa=\partial_i\xi^\kappa, \qquad \partial_i=\partial/\partial\eta^i.$$

The fundamental metric tensor of V_n is then given by

(3) $$g_{ji}=g_{\mu\lambda}B_j^\mu B_i^\lambda.$$

We denote by $\{^h_{ji}\}$ the Christoffel symbols formed with g_{ji} and by ∇_j the operator of covariant differentiation along V_n. The van der Waerden-Bortolotti covariant derivative of B_i^κ is then given by

(4) $$\nabla_j B_i^\kappa=\partial_j B_i^\kappa+\left\{ {\kappa \atop \mu\lambda} \right\} B_j^\mu B_i^\lambda-B_h^\kappa\left\{ {h \atop ji} \right\}.$$

From (3) and (4) we see that $\nabla_j B_i^\kappa$ are orthogonal to the submanifold V_n. We choose $m-n$ mutually orthogonal unit vectors C_y^κ which are normal to V_n. Then we have

(5) $$g_{\mu\lambda}B_j^\mu C_y^\lambda=0, \qquad g_{\mu\lambda}C_z^\mu C_y^\lambda=\delta_{zy},$$

where δ_{zy} are the Kronecker deltas and the indices x, y, z run over the range $\{1, 2, \cdots, m-n\}$.

The equations of Gauss are then given by

(6) $$\nabla_j B_i^\kappa=h_{ji}{}^x C_x^\kappa,$$

where $h_{ji}{}^x$ are the second fundamental tensors of V_n in the normal direction C_x^κ. The equations of Weingarten are given by

(7) $$\nabla_j C_y^\kappa=-h_j{}^i{}_y B_i^\kappa+l_{jy}{}^x C_x^\kappa,$$

where $h_j{}^i{}_y=h_{jt}{}^y g^{ti}$ and $l_{jy}{}^x=-l_{jx}{}^y$ are the third fundamental tensors. The mean curvature vector of V_n is given by

$$H^\kappa=\frac{1}{n}g^{ji}\nabla_j B_i^\kappa.$$

Since the curvature tensor of V_n is of the form (1), the equations of Gauss are given by

(8) $$K_{kji}{}^h=c(\delta_k^h g_{ji}-\delta_j^h g_{ki})+h_k{}^h{}_x h_{ji}{}^x-h_j{}^h{}_x h_{ki}{}^x,$$

the equations of Codazzi by

(9) $$\nabla_k h_{ji}{}^x-\nabla_j h_{ki}{}^x+l_{ky}{}^x h_{ji}{}^y-l_{jy}{}^x h_{ki}{}^y=0,$$

or

(10) $$\nabla_k h_j{}^h{}_y-\nabla_j h_k{}^h{}_y-l_{ky}{}^x h_j{}^h{}_x+l_{jy}{}^x h_k{}^h{}_x=0,$$

and the eqnations of Ricci by

(11) $$\nabla_k l_{jy}{}^x-\nabla_j l_{ky}{}^x-h_{kt}{}^x h_j{}^t{}_y+h_{jt}{}^x h_k{}^t{}_y+l_{kz}{}^x l_{jy}{}^z-l_{jz}{}^x l_{ky}{}^z=0.$$

If there exist, on the submanifold V_n, two functions α, β and a unit vector field u_j such that

$$h_{ji}{}^x = \alpha g_{ji} + \beta u_j u_i \tag{12}$$

for a fixed x, then V_n is said to be *quasi-umbilical* with respect to the normal direction $C_x{}^\kappa$. In particular, if $\alpha=0$ identically, then V_n is said to be *cylindrical* with respect to $C_x{}^\kappa$, if $\beta=0$ identically, then V_n is said to be *umbilical* with respect to $C_x{}^\kappa$, and if $\alpha=\beta=0$ identically, then V_n is said to be *geodesic* with respect to $C_x{}^\kappa$. If N is a normal subbundle, i.e., if N is a subbundle of the normal bundle, and the submanifold V_n is umbilical with respect to every normal direction in N, then V_n is said to be *umbilical with respect to the normal subbundle N.*

For a given normal subbundle N of V_n, if the covariant derivative of every unit normal direction in N has no component in the complementary normal subbundle $N^\perp$ orthogonal to N, then the subbundle N is said to be *parallel.* If there exists, in N, a unit normal direction C such that the covariant derivative of C has nonzero component everywhere in the complementary normal subbundle $N^\perp$ orthogonal to N, then the subbundle N is said to be *non-parallel.*

Let C and D be two unit normal directions of V_n in V_m. If the covariant derivative of C has no normal component except in the normal direction D, then C is said to be *quasi-parallel* with respect to D.

For a normal subbundle N of V_n in V_m, the dimension of the fibres of N is called the dimension of the subbundle N.

§ 2. Lemmas.

In this section, we prove the following lemmas.

LEMMA 1. *Let N be a normal subbundle of V_n in V_m. If N is non-parallel, then the complementary normal subbundle $N^\perp$ orthogonal to N is also non-parallel.*

Proof. Suppose that N is non-parallel, then there exists a unit normal direction C in N such that the covariant derivative of C has non-zero component in $N^\perp$. We choose unit normal $C_x{}^\kappa$ in such a way that we have

$$C_1{}^\kappa = C^\kappa,\ C_2{}^\kappa,\ \cdots,\ C_a{}^\kappa \in N, \qquad C_{a+1}{}^\kappa,\ \cdots,\ C_{m-n}{}^\kappa \in N^\perp,$$

where a denotes the dimension of the normal subbundle N. Then we have, putting $y=1$ in (7),

$$\nabla_j C_1{}^\kappa = -h_j{}^i{}_1 B_i{}^\kappa + l_{j1}{}^u C_u{}^\kappa + l_{j1}{}^r C_r{}^\kappa, \tag{13}$$

where

$$l_{j1}{}^r C_r{}^\kappa \neq 0,$$

and here and in the sequel the indices u, v, w run over the range $\{1, 2, \cdots, a\}$ and the indices r, s, t run over the range $\{a+1, a+2, \cdots, m-n\}$.

Since $l_{j1}{}^{r}C_{r}{}^{\kappa} \neq 0$, we have $l_{j1}{}^{r} \neq 0$ for some fixed j. We put, for that j,

$$D^{\kappa} = \frac{l_{j1}{}^{r}C_{r}{}^{\kappa}}{|l_{j1}{}^{r}C_{r}{}^{\kappa}|}, \tag{14}$$

where $|l_{j1}{}^{r}C_{r}{}^{\kappa}|$ denotes the length of $l_{j1}{}^{r}C_{r}{}^{\kappa}$. Then D^{κ} is a unit normal direction in $N^{\perp}$. From $g_{\mu\lambda}C_{1}{}^{\mu}D^{\lambda}=0$, we find

$$g_{\mu\lambda}(\nabla_{j}C_{1}{}^{\mu})D^{\lambda} + g_{\mu\lambda}C_{1}{}^{\mu}(\nabla_{j}D^{\lambda}) = 0, \tag{15}$$

or, using (13),

$$g_{\mu\lambda}C_{1}{}^{\mu}\nabla_{j}D^{\lambda} = -|l_{j1}{}^{r}C_{r}{}^{\kappa}| \neq 0. \tag{16}$$

This implies that the normal subbundle $N^{\perp}$ is also non-parallel.

LEMMA 2. *Let N be a normal subbundle of V_n in V_m. If N is parallel, then the complementary normal subbundle $N^{\perp}$ orthogonal to N is also parallel.*

This lemma follows immediately from Lemma 1.

LEMMA 3. *Let N be a non-parallel normal subbundle of V_n in V_m of dimension $m-n-1$. If the submanifold V_n is umbilical with respect to N, then the submanifold V_n is quasi-umbilical with respect to the normal direction in $N^{\perp}$.*

Proof. Since N is a non-parallel normal subbundle of dimension $m-n-1$, the subbundle $N^{\perp}$ is, by Lemma 1, also non-parallel and of dimension one. If we choose $C_{x}{}^{\kappa}$ in such a way that $C_{m-n}{}^{\kappa} = D^{\kappa}$ with D^{κ} as the unit normal direction in $N^{\perp}$, then by the umbilicity of V_n with respect to N, we have

$$h_{ji}{}^{u} = \alpha^{u} g_{ji} \tag{17}$$

for some functions α^{u} and

$$l_{jm-n}{}^{u} = l_{j}{}^{u} \tag{18}$$

do not vanish simultaneously.

Under these assumptions, (6) becomes

$$\nabla_{j}B_{i}{}^{\kappa} = \alpha^{u} g_{ji} C_{u}{}^{\kappa} + k_{ji} D^{\kappa}, \tag{19}$$

where $k_{ji} = k_{ji}{}^{m-n}$ and (7) becomes

$$\nabla_{j}C_{v}{}^{\kappa} = -\alpha_{v}B_{j}{}^{\kappa} + l_{jv}{}^{u}C_{u}{}^{\kappa} + l_{jv}D^{\kappa}, \tag{20}$$

where $\alpha_{v} = \alpha^{v}$ and

$$l_{jv} = l_{jv}{}^{m-n} = -l_{jm-n}{}^{v} = -l_{j}{}^{v}. \tag{21}$$

and

$$\nabla_{j}D^{\kappa} = -k_{j}{}^{i}B_{i}{}^{\kappa} + l_{j}{}^{u}C_{u}{}^{\kappa}. \tag{22}$$

Equations (9) of Codazzi beeome

(23) $$(\nabla_k\alpha^u)g_{ji}-(\nabla_j\alpha^u)g_{ki}+l_{kv}{}^u\alpha^v g_{ji}-l_{jv}{}^u\alpha^v g_{ki}+l_k{}^u k_{ji}-l_j{}^u k_{ki}=0$$

and

(24) $$\nabla_k k_{ji}-\nabla_j k_{ki}-l_k{}^u\alpha_u g_{ji}+l_j{}^u\alpha_u g_{ki}=0.$$

Equations (11) of Ricci becomes

(25) $$\nabla_k l_{jv}{}^u-\nabla_j l_{kv}{}^u+l_{kw}{}^u l_{jv}{}^w-l_{jw}{}^u l_{kv}{}^w-l_k{}^u l_j{}^v+l_j{}^u l_k{}^v=0$$

and

(26) $$\nabla_j l_k{}^u-\nabla_k l_j{}^u+l_{kv}{}^u l_j{}^v-l_{jv}{}^u l_k{}^v=0.$$

Since $C_{m-n}{}^\kappa=D^\kappa$ is non-parallel, without loss of generality, we can assume that

(27) $$l_i{}^1=l_i\neq 0.$$

Putting

(28) $$\nabla_k\alpha^1+l_{kv}{}^1\alpha^v=\alpha_k,$$

we have, from (23),

$$\alpha_k g_{ji}-\alpha_j g_{ki}+l_k k_{ji}-l_j k_{ki}=0.$$

Consequently, in exactly the same way as in the proof of Theorem 1 of [1], we can conclude that

(29) $$k_{ji}=\lambda g_{ji}+\mu l_j l_i,$$

where

(30) $$\lambda=-\frac{\alpha_t l^t}{l^2},\qquad l^2=l_t l^t,\qquad \mu=\frac{n\lambda-k_t{}^t}{l^2}.$$

This proves the lemma.

LEMMA 4. *Let N be a non-parallel normal subbundle of V_n in V_m of dimension $m-n-1$. If the submanifold V_n is geodesic with respect to N, then the submanifold V_n is cylindrical with respect to the normal direction of $N^\perp$. Consequently, the submanifold V_n is of constant curvature c. In particular, if V_n is complete and V_m is euclidean, then V_n is a cylinder.*

Proof. Since V_n is geodesic with respect to N, by (28), we have $\alpha_k=0$. Thus, by Lemma 3, we see that V_n is quasi-umbilical with respect to the normal direction $N^\perp$ satisying (29). This implies that V_n is cylindrical with respect to the normal direction of $N^\perp$. In particular, by equations (8) of Gauss, we see that V_n and V_m has the same constant curvature c. Hence if V_n is complete and V_m is complete and V_m is euclidean, then V_n is a cylinder.

LEMMA 5. *Let $C_{m-n}{}^\kappa=D^\kappa$ be a non-parallel unit normal direction of V_n in V_m*

and N be the $(m-n-1)$-dimensional normal subbundle generated by $C_1^\kappa, C_2^\kappa, \cdots, C_{m-n-1}^\kappa$. If the submanifold V_n is umbilical with respect to N, then all of the third fundametal tensors l_j^κ are proportional. In particular, if $l_j^1 \neq 0$, then we have

$$l_j^x = v^x l_j^1 \tag{31}$$

for some functions v^x, where $l_j^x = l_{jm-n}^x$.

Proof. By Lemma 3, we see that the submanifold V_n is quasi-umbilical with respect to $C_{m-n}^\kappa = D^\kappa$ and if we assume that $l_j = l_j^1 \neq 0$, then the second fundamental tensor k_{ji} is given by

$$k_{ji} = \lambda g_{ji} + \mu l_j l_i,$$

and consequently, this conclusion may be written as

$$k_{ji} = \lambda^1 g_{ji} + \mu^1 l_j^1 l_i^1.$$

Thus, if l_i^2 never vanishes, then we have

$$k_{ji} = \lambda^2 g_{ji} + \mu^2 l_j^2 l_i^2,$$

and consequently,

$$(\lambda^1 - \lambda^2) g_{ji} = -\mu^1 l_j^1 l_i^1 + \mu^2 l_j^2 l_i^2.$$

Thus

$$l_j^2 = v^2 l_j^1 = v^2 l_j \tag{32}$$

for some function v^2. This implies that all the third fundamental tensors l_j^x are proportional and proves the lemma.

From Lemma 5, we have immediately the following

PROPOSITION 1. *Let D^κ be a non-parallel unit normal direction of V_n in V_m and N be the $(m-n-1)$-dimensional normal subbundle orthogonal to D^κ. If the submanifold V_n is umbilical with respect to N, then the normal direction D^κ is quasi-parallel with respect to a normal direction in N.*

LEMMA 6. *Under the hypothesis of Lemma 5, we have*

$$\nabla_k l_j - \nabla_j l_k = 0. \tag{33}$$

Proof. Putting $u=1$ in equation (26), we find

$$\nabla_k l_j - \nabla_j l_k + l_{kv}{}^1 l_j^v - l_{jv}{}^1 l_k^v = 0, \tag{34}$$

that is,

$$\nabla_k l_j - \nabla_j l_k - l_k^v l_j^v + l_j^v l_k^v = 0. \tag{35}$$

By substituting (31) of Lemma 5 into (35), we obtain (33).

LEMMA 7. *Under the hypothesis of Lemma* 5, *we have*

$$\mu\nabla_j l_i = \frac{\lambda_t l^t - l^t l_t{}^u \alpha_u}{l^2} g_{ji} - (\mu_j l_i + \mu_i l_j) + 2r l_j l_i, \tag{36}$$

where $\lambda_k = \nabla_k \lambda$, $\mu_k = \nabla_k \mu$, *r is a function and* λ *and* μ *are given by* (29).

Proof. Substituting (29) into (24) and applying Lemma 6, we find

$$\lambda_k g_{ji} - \lambda_j g_{ki} + \mu_k l_j l_i - \mu_j l_k l_i + \mu l_j(\nabla_k l_i) - \mu l_k(\nabla_j l_i) - l_k{}^u \alpha_u g_{ji} + l_j{}^u \alpha_u g_{ki} = 0, \tag{37}$$

from which, by transvecting l^k,

$$\begin{aligned} &\lambda_t l^t g_{ji} - \lambda_j l_i + \mu_t l^t l_j l_i - \mu_j l^2 l_i + \mu l_j (l^t \nabla_t l_i) \\ &\qquad - \mu l^2 (\nabla_j l_i) - (l^t l_t{}^u \alpha_u) g_{ji} + l_j{}^u \alpha_u l_i = 0. \end{aligned} \tag{38}$$

Equation (38) shows that $\mu\nabla_j l_i$ is of the form

$$\mu\nabla_j l_i = p g_{ji} + q_j l_i + q_i l_j, \tag{39}$$

where

$$p = \frac{\lambda_t l^t - l^t l_t{}^u \alpha_u}{l^2},$$

$\mu\nabla_j l_i$ being symmetric by Lemma 6. Substituting (39) into (37), we find

$$(\lambda_k - p l_k - l_k{}^u \alpha_u) g_{ji} - (\lambda_j - p l_j - l_j{}^u \alpha_u) g_{ki} + (\mu_k l_j - \mu_j l_k + q_k l_j - q_j l_k) l_i = 0, \tag{40}$$

from which we obtain

$$\lambda_k = p l_k + l_k{}^u \alpha_u \tag{41}$$

and

$$\mu_j + q_j = r l_j, \tag{42}$$

r being a function. Thus, by (39) and (42), we obtain the lemma.

§ 3. Locus of $(n-1)$-spheres.

THEOREM 1. *Let* V_n *be an n-dimensional submanifold of a euclidean m-space* E_m *and N be an* $(m-n-1)$*-dimensional normal subbundle of the normal bundle of* V_n *in* E_m. *If the subbundle N is non-parallel and the submanifold* V_n *is umbilical with respect to N, then* V_n *is a locus of* $(n-1)$*-spheres, where an* $(n-1)$*-sphere means a hypersphere or a hyperplane of a euclidean n-space.*

Proof. Let D^κ denote the unit normal direction in the complementary normal subbundle $N^\perp$ orthogonal to N. Then D^κ is non-parallel. Thus the formulas in § 1 and § 2 hold. By Lemma 6, the distribution $l_j d\eta^i = 0$ is integrable. We represent one of the integral submanifolds V_{n-1} by

$$\eta^h=\eta^h(\zeta^a)$$

and put

$$B_b{}^h=\partial_b\eta^h, \qquad \partial_b=\frac{\partial}{\partial\zeta^b}, \qquad N^h=\frac{1}{l}l^h,$$

$$g_{cb}=g_{ji}B_c{}^jB_b{}^i$$

and

$$\nabla_cB_b{}^h=H_{cb}N^h,$$

$\nabla_cB_b{}^h$ denoting the van der Waerden-Bortolotti covariant differentiation of $B_b{}^h$ along V_{n-1} and H_{cb} the second fundamental tensor of V_{n-1}. Here and in the sequel, the indices b, c, d run over the range $\{1, 2, \cdots, n-1\}$. From

$$l_iB_b{}^i=0$$

and Lemma 7, we have

$$\mu l^3H_{cb}=\beta g_{cb}$$

with $\beta=\lambda_tl^t-l^tl_t{}^u\alpha_u$. Thus, on the open subset $U=\{p\in V_n;\ \mu\neq0 \text{ at } p\}$, we have

$$\nabla_cB_b{}^\kappa=\nabla_c(B_b{}^iB_i{}^\kappa)=H_{cb}N^iB_i{}^\kappa+B_c{}^jB_b{}^i(\nabla_jB_i{}^\kappa)$$

$$=\alpha^ug_{cb}C_u{}^\kappa+\frac{\beta}{\mu l^3}g_{cb}N^\kappa,$$

where $N^\kappa=N^iB_i{}^\kappa$. This shows that V_{n-1} is totally umbilical in E_m and hence the closure $\bar{U}$ of U is a locus of $(n-1)$-spheres. On the open subset $V_n-\bar{U}$, we have $\mu=0$. Hence every component of $V_n-\bar{U}$ is contained either in a hypersphere or in a hyperplane of a linear $(n+1)$-subspace of E_m. Thus, the subset $V_n-\bar{U}$ is also a locus of $(n-1)$-spheres. This completes the proof of the theorem.

BIBLIOGRAPHY

[1] CHEN, BANG-YEN, AND KENTARO YANO, Submanifolds umbilical with respect to a non-parallel normal direction. To appear in J. of Differential Geometry.

MICHIGAN STATE UNIVERSITY.