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BIEBERBACH CONJECTURE FOR THE EIGHTH COEFFICIENT, II

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§0. **Introduction.** Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

In 1916 Bieberbach [1] stated his famous conjecture which can be formulated in the present case in the following manner: $|a_8|\leqq 8$ with equality holding only for the Koebe function and its rotations. By a proper rotation this conjecture is reduced to the following form: $\Re a_8\leqq 8$ for $|\arg a_2|\leqq \pi/7$, with equality holding only for the Koebe function. In [4] the authors proved that $\Re a_8\leqq 8$ if $1.9\leqq \Re a_2\leqq 2$, $|\Im a_2/\Re a_2|\leqq 1/20$ and $\Re\{a_3-3a_2^2/4\}\geqq 0$, or if $1.8\leqq \Re a_2\leqq 2$, $|\Im a_2/\Re a_2|\leqq 1/10$ and $\Re\{a_3-3a_2^2/4\}\leqq 0$.

In this paper we shall prove the following

THEOREM. $\Re a_8<8$

if $1.7\leqq \Re a_2\leqq 1.9$, $|\Im a_2/\Re a_2|\leqq 1/20$ *and* $\Re\{a_3-3a_2^2/4\}\geqq 0$.

§1. We make use of the same notations as in [4]. By our assumption $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$ and $y\geqq 0$. Firstly we give several lemmas which were proved in [4].

LEMMA 1. $$11(\tau^2+\tau'^2)+9(\varphi^2+\varphi'^2)+7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)+3(y^2+y'^2)+x'^2\leqq 4x-x^2=4-p^2.$$

LEMMA 2. $$\eta+\left(2\beta-\frac{1}{2}p\right)y\leqq(2-p)\beta^2+\frac{1}{12}(8-p^3)-\frac{1}{2}x'y'+\frac{1}{4}px'^2.$$

LEMMA 3.
$$\begin{aligned}72p^3&\left\{\eta+\frac{1}{6}(\beta-3p)y\right\}\\ &\leqq 192+4\beta^2-\beta^2p^2-3p^6-6\beta p(4-p^2)y\\ &\quad+\{-45p^4-3(\beta-6p)^2-144+90p^2x'^2-108x'^2\}y^2\\ &\quad+\{-30(\beta-6p)p^2-72(\beta-6p)+30(\beta-6p)x'^2\}y\eta\end{aligned}$$

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$$+\{-5(\beta-6p)^2-432\}\eta^2+(-360p^2+360x'^2)y\xi-120(\beta-6p)\eta\xi-720\xi^2$$
$$+(-9p^4-\beta^2-9p^2x'^2-3x'^4)x'^2$$
$$+\{-36p^3-18(\beta-6p)p^2-24\beta+108px'^2+6(\beta-6p)x'^2\}x'y'$$
$$+\{-45p^4-3(\beta-6p)^2-144+90p^2x'^2-108x'^2\}y'^2+(-216p^2+72x'^2)x'\eta'$$
$$+\{-30(\beta-6p)p^2-72(\beta-6p)+30(\beta-6p)x'^2\}y'\eta'+\{-5(\beta-6p)^2-432\}\eta'^2$$
$$+(-360p^2+360x'^2)y'\xi'-120(\beta-6p)\eta'\xi'-720\xi'^2$$
$$+[\{18(\beta-6p)p+108p^2-36x'^2\}x'^2+180p^2y'^2+\{-60(\beta-6p)p-180p^2+432\}x'\eta'$$
$$+120(\beta-6p)y'\eta'-720px'\xi'+1440y'\xi']y$$
$$+[216px'^2+\{60(\beta-6p)p+180p^2-432\}x'y'-60(\beta-6p)y'^2+720x'\xi']\eta$$
$$+[720px'y'-720x'\eta'-720y'^2]\xi+180p^2y^3+60(\beta-6p)y^2\eta+720y^2\xi.$$

Lemma 3 was obtained by Golusin's inequality.

By Grunsky's inequality we have the following inequality, from which we start (see (B) in [4]).

$$\Re a_8\leqq U+\frac{\alpha^2-6\alpha}{64}p^6x+Q+R+Sy+T\eta+V\xi+\frac{5}{48}A(p^4+2p^3+4p^2)xy$$
$$+\frac{3-\alpha}{4}Bp^4xy+\frac{3-\alpha}{4}Cp^3x\eta+\frac{1}{128}(13+20\alpha)p^5y+\frac{1}{64}(3+16\alpha)p^4\eta$$
$$+\frac{1}{4}\alpha p^3\xi-\frac{5}{4}x'^2\varphi+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)py^3+(2A-A^2-2AC)y^2\eta$$
$$+\left(\frac{-51}{8}+A+\frac{1}{2}A^2\right)x'y'y^2+\frac{17}{8}x'y'^3,$$

$$U=8-\frac{31}{4}x-\frac{81}{8}x^2+\frac{1111}{48}x^3-\frac{863}{48}x^4+\frac{2291}{320}x^5$$
$$-\left(\frac{133}{128}+\frac{25}{96}+\frac{7}{40}\right)x^6+\left(\frac{9}{112}+\frac{25}{64\cdot 12}+\frac{1}{80}\right)x^7,$$

$$Q=\left\{\left(\frac{29}{16}-\frac{5-2\alpha}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$
$$\left.+\left(\frac{-21}{4}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$\text{(A)}\qquad +\left\{\left(\frac{37}{8}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\left(\frac{-39}{8}+\frac{A}{2}\right)x'^2\right\}y\eta$$

$$+\left\{\left(\frac{9}{4}-2C\right)p+C^2x\right\}\eta^2+\left(\frac{9}{2}-A-2B\right)py\xi+(4-2C)\eta\xi+(3-2A)y\varphi,$$

$$R=\left(\frac{-267}{256}p^5+\frac{131}{64}p^3x'^2-\frac{27}{64}px'^4\right)x'^2+\left(\frac{-319}{128}p^4+\frac{13}{2}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(\frac{-29}{16}p^3+\frac{21}{4}px'^2\right)y'^2+\left(\frac{-15}{4}p^3+\frac{15}{4}px'^2\right)x'\eta'+\left(\frac{-37}{8}p^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$-\frac{9}{4}p\eta'^2+\left(\frac{-17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'-\frac{9}{2}py'\xi'-4\eta'\xi'-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau',$$

$$S=\left\{\left(\frac{-99}{16}+\frac{5}{16}A\right)p^3+\frac{53}{16}px'^2\right\}x'^2+\left\{\left(\frac{-79}{8}+\frac{13}{8}A\right)p^2+3x'^2\right\}x'y'$$

$$+\left(\frac{-27}{8}-\frac{1}{2}A\right)py'^2-\frac{27}{4}px'\eta'-2Ay'\eta'-\left(\frac{3}{2}+A\right)x'\xi',$$

$$T=\left(\frac{-87}{16}p^2+\frac{7}{8}x'^2\right)x'^2-\frac{27}{4}px'y'-\frac{1}{2}x'\eta',$$

$$V=\frac{-29}{8}px'^2-\frac{3}{2}x'y'.$$

§2. In this section we are concerned with the case $\xi\geqq 0$. We divide this case into several subcases.

Case 1. $\eta\geqq 0$.

We start from (A) with $\alpha=0$. Applying Lemma 3 to the term $(3p^4/64)(\eta+13py/6)$ we have

$$\Re a_8\leqq U+\frac{1}{8}p+\frac{2}{3}p^3-\frac{1}{6}p^5-\frac{1}{32\cdot 16}p^7$$

$$+Q_1+R_1+S_1y+T_1\eta+V_1\xi$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{1}{16}(p^4+2p^3)xy+\frac{3}{4}Bp^4xy+\frac{3}{4}Cp^3x\eta$$

$$-\frac{5}{4}x'^2\varphi+\left\{\frac{15}{4\cdot 32}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{25}{64}p^2+2A-A^2-2AC\right)y^2\eta+\frac{15}{32}py^2\xi+\left(\frac{-51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,$$

$$Q_1=\left\{\frac{-15}{16\cdot 32}p^5+\left(\frac{207}{4\cdot 32}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2-\frac{3}{32}p\right.$$

$$+\frac{15}{8\cdot 32}p^3x'^2+\left(\frac{1}{4}A^2+\frac{1}{2}A-\frac{681}{128}\right)px'^2\Big\}y^2$$

$$(\mathrm{A}_1)\qquad +\left\{\frac{-25}{4\cdot 32}p^4+\left(\frac{133}{32}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\frac{75}{96\cdot 4}p^2x'^2\right.$$

$$\left.+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{\frac{-125}{96\cdot 4}p^3+\left(\frac{63}{32}-2C\right)p+C^2x\right\}\eta^2+\left\{\frac{-15}{64}p^3+\left(\frac{9}{2}-2B-A\right)p\right.$$

$$\left.+\frac{15}{64}px'^2\right\}y\xi+\left(\frac{-25}{32}p^2+4-2C\right)\eta\xi-\frac{15}{32}p\xi^2+(3-2A)y\varphi,$$

$$R_1=\left(\frac{-537}{64\cdot 8}p^5-\frac{1}{6}p^3+\frac{1045}{64\cdot 8}p^3x'^2-\frac{217}{64\cdot 8}px'^4\right)x'^2$$

$$+\left(\frac{-337}{64\cdot 2}p^4-\frac{1}{4}p^2+\frac{423}{64}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(\frac{-15}{16\cdot 32}p^5-\frac{257}{4\cdot 32}p^3-\frac{3}{32}p+\frac{15}{8\cdot 32}p^3x'^2+\frac{1989}{4\cdot 96}px'^2\right)y'^2$$

$$+\left(\frac{-249}{64}p^3+\frac{243}{64}px'^2\right)x'\eta'+\left(\frac{-25}{4\cdot 32}p^4-\frac{163}{32}p^2+\frac{75}{4\cdot 96}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(\frac{-125}{4\cdot 96}p^3-\frac{81}{32}p\right)\eta'^2+\left(\frac{-17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'$$

$$+\left(\frac{-15}{64}p^3-\frac{9}{2}p+\frac{15}{64}px'^2\right)y'\xi'$$

$$+\left(\frac{-25}{32}p^2-4\right)\eta'\xi'-\frac{15}{32}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau',$$

$$S_1=\left\{\left(\frac{5}{16}A-6\right)p^3+\frac{421}{128}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{15}{4\cdot 32}p^3-\left(\frac{1}{2}A+\frac{27}{8}\right)p\right\}y'^2+\left(\frac{-65}{4\cdot 32}p^3-\frac{207}{32}p\right)x'\eta'+\left(\frac{25}{32}p^2-2A\right)y'\eta'$$

$$+\left(\frac{-15}{32}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{15}{16}py'\xi',$$

$$T_1=\left(\frac{-339}{64}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{65}{4\cdot 32}p^3-\frac{225}{32}p\right)x'y'-\frac{25}{64}p^2y'^2-\frac{1}{2}x'\eta'$$

$$+\frac{15}{32}px'\xi',$$

$$V_1=\frac{-29}{8}px'^2+\left(\frac{15}{32}p^2-\frac{3}{2}\right)x'y'-\frac{15}{32}py'^2-\frac{15}{32}px'\eta'.$$

Here we put $A=3/2$, $B=9/8$, $C=1$. We remark the following facts:

$$\frac{1}{96}(30p^4+18p^3+60p^2)xy$$

$$\leqq\frac{\alpha_1}{96}(15p^4+9p^3+30p^2)x^2+\frac{1}{96}\cdot\frac{1}{\alpha_1}\cdot(15p^4+9p^3+30p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{1}{96}60\gamma_1\varphi^2+\frac{1}{96}\cdot\frac{1}{\gamma_1}60x'^4,$$

$$\left(\frac{25}{64}p^2-\frac{9}{4}\right)y^2\eta\leqq 0.$$

By Lemma 2

$$\frac{3}{4}xp^3\eta+\frac{5}{8}xp^4y\leqq\frac{3}{4}xp^3\left(\frac{4}{9}xp^2+\frac{2}{3}-\frac{1}{12}p^3-\frac{1}{2}x'y'+\frac{1}{4}px'^2\right).$$

Further by Lemma 1, for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$,

$$\left(\frac{15}{4\cdot 32}p^3-\frac{3}{8}p\right)y^3\leqq\left(\frac{15}{4\cdot 32}p^3-\frac{3}{8}p\right)\sqrt{\frac{4-p^2}{3}}y^2\leqq\frac{1}{96}(6.852p^3-19.8p)y^2,$$

$$\frac{15}{32}py^2\xi\leqq\frac{15\beta_1}{64}py^4+\frac{15}{64\beta_1}p\xi^2\leqq\frac{1}{96}(-7.5\beta_1p^3+30\beta_1p)y^2+\frac{1}{96}\cdot\frac{22.5}{\beta_1}p\xi^2,$$

$$-\frac{15}{4}y^2x'y'\leqq\frac{513}{128}px'^2y^2+\frac{49.536}{96}y^2y'^2\leqq\frac{513}{128}px'^2y^2+\frac{1}{96}(-16.512p^2+66.048)y'^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{17}{8}\cdot\frac{1}{20}p\sqrt{\frac{4-p^2}{3}}y'^2\leqq\frac{1}{96}\cdot 6.212py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(40-21.1x+299.31x^2)(4x-x^2)$ we have, with $\alpha_1=1.2$, $\beta_1=0.2$, $\gamma_1=6.132$,

$$\Re a_8\leqq 8-\frac{x^3}{96}\hat{P}_1(x)-\frac{1}{96}Q_1'-\frac{1}{96}R_1'-\frac{1}{96}S_1'y-\frac{1}{96}T_1'\eta-\frac{1}{96}V_1'\xi,$$

$$\hat{P}_1(x)=440.26-1766.99x+1075.75x^2-265.825x^3+25.77x^4,$$

$$\begin{aligned}Q_1'=&(2.8125p^5-12.5p^4+196.398p^3+629.93p^2-3505.62p+3441.12-5.625p^3x'^2)y^2\\&+2(9.375p^4+130.5p^2-216p-9.375p^2x'^2+198x'^2)y\eta\\&+(31.25p^3+1496.55p^2-5781.7p+5783.2)\eta^2+2(11.25p^3-36p-11.25px'^2)y\xi\\&+2(37.5p^2-96)\eta\xi+(2095.17p^2-8300.48p+8365.28)\xi^2,\end{aligned}$$

$$R_1'=(118.6875p^5-36p^4+16p^3+299.31p^2-1176.14p+1195.04$$
$$-195.9375p^3x'^2-9.84x'^2+40.6875px'^4)x'^2$$
$$+2(108.375p^4+36p^3+12p^2-317.25p^2x'^2+33x'^4)x'y'$$
$$+(2.8125p^5+192.75p^3+914.442p^2-3525.632p+3519.072$$
$$-5.625p^3x'^2-497.25px'^2)y'^2+2(186.75p^3-182.25px'^2)x'\eta'$$
$$+2(9.375p^4+244.5p^2-9.375p^2x'^2-234x'^2)y'\eta'$$
$$+(31.25p^3+1496.55p^2-5637.7p+5975.2)\eta'^2+2(204p^2-66x'^2)x'\xi'$$
$$+2(11.25p^3+216p-11.25px'^2)y'\xi'+2(37.5p^2+192)\eta'\xi'$$
$$+(2095.17p^2-8187.98p+8365.28)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$
$$+(2693.79p^2-10585.26p+10755.36)\varphi'^2+2\cdot 96x'\tau'$$
$$+(3292.41p^2-12937.54p+13145.44)\tau'^2,$$
$$S_1'=(531p^3-315.75px'^2)x'^2+2(357p^2-144x'^2)x'y'+(-11.25p^3+396p)y'^2$$
$$+2(24.375p^3+310.5p)x'\eta'+2(-37.5p^2+144)y'\eta'+2(22.5p^2+144)x'\xi'-2\cdot 45py'\xi',$$
$$T_1'=(508.5p^2-84x'^2)x'^2+2(-24.375p^3+337.5p)x'y'+37.5p^2y'^2+2\cdot 24x'\eta'-2\cdot 22.5px'\xi'$$
$$V_1'=348px'^2+2(-22.5p^2+72)x'y'+45py'^2+2\cdot 22.5px'\eta'.$$

Since $y\geqq 0$, $\eta\geqq 0$, $\xi\geqq 0$, we have, for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$

$$-S_1'y\leqq\{-2(24.375p^3+310.5p)x'\eta'-2(-37.5p^2+144)y'\eta'$$
$$-2(22.5p^2+144)x'\xi'+2\cdot 45py'\xi'\}y$$
$$\leqq(24.375\alpha_2p^3+310.5\alpha_2p)y^2+(24.375\alpha_2^{-1}p^3+310.5\alpha_2^{-1}p)x'^2\eta'^2$$
$$+(-37.5\alpha_3p^2+144\alpha_3)y^2+(6.25\alpha_3^{-1}p^4-49\alpha_3^{-1}p^2+96\alpha_3^{-1})y'^2$$
$$+(3.75\alpha_3^{-1}p^4-29.4\alpha_3^{-1}p^2+57.6\alpha_3^{-1})\eta'^2$$
$$+(22.5\alpha_4p^2+144\alpha_4)y^2+(22.5\alpha_4^{-1}p^2+144\alpha_4^{-1})x'^2\xi'^2$$
$$+45\alpha_5py^2+(-6.4286\alpha_5^{-1}p^3+25.7144\alpha_5^{-1}p)y'^2,$$
$$-T_1'\eta\leqq\{-2\cdot 32.36px'y'-2\cdot 24x'\eta'+2\cdot 22.5px'\xi'\}\eta$$
$$\leqq 32.36\beta_2p\eta^2+32.36\beta_2^{-1}px'^2y'^2+24\beta_3\eta^2+24\beta_3^{-1}x'^2\eta'^2$$
$$+22.5\beta_4p\eta^2+22.5\beta_4^{-1}px'^2\xi'^2,$$

and

$$-V_1'\xi \leqq -2\cdot 22.5px'\eta'\xi \leqq 22.5\gamma_2 p\xi^2 + 22.5\gamma_2^{-1}px'^2\eta'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=0.168$, $\alpha_5=0.672$, $\beta_2=\beta_3=\beta_4=3.059$, $\gamma_2=1$,

$$\Re a_8 \leqq 8 - \frac{x^3}{96}\hat{P}_1(x) - \frac{1}{96}\hat{Q}_1 - \frac{1}{96}\hat{R}_1,$$

$$\begin{aligned}\hat{Q}_1 =& (2.8125p^5 - 12.5p^4 + 192.303p^3 + 632.45p^2 - 3588.024p \\ &+ 3392.736 - 5.625p^3x'^2)y^2 + 2(9.375p^4 + 130.5p^2 - 216p \\ &- 9.375p^2x'^2 + 198x'^2)y\eta + (31.25p^3 + 1496.55p^2 - 5949.51674p \\ &+ 5709.784)\eta^2 + 2(11.25p^3 - 36p - 11.25px'^2)y\xi + 2(37.5p^2 - 96)\eta\xi \\ &+ (2095.17p^2 - 8322.98p + 8365.28)\xi^2,\end{aligned}$$

$$\begin{aligned}\hat{R}_1 =& (118.6875p^5 - 36p^4 + 16p^3 + 299.31p^2 - 1176.14p + 1195.04 \\ &- 195.9375p^3x'^2 - 9.84x'^2 + 40.6875px'^4)x'^2 \\ &+ 2(108.375p^4 + 36p^3 + 12p^2 - 317.25p^2x'^2 + 33x'^4)x'y' \\ &+ (2.8125p^5 - 37.20625p^4 + 202.3221p^3 + 1206.139p^2 - 3563.9208p \\ &\quad + 2947.584 - 5.625p^3x'^2 - 507.832px'^2)y'^2 \\ &+ 2(186.75p^3 - 182.25px'^2)x'\eta' + 2(9.375p^4 + 244.5p^2 \\ &\quad - 9.375p^2x'^2 - 234x'^2)y'\eta' \\ &+ (-22.32375p^4 + 31.25p^3 + 1671.5682p^2 - 5637.7p + 5632.3072 \\ &\quad - 145.1044p^3x'^2 - 1870.9065px'^2 - 7.848x'^2)\eta'^2 \\ &+ 2(204p^2 - 66x'^2)x'\xi' + 2(11.25p^3 + 216p - 11.25px'^2)y'\xi' \\ &+ 2(37.5p^2 + 192)\eta'\xi' + (2095.17p^2 - 8187.98p + 8365.28 \\ &\quad - 133.9425p^2x'^2 - 7.3575px'^2 - 857.232x'^2)\xi'^2 \\ &+ 2\cdot 168px'\varphi' + 2\cdot 144y'\varphi' + (2693.79p^2 - 10585.26p + 10755.36)\varphi'^2 \\ &+ 2\cdot 96x'\tau' + (3292.41p^2 - 12937.54p + 13145.44)\tau'^2.\end{aligned}$$

$\hat{P}_1(x)$ is monotone decreasing for $0.1 \leqq x \leqq 0.3$ and $\hat{P}_1(0.3) > 0$. Hence $\hat{P}_1(x) > 0$ for $0.1 \leqq x \leqq 0.3$.

Since $y\eta \geqq 0$, $\eta\xi \geqq 0$, we may consider $Q_1^* = \hat{Q}_1 - 2(9.375p^4 + 130.5p^2 - 216p - 9.375p^2x'^2 + 198x'^2)y\eta - 2(37.5p^2 - 96)\eta\xi$. It is easy to prove that Q_1^* is non-negative for $1.7 \leqq p \leqq 1.9$, $|x'/p'| \leqq 1/20$. Hence $\hat{Q}_1$ is non-negative for $1.7 \leqq p \leqq 1.9$, $|x'/p'| \leqq 1/20$.

In section 4 we shall prove the positive definiteness of $\hat{R}_1$ for $1.7 \leqq p \leqq 1.9$, $|x'/p'| \leqq 1/20$. Therefore we have $\Re a_8 < 8$ for $1.7 \leqq p \leqq 1.9$, $|x'/p'| \leqq 1/20$, $y \geqq 0$, $\eta \geqq 0$, $\xi \geqq 0$.

Case 2. $-2py/3 \leqq \eta \leqq 0$.

We start from (A_1) with $A=3/2$, $B=9/8$, $C=1/2$. We remark the following facts:

$$\frac{1}{96}(9p^4+18p^3+60p^2)xy \leqq \frac{\alpha_1}{96}(4.5p^4+9p^3+30p^2)x^2+\frac{1}{96\alpha_1}(4.5p^4+9p^3+30p^2)y^2,$$

$$\frac{27}{32}p^4xy \leqq \frac{27}{64}\beta_1 p^4x^2+\frac{27}{64\beta_1}p^4y^2,$$

$$-\frac{5}{4}x'^2\varphi \leqq \frac{60}{96}\gamma_1\varphi^2+\frac{60}{96\gamma_1}x'^4,$$

$$\left(\frac{25}{64}p^2-\frac{3}{4}\right)y^2\eta \leqq 0.$$

Further by Lemma 1, for $1.7 \leqq p \leqq 1.9$, $|x'/p| \leqq 1/20$

$$\frac{3}{8}xp^3\eta \leqq \frac{1}{96}(9p^3x'^2+27p^3y'^2+45p^3\eta'^2+63p^3\xi'^2)\eta,$$

$$\left(\frac{15}{4\cdot 32}p^3-\frac{3}{8}p\right)y^3 \leqq \frac{1}{96}(6.852p^3-19.8p)y^2,$$

$$\frac{15}{32}py^2\xi \leqq \frac{15\delta_1}{64}py^4+\frac{15}{64\delta_1}p\xi^2 \leqq \frac{1}{96}(-7.5\delta_1p^3+30\delta_1p)y^2+\frac{22.5}{96\delta_1}p\xi^2,$$

$$-\frac{15}{4}y^2x'y' \leqq \frac{513}{128}px'^2y^2+\frac{1}{96}(-16.512p^2+66.048)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{1}{96}\cdot 6.212py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(40+1.5x+193.88x^2)(4x-x^2)$ we have, with $\alpha_1=1.3$, $\beta_1=2.6$, $\gamma_1=6.313$, $\delta_1=0.2$,

$$\Re a_8 \leqq 8-\frac{x^3}{96}\hat{P}_2(x)-\frac{1}{96}Q_2'-\frac{1}{96}R_2'-\frac{1}{96}S_2'y-\frac{1}{96}T_2'\eta-\frac{1}{96}V_2'\xi,$$

$$\hat{P}_2(x)=176.18-644.42x+181.85x^2+33.025x^3-13x^4,$$

$$\begin{aligned}Q_2'=&(2.8125p^5-19.039p^4+196.9749p^3+315.563p^2-2308.26p\\&+2311.56-5.625p^3x'^2)y^2\\&+2(9.375p^4+46.5p^2-108p-9.375p^2x'^2+198x'^2)y\eta\\&+(31.25p^3+969.4p^2-3954.1p+4044.6)\eta^2+2(11.25p^3-36p-11.25px'^2)y\xi\\&+2(37.5p^2-144)\eta\xi+(1357.16p^2-5506.64p+5729.64)\xi^2,\end{aligned}$$

$$R_2' = (100.6875p^5 + 16p^3 + 193.88p^2 - 777.02p + 818.52$$
$$-195.9375p^3x'^2 - 9.54x'^2 + 40.6875px'^4)x'^2$$
$$+2(126.375p^4 + 12p^2 - 317.25p^2x'^2 + 33x'^4)x'y'$$
$$+(2.8125p^5 + 192.75p^3 + 598.152p^2 - 2328.272p + 2389.512$$
$$-5.625p^3x'^2 - 497.25px'^2)y'^2$$
$$+2(186.75p^3 - 182.25px'^2)x'\eta' + 2(9.375p^4 + 244.5p^2 - 9.375p^2x'^2 - 234x'^2)y'\eta'$$
$$+(31.25p^3 + 969.4p^2 - 3642.1p + 4092.6)\eta'^2 + 2(204p^2 - 66x'^2)x'\xi'$$
$$+2(11.25p^3 + 216p - 11.25px'^2)y'\xi' + 2(37.5p^2 + 192)\eta'\xi'$$
$$+(1357.16p^2 - 5394.14p + 5729.64)\xi'^2 + 2\cdot 168px'\varphi' + 2\cdot 144y'\varphi'$$
$$+(1744.92p^2 - 6993.18p + 7366.68)\varphi'^2 + 2\cdot 96x'\tau'$$
$$+(2132.68p^2 - 8547.22p + 9003.72)\tau'^2,$$
$$S_2' = (531p^3 - 315.75px'^2)x'^2 + 2(357p^2 - 144x'^2)x'y' + (-11.25p^3 + 396p)y'^2$$
$$+2(24.375p^3 + 310.5p)x'\eta' + 2(-37.5p^2 + 144)y'\eta'$$
$$+2(22.5p^2 + 144)x'\xi' - 2\cdot 45py'\xi',$$
$$T_2' = (-9p^3 + 508.5p^2 - 84x'^2)x'^2 + 2(-24.375p^3 + 337.5p)x'y'$$
$$+(-27p^3 + 37.5p^2)y'^2 + 2\cdot 24x'\eta' - 45p^3\eta'^2 - 2\cdot 22.5px'\xi' - 63p^3\xi'^2,$$
$$V_2' = 348px'^2 + 2(-22.5p^2 + 72)x'y' + 45py'^2 + 2\cdot 22.5px'\eta'.$$

Now, since $y \geqq 0$, $0 \geqq \eta \geqq -2py/3$, $\xi \geqq 0$, we have

$$-V_2'\xi \leqq -2\cdot 22.5px'\eta'\xi \leqq 22.5\gamma_2 p\xi^2 + 22.5p\gamma_2^{-1}x'^2\eta'^2,$$
$$-T_2'\eta \leqq -\{(-2.507p^3 + 508.5p^2 - 84x'^2)x'^2 + 2\cdot 127.81px'y'$$
$$+(-15p^3 + 37.5p^2)y'^2\}\eta - 2(-24.375p^3 + 187.31p)x'y'\eta$$
$$\leqq -T_2^*\eta + (-24.375\beta_2 p^3 + 187.31\beta_2 p)\eta^2 + (-24.375\beta_2^{-1}p^3 + 187.31\beta_2^{-1}p)x'^2\eta'^2,$$
$$T_2^* = (-2.507p^3 + 508.5p^2 - 84x'^2)x'^2 + 2\cdot 127.81px'y'$$
$$+(-15p^3 + 37.5p^2)y'^2 \geqq 0,$$
$$-\left(S_2' - \frac{2}{3}pT_2^*\right)y \leqq -\{2\cdot 17.014p^2x'y' + 2(24.375p^3 + 310.5p)x'\eta'$$
$$+2(-37.5p^2 + 144)y'\eta' + 2(22.5p^2 + 144)x'\xi' - 2\cdot 45py'\xi'\}y$$

$$\leqq 17.014\alpha_2 p^2 y^2 + 17.014\alpha_2^{-1} p^2 x'^2 y'^2$$
$$+(24.375\alpha_3 p^3 + 310.5\alpha_3 p) y^2 + (24.375\alpha_3^{-1} p^3 + 310.5\alpha_3^{-1} p) x'^2 \eta'^2$$
$$+(144\alpha_4 - 37.5\alpha_4 p^2) y^2 + (6.25\alpha_4^{-1} p^4 - 49\alpha_4^{-1} p^2 + 96\alpha_4^{-1}) y'^2$$
$$+(3.75\alpha_4^{-1} p^4 - 29.4\alpha_4^{-1} p^2 + 57.6\alpha_4^{-1}) \eta'^2$$
$$+(22.5\alpha_5 p^2 + 144\alpha_5) y^2 + (22.5\alpha_5^{-1} p^2 + 144\alpha_5^{-1}) x'^2 \xi'^2$$
$$+45\alpha_6 p y^2 + (-6.4286\alpha_6^{-1} p^3 + 25.7144\alpha_6^{-1} p) y'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0.112$, $\alpha_6=0.448$, $\beta_2=0.6$, $\gamma_2=1$,

$$\Re a_8 \leqq 8 - \frac{x^3}{96}\hat{P}_2(x) - \frac{1}{96}\hat{Q}_2 - \frac{1}{96}\hat{R}_2,$$

$$\hat{Q}_2 = (2.8125p^5 - 19.039p^4 + 194.2449p^3 + 315.3374p^2 - 2363.196p$$
$$+2279.304 - 5.625p^3 x'^2) y^2$$
$$+2(9.375p^4 + 46.5p^2 - 108p - 9.375p^2 x'^2 + 198x'^2) y\eta$$
$$+(45.875p^3 + 969.4p^2 - 4066.486p + 4044.6p)\eta^2$$
$$+2(11.25p^3 - 36p - 11.25px'^2) y\xi + 2(37.5p^2 - 144)\eta\xi$$
$$+(1357.16p^2 - 5529.14p + 5729.64)\xi^2,$$

$$\hat{R}_2 = (100.6875p^5 + 16p^3 + 193.88p^2 - 777.02p + 818.52$$
$$-195.9375p^3 x'^2 - 9.54x'^2 + 40.6875px'^4) x'^2$$
$$+2(126.375p^4 + 12p^2 - 317.25p^2 x'^2 + 33x'^4) x'y'$$
$$+(2.8125p^5 - 55.80625p^4 + 207.105p^3 + 1035.673p^2 - 2385.693p$$
$$+1532.328 + 35.008p^3 x'^2 - 151.919p^2 x'^2 - 809.496px'^2) y'^2$$
$$+2(186.75p^3 - 182.25px'^2) x'\eta' + 2(9.375p^4 + 244.5p^2 - 9.375p^2 x'^2 - 234x'^2) y'\eta'$$
$$+(-33.48375p^4 + 31.25p^3 + 1231.9126p^2 - 3642.1p$$
$$+3578.2896 - 217.6444p^3 x'^2 - 2794.9545px'^2)\eta'^2$$
$$+2(204p^2 - 66x'^2) x'\xi' + 2(11.25p^3 + 216p - 11.25px'^2) y'\xi' + 2(37.5p^2 + 192)\eta'\xi'$$
$$+(1357.16p^2 - 5394.14p + 5729.64 - 200.9025p^2 x'^2 - 1285.776x'^2)\xi'^2$$
$$+2\cdot 168px'\varphi' + 2\cdot 144y'\varphi' + (1744.92p^2 - 6993.18p + 7366.68)\varphi'^2$$
$$+2\cdot 96x'\tau' + (2132.68p^2 - 8547.22p + 9003.72)\tau'^2.$$

$\hat{P}_2(x)$ is monotone decreasing for $0.1\leqq x\leqq 0.3$ and $\hat{P}_2(0.3)>0$. Hence $\hat{P}_2(x)>0$ for $0.1\leqq x\leqq 0.3$.

Since $\eta\xi\leqq 0$ we may consider $Q_2^*=\hat{Q}_2-2(37.5p^2-144)\eta\xi$. We can prove the positive definiteness of the symmetric matrix associated with Q_2^* for $1.7\leqq p\leqq 1.9$, $|x'/p'|\leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_2$ is non-negative for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$.

In section 4 we shall prove the positive definiteness of $\hat{R}_2$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$. Therefore we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$, $y\geqq 0$, $0\geqq\eta\geqq -2py/3$, $\xi\geqq 0$.

Case 3. $\eta\leqq -2py/3$.

We start from (A) with $\alpha=0$ and apply Lemma 3 to the term $(27p^4/640)\cdot(\eta+7py/3)$. Then we have

$$\begin{aligned}\Re a_8\leqq\ & U+\frac{9}{80}p-\frac{9}{64\cdot 80}p^7+\frac{867}{64\cdot 20}p^3-\frac{867}{64\cdot 80}p^5\\ &+Q_3+R_3+S_3y+T_3\eta+V_3\xi\\ &+\frac{3}{640}p^4\left(\eta+\frac{2}{3}py\right)+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{153}{64\cdot 40}(p^4+2p^3)xy+\frac{3}{4}Bp^4xy\\ &+\frac{3}{4}Cp^3x\eta+\left\{\frac{27}{64\cdot 4}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3-\frac{5}{4}x'^2\varphi\\ &+\left(\frac{99}{64\cdot 4}p^2+2A-A^2-2AC\right)y^2\eta+\frac{27}{64}py^2\xi+\left(\frac{-51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,\end{aligned}$$

$$\begin{aligned}Q_3=&\left\{\frac{-27}{16\cdot 64}p^5+\left(\frac{8191}{64\cdot 80}-\frac{5}{8}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2-\frac{27}{320}p\right.\\ &\left.+\frac{27}{64\cdot 8}p^3x'^2+\left(\frac{-6801}{64.\ 20}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2\\ &+\left\{\frac{-99}{64\cdot 8}p^4+\left(\frac{2663}{640}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\frac{99}{64\cdot 8}p^2x'^2\right.\\ &\left.+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta\\ &+\left\{\frac{-363}{64\cdot 16}p^3+\left(\frac{639}{320}-2C\right)p+C^2x\right\}\eta^2+\left\{\frac{-27}{128}p^3+\left(\frac{9}{2}-2B-A\right)p+\frac{27}{128}px'^2\right\}y\xi\\ &+\left(\frac{-99}{128}p^2-2C+4\right)\eta\xi-\frac{27}{64}p\xi^2+(3-2A)y\varphi,\end{aligned}$$

$$R_3=\left(\frac{-5367}{64\cdot 80}p^5-\frac{867}{64\cdot 80}p^3+\frac{10453}{64\cdot 80}p^3x'^2-\frac{2169}{64\cdot 80}px'^4\right)x'^2$$

$$+\left(\frac{-6731}{64\cdot 40}p^4-\frac{153}{640}p^2+\frac{16901}{64\cdot 40}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(\frac{-27}{64\cdot 16}p^5-\frac{10369}{64\cdot 80}p^3-\frac{27}{320}p+\frac{27}{64\cdot 8}p^3x'^2+\frac{6639}{64\cdot 20}px'^2\right)y'^2$$

$$+\left(\frac{-2481}{640}p^3+\frac{2427}{640}px'^2\right)x'\eta'+\left(\frac{-99}{64\cdot 8}p^4-\frac{3257}{640}p^2+\frac{99}{64\cdot 8}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(\frac{-363}{64\cdot 16}p^3-\frac{801}{320}p\right)\eta'^2+\left(\frac{-17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'+\left(\frac{-27}{128}p^3-\frac{9}{2}p+\frac{27}{128}px'^2\right)y'\xi'$$

$$+\left(\frac{-99}{128}p^2-4\right)\eta'\xi'-\frac{27}{64}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau',$$

$$S_3=\left\{\left(\frac{5}{16}A-\frac{15381}{64\cdot 40}\right)p^3+\frac{4213}{64\cdot 20}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{27}{64\cdot 4}p^3-\left(\frac{1}{2}A+\frac{27}{8}\right)p\right\}y'^2+\left(\frac{-63}{128}p^3-\frac{2079}{320}p\right)x'\eta'+\left(\frac{99}{128}p^2-2A\right)y'\eta'$$

$$+\left(\frac{-27}{64}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{54}{64}py'\xi',$$

$$T_3=\left(\frac{-3399}{640}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{63}{128}p^3-\frac{2241}{320}p\right)x'y'-\frac{99}{64\cdot 4}p^2y'^2-\frac{1}{2}x'\eta'+\frac{27}{64}px'\xi',$$

$$V_3=\frac{-29}{8}px'^2+\left(\frac{27}{64}p^2-\frac{3}{2}\right)x'y'-\frac{27}{64}py'^2-\frac{27}{64}px'\eta'.$$

Here we put $A=3/2$, $B=19/16$, $C=19/32$. We remark the following facts:

$$\frac{3}{640}p^4\left(\eta+\frac{2}{3}py\right)\leqq\frac{1}{96}\cdot 180p^2x'^2\eta+\frac{1}{96}\cdot 120p^3x'^2y,$$

$$\frac{57}{128}p^3x\eta+\frac{19}{64}p^4xy=\frac{57}{128}p^3x\left(\eta+\frac{2}{3}py\right)\leqq 0,$$

$$\frac{1}{64\cdot 40}(1767p^4+494p^3+1600p^2)xy$$

$$\leqq\frac{\alpha_1}{64\cdot 80}(1767p^4+494p^3+1600p^2)x^2+\frac{1}{64\cdot 80\alpha_1}(1767p^4+494p^3+1600p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{1}{96}60\gamma_1\varphi^2+\frac{1}{96\gamma_1}60x'^4.$$

Further by Lemma 1

$$\left(\frac{27}{64\cdot 4}p^3-\frac{9}{16}\right)y^3+\left(\frac{99}{64\cdot 4}p^2-\frac{33}{32}\right)y^2\eta \leqq \frac{7}{64\cdot 4}p^3y^3+\frac{7.65}{64\cdot 4}p^2y^2\eta$$

$$\leqq \frac{1.9}{64\cdot 4}p^3y^3 \leqq \frac{0.434}{96}p^3y^2,$$

$$\frac{27}{64}py^2\xi \leqq \frac{27\beta_1}{128}py^4+\frac{27}{128\beta_1}p\xi^2 \leqq \frac{1}{96}(-6.75\beta_1p^3+27\beta_1p)y^2+\frac{20.25}{96\beta_1}p\xi^2,$$

$$-\frac{15}{4}y^2x'y' \leqq \frac{5121}{64\cdot 20}px'^2y^2+\frac{1}{96}(-16.5409p^2+66.1636)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{1}{96}6.212py'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(39.75+29.09x+116.82x^2)(4x-x^2)$ we have, with $\alpha_1=2.65$, $\beta_1=0.2$, $\gamma_1=6.57$,

$$\Re a_8 \leqq 8-\frac{x^3}{96}\hat{P}_3(x)-\frac{1}{96}Q_3'-\frac{1}{96}R_3'-\frac{1}{96}S_3'y-\frac{1}{96}T_3'\eta-\frac{1}{96}V_3'\xi,$$

$$\hat{P}_3(x)=82.1625-281.28875x+9.196875x^2+56.114x^3-12.209x^4,$$

$$\begin{aligned}Q_3'=&(2.53125p^5-12.5024p^4+229.7144p^3+68.3892p^2-1486.41p\\&+1551.63-5.0625p^3x'^2)y^2\\&+2(9.28125p^4+71.5875p^2-135.375p-9.28125p^2x'^2+198x'^2)y\eta\\&+(34.03125p^3+584.1p^2-2525.70625p+2758.3625)\eta^2\\&+2(10.125p^3-30p-10.125px'^2)y\xi+2(37.125p^2-135)\eta\xi\\&+(817.74p^2-3535.34p+3956.47)\xi^2,\end{aligned}$$

$$\begin{aligned}R_3'=&(100.63125p^5+16.25625p^3+116.82p^2-496.37p+565.21\\&-195.99375p^3x'^2-9.14x'^2+40.66875px'^4)x'^2\\&+2(126.20625p^4+11.475p^2-316.89375p^2x'^2+33x'^4)x'y'\\&+(2.53125p^5+194.41875p^3+367.0009p^2-1487.222p+1629.4664\\&-5.0625p^3x'^2-497.925px'^2)y'^2\\&+2(186.075p^3-182.025px'^2)x'\eta'+2(9.28125p^4+244.275p^2\\&-9.28125p^2x'^2-234x'^2)y'\eta'\\&+(34.03125p^3+584.1p^2-2241.55p+2826.05)\eta'^2+2(204p^2-66x'^2)x'\xi'\\&+2(10.125p^3+216p-10.125px'^2)y'\xi'+2(37.125p^2+192)\eta'\xi'\end{aligned}$$

$$+(817.74p^2-3434.09p+3956.47)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$

$$+(1051.38p^2-4467.33p+5086.89)\varphi'^2+2\cdot 96x'\tau'$$

$$+(1285.02p^2-5460.07p+6217.31)\tau'^2,$$

$$S_3'=(411.7875p^3-315.975px'^2)x'^2+2(357p^2-144x'^2)x'y'+(-10.125p^3+396p)y'^2$$

$$+2(23.625p^3+311.85p)x'\eta'+2(-37.125p^2+144)y'\eta'$$

$$+2(20.25p^2+144)x'\xi'-2\cdot 40.5py'\xi',$$

$$T_3'=(329.85p^2-84x'^2)x'^2+2(-23.625p^3+336.15p)x'y'+37.125p^2y'^2$$

$$+2\cdot 24x'\eta'-2\cdot 20.25px'\xi',$$

$$V_3'=348px'^2+2(-20.25p^2+72)x'y'+40.5py'^2+2\cdot 20.25px'\eta'.$$

Since $y\geqq 0$, $-2py/3\geqq\eta$, $\xi\geqq 0$, we have

$$-V_3'\xi\leqq -2\cdot 20.25px'\eta'\xi\leqq 20.25\gamma_2 p\xi^2+20.25\gamma_2^{-1}px'^2\eta'^2,$$

$$-T_3'\eta\leqq 164.925\beta_2p^2\eta^2+164.925\beta_2^{-1}p^2x'^4$$

$$+(-23.625p^3+336.15p)\beta_3\eta^2+(-23.625p^3+336.15p)\beta_3^{-1}x'^2y'^2$$

$$+18.5625\beta_4p^2\eta^2+(-6.1875p^4+24.75p^2)\beta_4^{-1}y'^2$$

$$+24\beta_5\eta^2+24\beta_5^{-1}x'^2\eta'^2+20.25\beta_6p\eta^2+20.25\beta_6^{-1}px'^2\xi'^2,$$

$$-S_3'y\leqq -\{2(23.625p^3+311.85p)x'\eta'+2(-37.125p^2+144)y'\eta'$$

$$+2(20.25p^2+144)x'\xi'-2\cdot 40.5py'\xi'\}y$$

$$\leqq(23.625p^3+311.85p)\alpha_2y^2+(23.625p^3+311.85p)\alpha_2^{-1}x'^2\eta'^2$$

$$+(-37.125p^2+144)\alpha_3y^2+(12.375p^4-97.5p^2+192)\alpha_3^{-1}\eta'^2$$

$$+(20.25p^2+144)\alpha_4y^2+(20.25p^2+144)\alpha_4^{-1}x'^2\xi'^2$$

$$+40.5\alpha_5py^2+(-5.7858p^3+23.1432p)\alpha_5^{-1}y'^2.$$

Hence we have, putting $\alpha_2=\alpha_3=\alpha_4=0.15$, $\alpha_5=0.6$, $\beta_2=\beta_3=\beta_5=\beta_6=0.181$, $\beta_4=0.543$, $\gamma_2=2$,

$$\Re a_8\leqq 8-\frac{x^3}{96}\hat{P}_3(x)-\frac{1}{96}\hat{Q}_3-\frac{1}{96}\hat{R}_3,$$

$$\hat{Q}_3=(2.53125p^5-12.5024p^4+226.17065p^3+70.92045p^2-1557.4875p$$

$$+1508.43-5.0625p^3x'^2)y^2$$

$$+2(9.28125p^4+71.5875p^2-135.375p-9.28125p^2x'^2+198x'^2)y\eta$$

$$+(38.30737p^3+544.16913p^2-2590.21465p+2754.0185)\eta^2$$

$$+2(10.125p^3-30p-10.125px'^2)y\xi+2(37.125p^2-135)\eta\xi$$

$$+(817.74p^2-3575.84p+3956.47)\xi^2,$$

$$\hat{R}_3=(100.63125p^5+16.25625p^3+116.82p^2-496.37p+565.21$$

$$-195.99375p^3x'^2-911.211p^2x'^2-9.18x'^2+40.66875px'^4)x'^2$$

$$+2(126.20625p^4+11.475p^2-316.89375p^2x'^2+33x'^4)x'y'$$

$$+(2.53125p^5+11.39737p^4+204.06367p^3+321.4114p^2-1525.80172p$$

$$+1629.4664+125.4656p^3x'^2-2355.15375px'^2)y'^2$$

$$+2(186.075p^3-182.025px'^2)x'\eta'$$

$$+2(9.28125p^4+244.275p^2-9.28125p^2x'^2-234x'^2)y'\eta'$$

$$+(-82.504125p^4+34.03125p^3+1234.1325p^2-2241.55p+1545.986$$

$$-157.507875p^3x'^2-2089.229px'^2-132.6x'^2)\eta'^2$$

$$+2(204p^2-66x'^2)x'\xi'+2(10.125p^3+216p-10.125px'^2)y'\xi'+2(37.125p^2+192)\eta'\xi'$$

$$+(817.74p^2-3434.09p+3956.47-135.0068p^2x'^2-111.8813px'^2-960.048x'^2)\xi'^2$$

$$+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'+(1051.38p^2-4467.33p+5086.89)\varphi'^2$$

$$+2\cdot 96x'\tau'+(1285.02p^2-5460.07p+6217.31)\tau'^2.$$

$\hat{P}_3(x)$ is monotone decreasing for $0.1\leqq x\leqq 0.3$ and $\hat{P}_3(0.3)>0$. Hence $\hat{P}_3(x)>0$ for $0.1\leqq x\leqq 0.3$.

Since $\eta\xi\leqq 0$, we may consider $Q_3^*=\hat{Q}_3-2(37.125p^2-135)\eta\xi$. We can prove the positive definiteness of the symmetric matrix associated with Q_3^* for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_3$ is non-negative for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$.

In section 4 we shall prove the positive definiteness of $\hat{R}_3$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$. Therefore we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$, $y\geqq 0$, $-2py/3\geqq\eta$, $\xi\geqq 0$.

§ **3.** In this section we are concerned with the case $\xi\leqq 0$. We divide this case into several subcases.

Case 1. $\eta\geqq 0$.

We start from (A) with $\alpha=7/160$. Applying Lemma 3 to the term $(37p^4/640)\cdot(\eta+15py/8)$ we have

$$\Re a_8 \leqq U+\frac{37}{240}p+\frac{13357}{640\cdot 32}p^3-\frac{13357}{640\cdot 128}p^5-\frac{37}{240\cdot 64}p^7-\frac{6671}{64\cdot 25600}p^6x$$

$$+Q_4+R_4+S_4y+T_4\eta+V_4\xi$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{703}{640\cdot 16}(p^4+2p^3)xy+\frac{473}{640}Bp^4xy+\frac{473}{640}Cp^3x\eta$$

$$+\frac{7}{640}p^3\xi-\frac{5}{4}x'^2\varphi+\left\{\frac{37}{32\cdot 8}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3$$

$$+\left(\frac{407}{32\cdot 32}p^2+2A-A^2-2AC\right)y^2\eta+\frac{37}{64}py^2\xi+\left(\frac{-51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,$$

(A₂)

$$Q_4=\left\{\frac{-37}{128\cdot 8}p^5+\left(\frac{135049}{128\cdot 640}-\frac{393}{640}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$

$$\left.-\frac{37}{320}p+\frac{37}{64\cdot 8}p^3x'^2+\left(\frac{-6831}{320\cdot 4}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{\frac{-407}{128\cdot 16}p^4+\left(\frac{10619}{128\cdot 20}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp+\frac{407}{128\cdot 16}p^2x'^2\right.$$

$$\left.+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{\frac{-4477}{128\cdot 128}p^3+\left(\frac{609}{320}-2C\right)p+C^2x\right\}\eta^2+\left\{\frac{-37}{128}p^3+\left(\frac{9}{2}-A-2B\right)p\right.$$

$$\left.+\frac{37}{128}px'^2\right\}y\xi$$

$$+\left\{\frac{-407}{32\cdot 16}p^2+(4-2C)\right\}\eta\xi+(3-2A)y\varphi-\frac{37}{64}p\xi^2,$$

$$R_4=\left(\frac{-5377}{640\cdot 8}p^5-\frac{13357}{640\cdot 128}p^3+\frac{10443}{640\cdot 8}p^3x'^2-\frac{6517}{640\cdot 24}px'^4\right)x'^2$$

$$+\left(\frac{-27037}{640\cdot 16}p^4-\frac{703}{640\cdot 4}p^2+\frac{13571}{128\cdot 16}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(\frac{-37}{128\cdot 8}p^5-\frac{161911}{128\cdot 640}p^3-\frac{37}{320}p+\frac{37}{64\cdot 8}p^3x'^2+\frac{6609}{640\cdot 2}px'^2\right)y'^2$$

$$+\left(\frac{-2511}{640}p^3+\frac{2437}{640}px'^2\right)x'\eta'+\left(\frac{-407}{128\cdot 16}p^4-\frac{13061}{128\cdot 20}p^2+\frac{407}{128\cdot 16}p^2x'^2\right.$$

$$\left.+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(\frac{-4477}{128\cdot 128}p^3-\frac{831}{320}p\right)\eta'^2+\left(\frac{-17}{4}p^2+\frac{11}{8}x'^2\right)x'\xi'$$

$$+\left(\frac{-37}{128}p^3-\frac{9}{2}p+\frac{37}{128}px'^2\right)y'\xi'$$

$$+\left(\frac{-407}{32\cdot 16}p^2-4\right)\eta'\xi'-\frac{37}{64}p\xi'^2-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau',$$

$$S_4=\left\{\left(\frac{-61251}{640\cdot 16}+\frac{5}{16}A\right)p^3+\frac{4203}{640\cdot 2}px'^2\right\}x'^2+\left\{\left(\frac{13}{8}A-\frac{79}{8}\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{37}{128\cdot 2}p^3-\left(\frac{1}{2}A+\frac{27}{8}\right)p\right\}y'^2+\left(\frac{-555}{128\cdot 8}p^3-\frac{2049}{320}p\right)x'\eta'$$

$$+\left(\frac{407}{32\cdot 16}p^2-2A\right)y'\eta'+\left(\frac{-37}{64}p^2-\frac{3}{2}-A\right)x'\xi'+\frac{37}{32}py'\xi',$$

$$T_4=\left(\frac{-3369}{640}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{555}{128\cdot 8}p^3-\frac{2271}{320}p\right)x'y'-\frac{407}{32\cdot 32}p^2y'^2$$

$$-\frac{1}{2}x'\eta'+\frac{37}{64}px'\xi',$$

$$V_4=\frac{-29}{8}px'^2+\left(\frac{37}{64}p^2-\frac{3}{2}\right)x'y'-\frac{37}{64}px'\eta'-\frac{37}{64}py'^2.$$

Here we put $A=3/2$, $B=9/8$, $C=1$. We remark the following facts:

$$\frac{1}{640\cdot 48}(9313p^4+5382p^3+19200p^2)xy$$

$$\leqq\frac{\alpha_1}{640\cdot 96}(9313p^4+5382p^3+19200p^2)x^2$$

$$+\frac{1}{640\cdot 96\alpha_1}(9313p^4+5382p^3+19200p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi\leqq\frac{1}{96\cdot 32}1920\gamma_1\varphi^2+\frac{1}{96\cdot 32\gamma_1}1920x'^4,$$

$$\left(\frac{407}{32\cdot 32}p^2-\frac{9}{4}\right)y^2\eta\leqq 0,$$

$$\frac{37}{64}py^2\xi\leqq 0.$$

By Lemma 2,

$$\frac{473}{640}p^3x\left(\eta+\frac{5}{6}py\right)\leqq\frac{473}{640}p^3x\left(\frac{4}{9}xp^2+\frac{2}{3}-\frac{1}{12}p^3-\frac{1}{2}x'y'+\frac{1}{4}px'^2\right).$$

By Lemma 1,

$$\left(\frac{37}{32\cdot 8}p^3-\frac{3}{8}p\right)y^3\leqq\frac{1}{96\cdot 32}(270.396p^3-701.568p)y^2,$$

$$-\frac{15}{4}y^2x'y'\leqq\frac{12362.4}{96\cdot 32}px'^2y^2+\frac{1578.677}{96\cdot 32}y'^2y^2$$

$$\leqq\frac{12362.4}{96\cdot 32}px'^2y^2+\frac{1}{96\cdot 32}(-526.226p^2+2104.904)y'^2,$$

$$\frac{17}{8}x'y'^3\leqq\frac{1}{96\cdot 32}198.778py'^2.$$

Further, since $y\geqq 0$, $\eta\geqq 0$, $\xi\leqq 0$, we have

$$S_4y\leqq-\frac{1}{96\cdot 32}\{2(832.5p^3+9835.2p)x'\eta'+2(-1221p^2+4608)y'\eta'$$

$$+2(888p^2+4608)x'\xi'-2\cdot 1776py'\xi'\}y$$

$$\leqq\frac{\alpha_2}{96\cdot 32}(832.5p^3+9835.2p)y^2+\frac{\alpha_2^{-1}}{96\cdot 32}(832.5p^3+9835.2p)x'^2\eta'^2$$

$$+\frac{\alpha_3}{96\cdot 32}(-1221p^2+4608)y^2+\frac{\alpha_3^{-1}}{96\cdot 32}(203.5p^4-1582p^2+3072)y'^2$$

$$+\frac{\alpha_3^{-1}}{96\cdot 32}(122.1p^4-949.2p^2+1843.2)\eta'^2$$

$$+\frac{\alpha_4}{96\cdot 32}(888p^2+4608)y^2+\frac{\alpha_4^{-1}}{96\cdot 32}(888p^2+4608)x'^2\xi'^2$$

$$+\frac{\alpha_5}{96\cdot 32}1776py^2+\frac{\alpha_5^{-1}}{96\cdot 32}(-253.715p^3+1014.86p)y'^2,$$

$$T_4\eta\leqq-\frac{1}{96\cdot 32}\{2\cdot 942.5px'y'+2\cdot 768x'\eta'-2\cdot 888px'\xi'\}\eta$$

$$\leqq\frac{\beta_2}{96\cdot 32}\cdot 942.5p\eta^2+\frac{\beta_2^{-1}}{96\cdot 32}942.5px'^2y'^2$$

$$+\frac{\beta_3}{96\cdot 32}\cdot 768\eta^2+\frac{\beta_3^{-1}}{96\cdot 32}768x'^2\eta'^2+\frac{\beta_4}{96\cdot 32}888p\eta^2+\frac{\beta_4^{-1}}{96\cdot 32}888px'^2\xi'^2,$$

$$\frac{7}{640}p^3\xi+V_4\xi\leqq\frac{-1}{96\cdot 32}\{-2(888p^2-2304)x'y'+2\cdot 888px'\eta'+1776py'^2\}\xi$$

$$\leqq\frac{\gamma_2}{96\cdot 32}(888p^2-2304)\xi^2+\frac{\gamma_2^{-1}}{96\cdot 32}(888p^2-2304)x'^2y'^2$$

$$+\frac{\gamma_3}{96\cdot 32}\cdot 888p\xi^2+\frac{\gamma_3^{-1}}{96\cdot 32}888px'^2\eta'^2$$

$$+\frac{\gamma_4}{96\cdot 32}888p\xi^2+\frac{\gamma_4^{-1}}{96\cdot 32}(-296p^3+1184p)y'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(1434.63-856.41x+9576.01x^2)(4x-x^2)$ we have, with $\alpha_1=1.2$, $\alpha_2=\alpha_3=\alpha_4=0.157$, $\alpha_5=0.628$, $\beta_2=\beta_3=\beta_4=1$, $\gamma_1=6.772$, $\gamma_2=\gamma_3=\gamma_4=1$,

$$\Re a_8\leqq 8-\frac{x^3}{96\cdot 32}\hat{P}_4(x)-\frac{1}{96\cdot 32}\hat{Q}_4-\frac{1}{96\cdot 32}\hat{R}_4,$$

$$\hat{P}_4(x)=13977.2833-56096.3884x+34142.2266x^2-8436.7442x^3+818.1175x^4,$$

$$\begin{aligned}\hat{Q}_4=&(111p^5-388.042p^4+5923.914p^3+20204.311p^2-113945.577p\\&+108022.638-222p^3x'^2)y^2\\&+2(305.25p^4+4188.6p^2-6912p-305.25p^2x'^2+6336x'^2)y\eta\\&+(839.4375p^3+47880.05p^2-185699.05p+183217.25)\eta^2\\&+2(444p^3-1152p-444px'^2)y\xi+2(1221p^2-3072)\eta\xi\\&+(66144.07p^2-262133.41p+268484.95)\xi^2,\end{aligned}$$

$$\begin{aligned}\hat{R}_4=&(3793.8p^5-1135.2p^4+500.8875p^3+9576.01p^2-37447.63p\\&+38025.85-6265.8p^3x'^2-284.16x'^2+1303.4px'^4)x'^2\\&+2(3487.95p^4+1135.2p^3+421.8p^2-10178.25p^2x'^2+1056x'^4)x'y'\\&+(111p^5-1296.295p^4+6771.8304p^3+39331.596p^2-114987.14p\\&+92404.006-222p^3x'^2-888p^2x'^2-16804.1px'^2+2304x'^2)y'^2\\&+2(6026.4p^3-5848.8px'^2)x'\eta'\\&+2(305.25p^4+7836.6p^2-305.25p^2x'^2-7488x'^2)y'\eta'\\&+(-777.777p^4+839.4375p^3+53926.454p^2-179260.55p+178388.066\\&-5303.025p^3x'^2-63538.224px'^2-768x'^2)\eta'^2\\&+2(6528p^2-2112x'^2)x'\xi'+2(444p^3+6912p-444px'^2)y'\xi'+2(1221p^2+6144)\eta'\xi'\\&+(67032.07p^2-260357.41p+266180.95-5656.56p^2x'^2-888px'^2-29352.96x'^2)\xi'^2\\&+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'+(86184.09p^2-337028.67p+342232.65)\varphi'^2\\&+2\cdot 3072x'\tau'+(105336.11p^2-411923.93p+418284.35)\tau'^2.\end{aligned}$$

$\hat{P}_4(x)$ is monotone decreasing for $0.1 \leqq x \leqq 0.3$ and $\hat{P}_4(0.3) > 0$. Hence $\hat{P}_4(x) > 0$ for $0.1 \leqq x \leqq 0.3$.

Since $y\eta \geqq 0$, we may consider $Q_4^* = \hat{Q}_4 - 2(305.25p^4 + 4188.6p^2 - 6912p - 305.25p^2x'^2 + 6336x'^2)y\eta$. It is not so difficult to prove that Q_4^* is non-negative for $1.7 \leqq p \leqq 1.9$, $|x'/p| \leqq 1/20$. Hence $\hat{Q}_4$ is non-negative for $1.7 \leqq p \leqq 1.9$, $|x'/p| \leqq 1/20$.

In section 4 we shall prove the positive definiteness of $\hat{R}_4$ for $1.7 \leqq p \leqq 1.9$, $|x'/p| \leqq 1/20$. Therefore we have $\Re a_8 < 8$ for $1.7 \leqq p \leqq 1.9$, $|x'/p| \leqq 1/20$, $y \geqq 0$, $\eta \geqq 0$, $\xi \leqq 0$.

Case 2. $\quad -2py/3 \leqq \eta \leqq 0.$

In this case we start from (A_2) with $A = 3/2$, $B = 9/8$, $C = 1/2$. We remark the following facts:

$$\frac{473 \cdot 9}{640 \cdot 8} p^4 xy \leqq \frac{473 \cdot 9}{640 \cdot 16} \beta_1^{-1} p^4 y^2 + \frac{473 \cdot 9}{640 \cdot 16} \beta_1 p^4 x^2,$$

$$\frac{1}{640 \cdot 16}(897p^4 + 1794p^3 + 6400p^2)xy$$

$$\leqq \frac{\alpha_1}{640 \cdot 32}(897p^4 + 1794p^3 + 6400p^2)x^2 + \frac{\alpha_1^{-1}}{640 \cdot 32}(897p^4 + 1794p^3 + 6400p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi \leqq \frac{1}{96 \cdot 32} 1920\gamma_1\varphi^2 + \frac{1}{96 \cdot 32} 1920\gamma_1^{-1}x'^4,$$

$$\left(\frac{407}{32 \cdot 32}p^2 - \frac{3}{4}\right)y^2\eta \leqq 0,$$

$$\frac{37}{64}py^2\xi \leqq 0.$$

By Lemma 1

$$\left(\frac{37}{32 \cdot 8}p^3 - \frac{3}{8}p\right)y^3 \leqq \frac{1}{96 \cdot 32}(270.396p^3 - 701.568p)y^2,$$

$$-\frac{15}{4}y^2x'y' \leqq \frac{5151}{320 \cdot 4}px'^2y^2 + \frac{1}{96 \cdot 32}(-526.232p^2 + 2104.928)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{1}{96 \cdot 32} 198.778py'^2.$$

Further, since $y \geqq 0$, $0 \geqq \eta \geqq -2py/3$, $\xi \leqq 0$, we have

$$\frac{7}{640}p^3\xi + V_4\xi \leqq \frac{-1}{96 \cdot 32}\{-2(888p^2 - 2304)x'y' + 2 \cdot 888px'\eta' + 1776py'^2\}\xi$$

$$\leqq \frac{\gamma_2}{96 \cdot 32}(888p^2 - 2304)\xi^2 + \frac{\gamma_2^{-1}}{96 \cdot 32}(888p^2 - 2304)x'^2y'^2$$

$$+\frac{\gamma_3}{96\cdot 32}888p\xi^2+\frac{\gamma_3^{-1}}{96\cdot 32}888px'^2\eta'^2$$

$$+\frac{\gamma_4}{96\cdot 32}888p\xi^2+\frac{\gamma_4^{-1}}{96\cdot 32}(-296p^3+1184p)y'^2,$$

$$\frac{473}{1280}p^3x\eta+T_4\eta\leqq\frac{473}{1280\cdot 4}p^3\eta(x'^2+3y'^2+5\eta'^2+7\xi'^2)+T_4\eta$$

$$\leqq\frac{-1}{96\cdot 32}T_4{}^*\eta-\frac{1}{96\cdot 32}2(-832.5p^3+6272.81p)x'y'\eta$$

$$\leqq\frac{-1}{96\cdot 32}T_4{}^*\eta+\frac{\beta_2}{96\cdot 32}(-832.5p^3+6272.81p)\eta^2$$

$$+\frac{\beta_2^{-1}}{96\cdot 32}(-832.5p^3+6272.81p)x'^2y'^2,$$

$$T_4{}^*=(-59.05p^3+16171.2p^2)x'^2+2\cdot 3944.1px'y'+(-501.4p^3+1221p^2)y'^2\geqq 0,$$

$$S_4y+\frac{1}{96\cdot 48}pT_4{}^*y\leqq\frac{-1}{96\cdot 32}\{2\cdot 753.01p^2x'y'+2(832.5p^3+9835.2p)x'\eta'$$

$$+2(-1221p^2+4608)y'\eta'+2(888p^2+4608)x'\xi'-2\cdot 1776py'\xi'\}y$$

$$\leqq\frac{\alpha_2}{96\cdot 32}753.01p^2y^2+\frac{\alpha_2^{-1}}{96\cdot 32}753.01p^2x'^2y'^2$$

$$+\frac{\alpha_3}{96\cdot 32}(832.5p^3+9835.2p)y^2+\frac{\alpha_3^{-1}}{96\cdot 32}(832.5p^3+9835.2p)x'^2\eta'^2$$

$$+\frac{\alpha_4}{96\cdot 32}(-1221p^2+4608)y^2+\frac{\alpha_4^{-1}}{96\cdot 32}(203.5p^4-1582p^2+3072)y'^2$$

$$+\frac{\alpha_4^{-1}}{96\cdot 32}(122.1p^4-949.2p^2+1843.2)\eta'^2$$

$$+\frac{\alpha_5}{96\cdot 32}(888p^2+4608)y^2+\frac{\alpha_5^{-1}}{96\cdot 32}(888p^2+4608)x'^2\xi'^2$$

$$+\frac{\alpha_6}{96\cdot 32}1776py^2+\frac{\alpha_6^{-1}}{96\cdot 32}(-253.715p^3+1014.86p)y'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(1434.63-139.27x+6244.92x^2)(4x-x^2)$ we have, with $\alpha_1=1.3$, $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0.108$, $\alpha_6=0.432$, $\beta_1=2.6$, $\beta_2=0.4$, $\gamma_1=6.952$, $\gamma_2=\gamma_3=0.8$, $\gamma_4=1.6$,

$$\Re a_8\leqq 8-\frac{x^3}{96\cdot 32}\hat{P}_5(x)-\frac{1}{96\cdot 32}\hat{Q}_5-\frac{1}{96\cdot 32}\hat{R}_5,$$

$$\hat{P}_5(x)=5645.01-20679.485x+5947.23x^2+987.7275x^3-380.15x^4,$$

$$\begin{aligned}\hat{Q}_5=&(110.445p^5-594.6924p^4+5981.9565p^3+10174.9369p^2\\&-75293.8956p+72803.982)y^2\\&+2(305.25p^4+1500.6p^2-3456p-305.25p^2x'^2+6336x'^2)y\eta\\&+(1172.4375p^3+31224.6p^2-128717.574p+129142.85)\eta^2\\&+2(444p^3-1152p-444px'^2)y\xi+2(1221p^2-4608)\eta\xi\\&+(43004.04p^2-174238.07p+184793.59)\xi^2,\end{aligned}$$

$$\begin{aligned}\hat{R}_5=&(3226.2p^5+500.8875p^3+6244.92p^2-24840.41p+26135.77\\&-6265.8p^3x'^2-276.48x'^2+1303.4px'^4)x'^2\\&+2(4055.55p^4+421.8p^2-10178.25p^2x'^2+1056x'^4)x'y'\\&+(111p^5-1884.41p^4+6844.0127p^3+33910.312p^2-77454.2089p\\&+47855.662+1859.25p^3x'^2-8082.873p^2x'^2-31543.625px'^2+2880x'^2)y'^2\\&+2(6026.4p^3-5848.8px'^2)x'\eta'+2(305.25p^4+7836.6p^2-305.25p^2x'^2\\&-7488x'^2)y'\eta'\\&+(-1130.646p^4+839.4375p^3+40014.192p^2-116224.45p\\&+113610.818-7708.95p^3x'^2-92183.952px'^2)\eta'^2\\&+2(6528p^2-2112x'^2)x'\xi'+2(444p^3+6912p-444px'^2)y'\xi'\\&+2(1221p^2+6144)\eta'\xi'\\&+(43714.44p^2-172106.87p+182950.39-8222.88p^2x'^2-42670.08x'^2)\xi'^2\\&+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'+(56204.28p^2-223563.69p+235221.93)\varphi'^2\\&+2\cdot 3072x'\tau'+(68694.12p^2-273244.51p+287493.47)\tau'^2.\end{aligned}$$

$\hat{P}_5(x)$ is monotone decreasing for $0.1\leqq x\leqq 0.3$ and $\hat{P}_5(0.3)>0$. Hence $\hat{P}_5(x)>0$ for $0.1\leqq x\leqq 0.3$.

We can prove the positive definiteness of the symmetric matrix associated with $\hat{Q}_5$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$ by taking its principal diagonal minor determinants. Hence $\hat{Q}_5$ is non-negative for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$.

In section 4 we shall prove the positive definiteness of $\hat{R}_5$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$. Therefore we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$, $y\geqq 0$, $0\geqq\eta\geqq -2py/3$, $\xi\leqq 0$.

Case 6. $\eta\leqq -2py/3$.

We start from (A) with $\alpha=7/160$. Applying Lemma 3 to the term $(37\cdot 29p^4/640\cdot 32)(\eta+2py)$ we have

$$\Re a_8 \leqq U+\frac{1073}{640\cdot 12}p+\frac{5365}{128\cdot 64}p^3-\frac{5365}{128\cdot 256}p^5-\frac{1073}{640\cdot 12\cdot 64}p^7-\frac{6671}{64\cdot 25600}p^6x$$

$$+Q_6+R_6+S_6y+T_6\eta+V_6\xi$$

$$+\frac{5}{48}A(p^4+2p^3+4p^2)xy-\frac{1073}{128\cdot 128}(p^4+2p^3)xy+\frac{473}{640}Bp^4xy+\frac{473}{640}Cp^3x\eta$$

$$+\frac{37\cdot 3}{640\cdot 32}p^4\left(\eta+\frac{2}{3}py\right)+\frac{7}{640}p^3\xi-\frac{5}{4}x'^2\varphi$$

$$+\left\{\frac{1073}{64\cdot 128}p^3+\left(\frac{9}{8}+\frac{1}{2}A+\frac{1}{2}A^2-2AB\right)p\right\}y^3+\left(\frac{3219}{128\cdot 64}p^2+2A-A^2-2AC\right)y^2\eta$$

$$+\frac{1073}{64\cdot 32}py^2\xi+\left(\frac{-51}{8}+A+\frac{1}{2}A^2\right)y^2x'y'+\frac{17}{8}x'y'^3,$$

$$Q_6=\left\{\frac{-1073}{128\cdot 256}p^5+\left(\frac{267989}{640\cdot 256}-\frac{393}{640}A-\frac{5}{4}B\right)p^3+\frac{1}{12}A^2x(p^2+2p+4)+B^2xp^2\right.$$

$$\left.-\frac{1073}{640\cdot 16}p+\frac{1073}{128\cdot 128}p^3x'^2+\left(\frac{-218259}{640\cdot 64}+\frac{1}{2}A+\frac{1}{4}A^2\right)px'^2\right\}y^2$$

$$+\left\{\frac{-3219}{128\cdot 128}p^4+\left(\frac{85063}{640\cdot 32}-\frac{3}{4}A-2B-\frac{5}{4}C\right)p^2+2BCxp\right.$$

$$\left.+\frac{3219}{128\cdot 128}p^2x'^2+\left(\frac{A}{2}-\frac{39}{8}\right)x'^2\right\}y\eta$$

$$+\left\{\frac{-9657}{128\cdot 256}p^3+\left(\frac{19821}{640\cdot 16}-2C\right)p+C^2x\right\}\eta^2$$

$$+\left\{\frac{-1073}{128\cdot 32}p^3+\left(\frac{9}{2}-A-2B\right)p+\frac{1073}{128\cdot 32}px'^2\right\}y\xi$$

$$+\left\{\frac{-3219}{128\cdot 32}p^2+(4-2C)\right\}\eta\xi-\frac{1073}{64\cdot 32}p\xi^2+(3-2A)y\varphi,$$

$$R_6=\left(\frac{-171953}{640\cdot 256}p^5-\frac{5365}{128\cdot 256}p^3+\frac{334287}{640\cdot 256}p^3x'^2-\frac{208433}{640\cdot 768}px'^4\right)x'^2$$

$$+\left(\frac{-215963}{640\cdot 128}p^4-\frac{1073}{128\cdot 32}p^2+\frac{542137}{640\cdot 128}p^2x'^2-\frac{11}{16}x'^4\right)x'y'$$

$$+\left(\frac{-1073}{128\cdot 256}p^5-\frac{325931}{640\cdot 256}p^3-\frac{1073}{640\cdot 16}p+\frac{1073}{128\cdot 128}p^3x'^2+\frac{211821}{640\cdot 64}px'^2\right)y'^2$$

$$+\left(\frac{-80019}{640\cdot 32}p^3+\frac{77873}{640\cdot 32}px'^2\right)x'\eta'$$

$$+\left(\frac{-3219}{128\cdot 128}p^4-\frac{104377}{640\cdot 32}p^2+\frac{3219}{128\cdot 128}p^2x'^2+\frac{39}{8}x'^2\right)y'\eta'$$

$$+\left(\frac{-9657}{128\cdot 256}p^3-\frac{26259}{640\cdot 16}p\right)\eta'^2+\left(\frac{-34}{8}p^2+\frac{11}{8}x'^2\right)x'\xi'$$

$$+\left(\frac{-1073}{128\cdot 32}p^3-\frac{9}{2}p+\frac{1073}{128\cdot 32}px'^2\right)y'\xi'+\left(\frac{-3219}{128\cdot 32}p^2-4\right)\eta'\xi'-\frac{1073}{64\cdot 32}p\xi'^2$$

$$-\frac{7}{2}px'\varphi'-3y'\varphi'-2x'\tau',$$

$$S_6=\left\{\left(\frac{-98157}{64\cdot 256}+\frac{5}{16}A\right)p^3+\frac{134607}{640\cdot 64}px'^2\right\}x'^2+\left\{\left(\frac{-79}{8}+\frac{13}{8}A\right)p^2+3x'^2\right\}x'y'$$

$$+\left\{\frac{1073}{64\cdot 128}p^3-\left(\frac{27}{8}+\frac{1}{2}A\right)p\right\}y'^2+\left(\frac{-1073}{64\cdot 32}p^3-\frac{65901}{640\cdot 16}p\right)x'\eta'$$

$$+\left(\frac{3219}{128\cdot 32}p^2-2A\right)y'\eta'+\left\{\frac{-1073}{64\cdot 32}p^2-\left(\frac{3}{2}+A\right)\right\}x'\xi'+\frac{1073}{32\cdot 32}py'\xi',$$

$$T_6=\left(\frac{-108141}{640\cdot 32}p^2+\frac{7}{8}x'^2\right)x'^2+\left(\frac{1073}{64\cdot 32}p^3-\frac{72339}{640\cdot 16}p\right)x'y'-\frac{3219}{128\cdot 64}p^2y'^2$$

$$-\frac{1}{2}x'\eta'+\frac{1073}{64\cdot 32}px'\xi',$$

$$V_6=\frac{-29}{8}px'^2+\left(\frac{1073}{64\cdot 32}p^2-\frac{3}{2}\right)x'y'-\frac{1073}{64\cdot 32}px'\eta'-\frac{1073}{64\cdot 32}py'^2.$$

Here we put $A=3/2$, $B=1$, $C=1/2$. We remark the following facts:

$$\frac{473}{1280}p^3x\eta+\frac{473}{1920}p^4xy=\frac{473}{1280}p^3x\left(\eta+\frac{2}{3}py\right)\leqq 0,$$

$$\frac{946}{1920}p^4xy\leqq\frac{473}{1920}\beta_1p^4x^2+\frac{473}{1920}\beta_1^{-1}p^4y^2,$$

$$\frac{1}{128\cdot 128}(1487p^4+2974p^3+10240p^2)xy$$

$$\leqq\frac{\alpha_1}{128\cdot 256}(1487p^4+2974p^3+10240p^2)x^2+\frac{\alpha_1^{-1}}{128\cdot 256}(1487p^4+2974p^3$$

$$+10240p^2)y^2,$$

$$-\frac{5}{4}x'^2\varphi \leqq \frac{\gamma_1}{96\cdot 32}1920\varphi^2 + \frac{\gamma_1^{-1}}{96\cdot 32}1920x'^4,$$

$$\frac{1073}{64\cdot 32}py^2\xi \leqq 0.$$

By Lemma 1,

$$\frac{1073}{64\cdot 128}p^3y^3 + \left(\frac{3219}{128\cdot 64}p^2 - \frac{3}{4}\right)y^2\eta \leqq \frac{1093}{64\cdot 128}p^2y^2\eta + \frac{1073}{64\cdot 128}p^3y^3$$

$$= \frac{1093}{64\cdot 128}p^2y^2\left(\eta + \frac{2}{3}py\right) + \frac{344.334}{64\cdot 128}p^3y^3 \leqq \frac{1}{96\cdot 32}129.126p^3y^2,$$

$$-\frac{15}{4}y^2x'y' \leqq \frac{164499}{640\cdot 64}px'^2y^2 + \frac{1}{96\cdot 32}(-527.291p^2 + 2109.164)y'^2,$$

$$\frac{17}{8}x'y'^3 \leqq \frac{1}{96\cdot 32}198.778py'^2.$$

Further, since $y \geqq 0$, $-2py/3 \geqq \eta$, $\xi \leqq 0$, we have

$$\frac{37\cdot 3}{640\cdot 32}p^4\left(\eta + \frac{2}{3}py\right) + S_6y + T_6\eta \leqq \frac{6660}{96\cdot 32}p^2x'^2\eta + \frac{4440}{96\cdot 32}p^3x'^2y + S_6y + T_6\eta$$

$$\leqq \frac{-1}{96\cdot 32}\{2(804.75p^3 + 9885.15p)x'\eta' + 2(-1207.125p^2 + 4608)y'\eta'$$

$$+ 2(804.75p^2 + 4608)x'\xi' - 2\cdot 1609.5py'\xi'\}y$$

$$- \frac{1}{96\cdot 32}\{9561.15p^2x'^2 + 2(-804.75p^3 + 10850.85p)x'y' + 1207.125p^2y'^2$$

$$+ 2\cdot 768x'\eta' - 2\cdot 804.75px'\xi'\}\eta$$

$$\leqq \frac{\alpha_2}{96\cdot 32}(804.75p^3 + 9885.15p)y^2 + \frac{\alpha_2^{-1}}{96\cdot 32}(804.75p^3 + 9885.15p)x'^2\eta'^2$$

$$+ \frac{\alpha_3}{96\cdot 32}(-1207.125p^2 + 4608)y^2$$

$$+ \frac{\alpha_3^{-1}}{96\cdot 32}(402.375p^4 - 3145.5p^2 + 6144)\eta'^2$$

$$+ \frac{\alpha_4}{96\cdot 32}(804.75p^2 + 4608)y^2 + \frac{\alpha_4^{-1}}{96\cdot 32}(804.75p^2 + 4608)x'^2\xi'^2$$

$$+ \frac{\alpha_5}{96\cdot 32}1609.5py^2 + \frac{\alpha_5^{-1}}{96\cdot 32}(-229.93p^3 + 919.72p)y'^2,$$

$$+\frac{\beta_2}{96\cdot 32}4780.575p^2\eta^2+\frac{\beta_2^{-1}}{96\cdot 32}4780.575p^2x'^4$$

$$+\frac{\beta_3}{96\cdot 32}(-804.75p^3+10850.85p)\eta^2+\frac{\beta_3^{-1}}{96\cdot 32}(-804.75p^3+10850.85p)x'^2y'^2$$

$$+\frac{\beta_4}{96\cdot 32}603.5625p^2\eta^2+\frac{\beta_4^{-1}}{96\cdot 32}(-201.1875p^4+804.75p^2)y'^2$$

$$+\frac{\beta_5}{96\cdot 32}768\eta^2+\frac{\beta_5^{-1}}{96\cdot 32}768x'^2\eta'^2+\frac{\beta_6}{96\cdot 32}\cdot 804.75p\eta^2$$

$$+\frac{\beta_6^{-1}}{96\cdot 32}\cdot 804.75px'^2\xi'^2,$$

$$\frac{7}{640}p^3\xi+V_6\xi\leqq\frac{-1}{96\cdot 32}\{2(-804.75p^2+2304)x'y'+2\cdot 804.75px'\eta'+1609.5py'^2\}\xi$$

$$\leqq\frac{\gamma_2}{96\cdot 32}(804.75p^2-2304)\xi^2+\frac{\gamma_2^{-1}}{96\cdot 32}(804.75p^2-2304)x'^2y'^2$$

$$+\frac{\gamma_3}{96\cdot 32}\cdot 804.75p\xi^2+\frac{\gamma_3^{-1}}{96\cdot 32}\cdot 804.75px'^2\eta'^2$$

$$+\frac{\gamma_4}{96\cdot 32}\cdot 804.75p\xi^2+\frac{\gamma_4^{-1}}{96\cdot 32}(-268.25p^3+1073p)y'^2.$$

Making use of these remarks and applying Lemma 1 to the term $-(1484.58+1172.98x+2911.77x^2)(4x-x^2)$ we have, with $\alpha_1=2.85$, $\alpha_2=\alpha_3=\alpha_4=0.175$, $\alpha_5=0.7$, $\beta_1=2.85$, $\beta_2=\beta_3=\beta_5=\beta_6=0.19$, $\beta_4=0.57$, $\gamma_1=7.645$, $\gamma_2=\gamma_3=1$, $\gamma_4=3$,

$$\Re a_8\leqq 8-\frac{x^3}{96\cdot 32}\hat{P}_6(x)-\frac{1}{96\cdot 32}\hat{Q}_6-\frac{1}{96\cdot 32}\hat{R}_6,$$

$$\hat{P}_6(x)=1774.8725-5754.29375x-1081.288125x^2+1919.2021875x^3-379.46x^4,$$

$$\begin{aligned}\hat{Q}_6=&(100.59375p^5-314.45834p^4+4925.02p^3+2324.882625p^2-40994.83125p\\&+40212.06-201.1875p^3x'^2)y^2\\&+2(301.78125p^4+916.275p^2-3072p-301.78125p^2x'^2+6336x'^2)y\eta\\&+(1058.24625p^3+13306.510125p^2-68421.164p+75706.18)\eta^2\\&+2(402.375p^3-1536p-402.375px'^2)y\xi+2(1207.125p^2-4608)\eta\xi\\&+(19577.64p^2-91349.92p+110647.34)\xi^2,\end{aligned}$$

$$\hat{R}_6=(3224.11875p^5+502.96875p^3+2911.77p^2-12820.06p+15477.62$$

$$-6267.88125p^3x'^2-25164.9468p^2x'^2-251.52x'^2+1302.70625px'^4)x'^2$$

$$+2(4049.30625p^4+402.375p^2-10165.06875p^2x'^2+1056x'^4)x'y'$$

$$+(100.59375p^5+353.084p^4+6529.3717p^3+7850.26475p^2-40009.7199p$$

$$+44323.696+4035.0165p^3x'^2-804.75p^2x'^2-73005.4494px'^2+2304x'^2)y'^2$$

$$+2(6001.425p^3-5840.475px'^2)x'\eta'$$

$$+2(301.78125p^4+7828.275p^2-301.78125p^2x'^2-7488x'^2)y'\eta'$$

$$+(-2299.5732p^4+905.34375p^3+32535.3825p^2-56222.6p+42275.14$$

$$-4599.14625p^3x'^2-57298.38225px'^2-4042.752x'^2)\eta'^2$$

$$+2(6528p^2-2112x'^2)x'\xi'+2(402.375p^3+6912p-402.375px'^2)y'\xi'$$

$$+2(1207.125p^2+6144)\eta'\xi'$$

$$+(20382.39p^2-88130.92p+108343.34-4599.14625p^2x'^2$$

$$-4236.204px'^2-26334.72x'^2)\xi'^2$$

$$+2\cdot 5376px'\varphi'+2\cdot 4608y'\varphi'+(26205.93p^2-115380.54p+139298.58)\varphi'^2$$

$$+2\cdot 3072x'\tau'+(32029.47p^2-141020.66p+170253.82)\tau'^2.$$

$\hat{P}_6(x)$ is monotone decreasing for $0.1\leqq x\leqq 0.3$ and $\hat{P}_6(0.3)>0$. Hence $\hat{P}_6(x)>0$ for $0.1\leqq x\leqq 0.3$.

Since $y\xi\leqq 0$, we may consider $Q_6{}^*=\hat{Q}_6-2(402.375p^3-1536p-402.375px'^2)y\xi$. It is not so difficult to prove that $Q_6{}^*$ is non-negative for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$. Hence $\hat{Q}_6$ is non-negative for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$.

In the following section we shall prove the positive definiteness of $\hat{R}_6$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$. Therefore we have $\Re a_8<8$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$, $y\geqq 0$, $-2py/3\geqq\eta$, $\xi\leqq 0$.

§4. In this section we prove the positive definiteness of $\hat{R}_i$ $(i=1, 2, \cdots 6)$. We need laborious calculations here.

We consider the following modified quadratic forms:

$$R_1{}^*=(119.377p^5-36p^4+16p^3+299.285p^2-1176.14p+1139.54)x'^2$$

$$+2(107.582p^4+36p^3+12p^2-9)x'y'$$

$$+(2.9p^5-56.187p^4+205.081p^3+1035.673p^2-2385.693p+1524.328)y'^2$$

$$+2\cdot 186.295p^3x'\eta'+2(9.352p^4+243.915p^2)y'\eta'$$

$$+(-34.518p^4+24.262p^3+1231.913p^2-3642.1p+3578.289)\eta'^2$$

$$+2\cdot 203.835p^2x'\xi'+2(11.222p^3+216p)y'\xi'+2(37.5p^2+192)\eta'\xi'$$

$$+(1351.132p^2-5394.14p+5729.64)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$

$$+(1744.92p^2-6993.18p+7355.36)\varphi'^2,$$

$$R_2^*=(118.065p^5-35.475p^4+15.652p^3+299.228p^2-1170.239p+1144.75)x'^2$$

$$+2(122.941p^4+25.182p^2)x'y'$$

$$+(3.613p^5-59.52p^4+211.411p^3+1059.922p^2-2420.444p+1492.489)y'^2$$

$$+2\cdot 187.868p^3x'\eta'+2(9.515p^4+244.309p^2)y'\eta'$$

$$+(-36.477p^4+19.03p^3+1250.443p^2-3632.014p+3550.338)\eta'^2$$

$$+2\cdot 203.835p^2x'\xi'+2(13.84p^3+216p)y'\xi'+2(38.157p^2+192)\eta'\xi'$$

$$+(1360.423p^2-5378.34p+5713.294)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$

$$+(1737.633p^2-6986.366p+7404.872)\varphi'^2.$$

$$R_3^*=(100.141p^5+11.928p^3+116.797p^2-496.37p+546.09)x'^2$$

$$+2(125.414p^4+11.475p^2)x'y'$$

$$+(2.844p^5+11.397p^4+198.175p^3+321.411p^2-1525.802p+1629.466)y'^2$$

$$+2\cdot 185.62p^3x'\eta'+2(9.258p^4+243.69p^2)y'\eta'$$

$$+(-83.253p^4+28.808p^3+1233.801p^2-2241.55p+1545.986)\eta'^2$$

$$+2\cdot 203.835p^2x'\xi'+2(10.1p^3+216p)y'\xi'+2(37.125p^2+192)\eta'\xi'$$

$$+(813.59p^2-3434.09p+3956.47)\xi'^2+2\cdot 168px'\varphi'+2\cdot 144y'\varphi'$$

$$+(1051.37p^2-4467.33p+5086.89)\varphi'^2.$$

Then we have $\hat{R}_1 \geqq R_1^*$ for $1.7 \leqq p \leqq 1.9$, $|x'/p| \leqq 1/20$. Indeed

$$\hat{R}_1-R_1^*=(-0.6895p^5+0.025p^2+55.5-195.9375p^3x'^2-9.84x'^2)x'^2$$

$$+2(0.793p^4+9-317.25p^2x'^2+33x'^4)x'y'$$

$$+(-0.0875p^5+18.98075p^4-2.7589p^3+170.466p^2$$

$$-1178.2278p+1423.256-5.625p^3x'^2-507.832px'^2)y'^2$$

$$+2(0.455p^3-182.25px'^2)x'\eta'+2(0.023p^4+0.585p^2-9.375p^2x'^2-234x'^2)y'\eta'$$

$$+(12.19425p^4+6.988p^3+439.6552p^2-1995.6p+2054.0182$$

$$-145.1044p^3x'^2-1870.9065px'^2-7.848x'^2)\eta'^2$$

$$+2(0.165p^2-66x'^2)x'\xi'+2(0.028p^3-11.25px'^2)y'\xi'$$
$$+(744.038p^2-2793.84p+2635.64-133.9425p^2x'^2$$
$$-7.3575px'^2-857.232x'^2)\xi'^2$$
$$+(948.87p^2-3592.08p+3400)\varphi'^2+2\cdot 96x'\tau'$$
$$+(3292.41p^2-12937.54p+13145.44)\tau'^2.$$

Since for $1.7\leqq p\leqq 1.9$,

$$x'^2\{0.54py'^2+2\cdot 11.25py'\xi'+(133.9425p^2+7.3575p)\xi'^2\}\geqq 0,$$
$$x'^2(5.09x'^2+2\cdot 66x'\xi'+857.232\xi'^2)\geqq 0,$$
$$x'^2\{8.81p^3x'^2+2\cdot 182.25px'\eta'+(1300p+7.848)\eta'^2\}\geqq 0,$$
$$x'^2\{23.821py'^2+2(9.375p^2+234)y'\eta'+(145.1044p^3+570.9065p)\eta'^2\}\geqq 0,$$
$$x'^2\{202p^3x'^2+2\cdot 317.25p^2x'y'+(5.625p^3+483.471p)y'^2\}\geqq 0,$$

and

$$400x'^2\leqq p^2,$$

we have

$$\hat{R}_1-R_1^*\geqq(-1.2166p^5+0.01227p^2+55.5)x'^2+2(-0.000125p^4+9)x'y'$$
$$+(-0.1015625p^5+18.98075p^4-4.02848p^3+170.466p^2$$
$$-1178.2278p+1423.256)y'^2$$
$$+2(-0.000625p^3)x'\eta'+2(-0.0004375p^4)y'\eta'$$
$$+(11.5050041p^4+2.31073p^3+439.63558p^2-1995.6p+2054.0182)\eta'^2$$
$$+2(-0.000125p^3)y'\xi'+(740.65113p^2-2793.84p+2635.64)\xi'^2$$
$$+(948.87p^2-3592.08p+3400)\varphi'^2+2\cdot 96x'\tau'$$
$$+(3292.41p^2-12937.54p+13145.44)\tau'^2.$$

Further, since

$$(-1.217p^5+0.01227p^2+35)x'^2+2(-0.000125p^4+9)x'y'$$
$$+(-0.102p^5+18.98p^4-4.02848p^3+170.466p^2-1178.2278p+1423.256)y'^2\geqq 0,$$
$$0.0004p^5x'^2-2\cdot 0.000625p^3x'\eta'+0.00073p^3\eta'^2\geqq 0,$$
$$0.0004375p^5y'^2-2\cdot 0.0004375p^4y'\eta'+0.0010041p^4\eta'^2\geqq 0,$$
$$0.00075p^4y'^2-2\cdot 0.000125p^3y'\xi'+0.00113p^2\xi'^2\geqq 0,$$

$$(11.504p^4+2.31p^3+439.635p^2-1995.6p+2054.0182)\eta'^2\geqq 0,$$

$$(740.65p^2-2793.84p+2635.64)\xi'^2\geqq 0,$$

$$(948.87p^2-3592.08p+3400)\varphi'^2\geqq 0,$$

$$20.5x'^2+2\cdot 96x'\tau'+(3292.41p^2-12937.54p+13145.44)\tau'^2\geqq 0,$$

we have the desired result: $\hat{R}_1\geqq R_1^*$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$. Similarly we have $\hat{R}_2\geqq R_1^*$, $\hat{R}_4\geqq 32R_2^*$, $\hat{R}_5\geqq 32R_2^*$, $\hat{R}_3\geqq R_3^*$ and $\hat{R}_6\geqq 32R_3^*$ for $1.7\leqq p\leqq 1.9$, $|x'/p|\leqq 1/20$.

Firstly we calculate the principal diagonal minor determinants of the symmetric matrix associated with R_1^*. Then they are larger than

$$119p^5-36p^4+16p^3+299p^2-1177p+1139,$$

$$346p^{10}-6812p^9+14977p^8+108475p^7-342905p^6+414327p^5-86525p^4$$
$$-1673364p^3+4442524p^2-4511416p+1736951,$$

$$-11950p^{14}+233089p^{13}-252624p^{12}-13305246p^{11}+61180507p^{10}$$
$$+27717064p^9-782871326p^8+2291397608p^7-3112478036p^6$$
$$+68941413p^5+11020542454p^4-27683425018p^3+34467436431p^2$$
$$-22469301193p+6215315265,$$

$$-16145893p^{16}+379913151p^{15}-1667182112p^{14}-15280837487p^{13}$$
$$+152947663551p^{12}-369147132121p^{11}-850771984488p^{10}$$
$$+7457090506635p^9-21069837921842p^8+30277599474286p^7$$
$$-3972981934815p^6-95662234365115p^5+258396531774826p^4$$
$$-374360366693855p^3+326707244164697p^2-162100978740764p$$
$$+35547487971231,$$

$$-28173291146p^{18}+775829190591p^{17}-5684659316693p^{16}-12210536350876p^{15}$$
$$+3614803595638 95p^{14}-1826114999058441p^{13}+2221879746091551p^{12}$$
$$+16246785164487863p^{11}-95167985323311396p^{10}+254992696509034884p^9$$
$$-373575275469942257p^8+83784363636736221p^7+1089123729791597286p^6$$
$$-3160141232572236258p^5+5083527883690283885p^4-5316745337144062814p^3$$
$$+3595963656627195344p^2-1439461638186389166p+260980982890665870,$$

respectively. All of them are positive for $1.7\leqq p\leqq 1.9$. Hence R_1^* is positive definite there.

Next we calculate the principal diagonal minor determinants of the symmetric matrix associated with R_2^*. Then they are larger than

$$118p^5-36p^4+15p^3+299p^2-1171p+1144,$$

$$426p^{10}-7156p^9+12013p^8+117789p^7-348292p^6+415714p^5-89844p^4$$
$$-1699254p^3+4492438p^2-4517373p+1708526,$$

$$-15560p^{14}+258436p^{13}-37782p^{12}-14803019p^{11}+59710232p^{10}$$
$$+53358739p^9-846276711p^8+2355538739p^7-3130843544p^6$$
$$-2460213p^5+11253661318p^4-27965725671p^3+34493248042p^2$$
$$-22243591035p+6065847560,$$

$$-21168117p^{16}+436094031p^{15}-1530367872p^{14}-18470698299p^{13}$$
$$+160622391100p^{12}-333524805319p^{11}-1091043959648p^{10}$$
$$+8042088378105p^9-21794300140323p^8+30589596591678p^7$$
$$-3254394639010p^6-97778918152242p^5+260982237205863p^4$$
$$-375013665283927p^3+324575516335487p^2-159541837362780p$$
$$+34592987342800,$$

$$-36782417348p^{18}+905659587380p^{17}-5862677418586p^{16}-18174364353250p^{15}$$
$$+396813647705025p^{14}-1838478616376548p^{13}+1623577887863074p^{12}$$
$$+19127261998285411p^{11}-102130015744446423p^{10}+264930853062103376p^9$$
$$-380685766619014326p^8+79618121827136448p^7+1110866454970045690p^6$$
$$-3195114767337193006p^5+5111332134579807948p^4-5317391148875246430p^3$$
$$+3575474730972620635p^2-1421628994889791211p+255676023289070625,$$

respectively. All of them are positive for $1.7\leqq p\leqq 1.9$. Hence R_2^* is positive definite there.

Finally we calculate the principal diagonal minor determinants of the symmetric matrix associated with R_3^*. Then they are larger than

$$100p^5+11p^3+116p^2-497p+546,$$

$$284p^{10}+1141p^9+4150p^8+32654p^7-153391p^6+186052p^5-72936p^4$$
$$-210091p^3+1123198p^2-1642044p+889835,$$

$$-23711p^{14}-95396p^{13}+38708p^{12}-1949089p^{11}+16311355p^{10}+11440485p^9$$

$$-256248986p^8+694397196p^7-909278147p^6+390453314p^5$$

$$+1590158236p^4-4842822175p^3+6515050494p^2-4533185722p+1375672588,$$

$$-19290657p^{16}+4661494p^{15}+265289320p^{14}-2106971724p^{13}$$

$$+20179440686p^{12}-54374550943p^{11}-182353201869p^{10}$$

$$+1483439845906p^9-4145593840903p^8+6299849312779p^7$$

$$-3924514158548p^6-7529286014841p^5+27997976227875p^4$$

$$-45086799343824p^3+42369722758591p^2-22599064506865p+5410004444607,$$

$$-20281617630p^{18}+91078684149p^{17}+159963351518p^{16}-3376629292306p^{15}$$

$$+31978094136427p^{14}-158028488850288p^{13}+153837658039836p^{12}$$

$$+2097633207859323p^{11}-11912720253711391p^{10}+32683282266721222p^9$$

$$-53339728802112426p^8+41707014096901506p^7+42722957211144874p^6$$

$$-209771209691609249p^5+386979603825481135p^4-441231426168410081p^3$$

$$+321578728354435090p^2-138904108733065204p+27451251732657961,$$

respectively. All of them are positive for $1.7\leqq p\leqq 1.9$. Hence $R_3{}^*$ is positive definite there.

Summing up the results we have completed the proof of our theorem.

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