NAKAGAWA H. AND I. YOKOTE
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COMPACT HYPERSURFACES IN AN ODD DIMENSIONAL SPHERE

BY HISAO NAKAGAWA AND ICHIRO YOKOTE

Introduction.

As is well known, a $(2n+1)$-dimensional sphere $S^{2n+1}(c)$ of constant curvature c is naturally endowed with a normal contact metric structure and any hypersurface M in $S^{2n+1}(c)$ admits also an (f, g, u, v, λ)-structure, which is defined by Yano and Okumura [8], induced from the Sasakian structure in $S^{2n+1}(c)$. For an (f, g, u, v, λ)-structure, the exterior derivatives of the dual 1-form of the vector field u is equal to twice of the fundamental 2-form induced from f. It might be interesting to study the manifold structure of the hypersurfaces of an odd-dimensional sphere, when the exterior derivatives of the dual 1-form of v is proportional to the fundamental 2-form induced from f. Recently, in this sense, taking in connection with the paper due to Blair, Ludden and Yano [1], the present authors [4] have proved the following

THEOREM. *Let M be a complete orientable hypersurface with constant scalar curvature in $S^{2n+1}(1)$. We assume that, for an (f, g, u, v, λ)-structure on M, there exists a constant ϕ such that*

$$H_k{}^i f_j{}^k + f_k{}^i H_j{}^k = 2\phi f_j{}^i, \tag{0.1}$$

or equivalently

$$\nabla_j v_i - \nabla_i v_j = 2\phi f_{ji}, \tag{0.2}$$

where $H_j{}^i$ denotes components of the second fundamental tensor in M. Then either of the following two assertions (a) *and* (b) *is true*:

(a) *M is isometric to one of the following spaces*:

(1) *the great sphere $S^{2n}(1)$;*

(2) *the small sphere $S^{2n}(c)$, where $c=1+\phi^2$;*

(3) *the product manifold $S^{2n-1}(c_1)\times S^1(c_2)$, where $c_1=1+\phi^2$ and $c_2=1+1/\phi^2$;*

(4) *the product manifold $S^n(c_1)\times S^n(c_2)$, where $c_1=2(1+\phi^2+\phi\sqrt{1+\phi^2})$ and $c_2=2(1+\phi^2-\phi\sqrt{1+\phi^2})$.*

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(b) *M has exactly four distinct constant principal curvatures of multiplicities $n-1$, $n-1$, 1 and 1, respectively.*

The main purpose of the present paper is to show that this theorem will be established under some weaker conditions.

In §1, as preliminaries, we recall the definition and some properties of an (f, g, u, v, λ)-structure on a hypersurface naturally induced from a normal contact structure of $S^{2n+1}(1)$. In §2, we prove some lemmas and properties concerning a hypersurface satisfying the condition (0.1) with a differentiable function ϕ. In §3, we prove a theorem concerning a hypersurface satisfying the condition (0.1) with a function ϕ (cf. Theorem 3.5) and, in the last §4, another theorem concerning a compact hypersurface without the assumption that the scalar curvature is constant (cf. Theorem 4.1).

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§1. Hypersurfaces in an odd dimensional sphere.

Let M be a $2n$-dimensional Riemannian manifold of class C^∞ covered by a system of local coordinate neighborhoods $\{U; x^h\}$. Throughout this paper, indices $i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$. Let there be given in M a tensor field f of type (1, 1), vector fields u and v, a scalar function λ satisfying the following conditions:

$$
\begin{aligned}
&f_k{}^h f_j{}^k = -\delta_j^h + u^h u_j + v^h v_j,\\
&f_k{}^h u^k = \lambda v^h, \qquad f_k{}^h v^k = -\lambda u^h,\\
&u_k f_j{}^k = -\lambda v_j, \qquad v_k f_j{}^k = \lambda u_j,\\
&u_k u^k = v_k v^k = 1-\lambda^2, \qquad u_k v^k = v_k u^k = 0,\\
&g_{kh} f_j{}^k f_i{}^h = g_{ji} - u_j u_i - v_j v_i,
\end{aligned}
\tag{1.1}
$$

where $f_j{}^h, u^h, v^h$ and g_{ji} are components of the tensor field f, vector fields u, v and the Riemannian metric tensor g, and $u_j = g_{jk}u^k$, $v_j = g_{jk}v^k$. The set of these tensor fields is called an *(f, g, u, v, λ)-structure* [8].

Now, let $S^m(c)$ be an m-dimensional sphere of constant curvature c in an $(m+1)$-dimensional Euclidean space E^{m+1}. As is well known, $S^{2n+1}(1)$ admits a normal contact metric structure $(\bar\phi, \bar\xi, \bar\eta, \bar g)$, which is induced from the natural Kaehlerian structure equipped on E^{2n+2}. Let $S^{2n+1}(1)$ be covered by a system of local coordinate neighborhoods $\{\bar U; y^\kappa\}$, where indices $\kappa, \lambda, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$. Let M be an orientable and connected hypersurface in $S^{2n+1}(1)$. We put

$$B_j{}^\kappa = \partial y^\kappa / \partial x^j,$$

then B_j is a local vector field with components B_j^{κ} of $S^{2n+1}(1)$ tangent to M for each j. We choose a unit normal vector C of M such that B_j and C give the positive orientation of $S^{2n+1}(1)$. The transforms $\bar{\phi}_{\lambda}{}^{\kappa}B_j{}^{\lambda}$ of B_j by $\bar{\phi}$ can be expressed as a linear combination of B_j and C, that is,

$$\bar{\phi}_{\lambda}{}^{\kappa}B_j{}^{\lambda}=f_j{}^kB_k{}^{\kappa}+v_jC^{\kappa}, \tag{1.2}$$

where $\bar{\phi}_{\lambda}{}^{\kappa}$ are components of the tensor field $\bar{\phi}$ of type (1,1). Then $f_j{}^k$ is a tensor field of type (1,1) and v_j is a 1-form on M. Similarly, since the transforms $\bar{\phi}_{\lambda}{}^{\kappa}C^{\lambda}$ of the normal vector C with components C^{λ} by $\bar{\phi}$ is tangent to M, it is written as

$$\bar{\phi}_{\lambda}{}^{\kappa}C^{\lambda}=-B_j{}^{\kappa}v^j. \tag{1.3}$$

Moreover the vector field $\bar{\xi}$ with components $\bar{\xi}^{\kappa}$ of $S^{2n+1}(1)$ on M is also a linear combination of B_j and C, and hence we can express $\bar{\xi}$ as follows:

$$\bar{\xi}^{\kappa}=B_j{}^{\kappa}u^j+\lambda C^{\kappa}, \tag{1.4}$$

where u^j is a vector field on M and λ is a differentiable function. Then it is seen that the set $(f_i{}^j, g_{ji}, u^j, v^j, \lambda)$ satisfies the equation (1.1) and hence it is an (f, g, u, v, λ)-structure. Furthermore, by making use of the property of the normal contact metric structure on $S^{2n+1}(1)$, the (f, g, u, v, λ)-structure satisfies the following conditions:

$$\begin{aligned} &\nabla_j f_i{}^h=\delta_j^h u_i-g_{ji}u^h-H_{ji}v^h+H_j{}^hv_i,\\ &\nabla_j u^h=f_j{}^h+\lambda H_j{}^h, \qquad \nabla_j v^h=f_k{}^hH_j{}^k-\lambda\delta_j^h,\\ &\lambda_j=v_j-H_{jk}u^k, \end{aligned} \tag{1.5}$$

where $\lambda_j=\nabla_j\lambda$ and $H_j{}^h$ are components of the second fundamental tensor H of M in $S^{2n+1}(1)$ (cf. [6]). Throughout this paper, we concern with hypersurfaces in $S^{2n+1}(1)$ and with their induced (f, g, u, v, λ)-structures.

We now denote by $K_{kji}{}^h$, K_{ji} and K components of the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The equation of Gauss for the hypersurface M is written as

$$K_{kji}{}^h=\delta_k^hg_{ji}-\delta_j^hg_{ki}+H_k{}^hH_{ji}-H_j{}^hH_{ki}, \tag{1.6}$$

where $H_{ji}=H_j{}^kg_{ki}$, and the equation of Codazzi is given by

$$\nabla_kH_{ji}-\nabla_jH_{ki}=0, \tag{1.7}$$

where ∇_j means the covariant derivation with respect to the induced Riemannian connection of M. From (1.6), we have easily

$$K_{ji}=(2n-1)g_{ji}+H_k{}^kH_{ji}-H_{jk}H_i{}^k, \tag{1.8}$$

and

$$K=2n(2n-1)+(H_j{}^j)^2-H_{ji}H^{ji}. \tag{1.9}$$

§ 2. The second fundamental tensor.

In the sequel, we assume that on the hypersurface M in $S^{2n+1}(1)$, the linear transformation f and the second fundamental tensor H satisfy the following condition

$$H_k{}^i f_j{}^k + f_k{}^i H_j{}^k = 2\phi f_j{}^i, \tag{2.1}$$

where ϕ is a certain differentiable function, or equivalently,

$$f_j{}^k H_{ki} - f_i{}^k H_{kj} = 2\phi f_{ji}, \tag{2.2}$$

because $f_{ji}=f_j{}^k g_{ki}$ is skew-symmetric. Taking account of the third equation of (1.5), we see that (2.2) is also equivalent to

$$\nabla_j v_i - \nabla_i v_j = 2\phi f_{ji}. \tag{2.3}$$

If we now put $N_0=\{x\in M|\lambda(x)=0\}$, $N_1=\{x\in M|\lambda^2(x)=1\}$ and $N=M-N_0\cup N_1$, then we have $M=N\cup N_0\cup N_1$. Then the sets N_0 and N_1 are geometrically characterised as follows: the vector field $\bar{\xi}$, i.e., the Sasakian structure $\bar{\xi}$ in the ambient space is tangent to the hypersurface M at any point in the set N_0 and the vector $\bar{\xi}$ is orthogonal to M at each point in N_1 (see (1.2) and (1.4)).

The second equation of (1.5) implies that N_1 is a bordered set. In fact, if we suppose that there is an open subset U contained in N_1, then we have, in U, $f_{ji}\pm H_{ji}=0$, because $u_i u^i=1-\lambda^2=0$ in U, and hence $u=0$ in U. This implies f_{ji} vanishes in U, because f_{ji} is skew-symmetric and H_{ji} is symmetric. This contradicts the fact that f_{ji} is of rank $2n-2$ or of rank $2n$ in M. Consequently N_1 is a bordered set. Thus we may discuss properties of principal curvatures only on $N\cup N_0$, since they are continuous. In the sequel, we consider only hypersurfaces in $S^{2n+1}(1)$ satisfying the condition (2.1). First we prove

LEMMA 2.1. *On the set $N\cup N_0$, the transforms Hu and Hv of the vector fields u and v by the linear transformation H are linear combination of u and v, that is,*

$$H_k{}^j u^k = \alpha u^j + \beta v^j, \tag{2.4}$$

$$H_k{}^j v^k = \beta u^j + \gamma v^j, \tag{2.5}$$

where $\alpha=H(u,u)/(1-\lambda^2)$, $\beta=H(u,v)/(1-\lambda^2)$, $\gamma=H(v,v)/(1-\lambda^2)$, $H(u,u)=H_{ji}u^j u^i$, $H(u,v)=H_{ji}u^j v^i$ *and* $H(v,v)=H_{ji}v^j v^i$.

Proof. Transvecting $f_h{}^j$ with equation (2.1) and taking account of (2.1) and the first equation of (1.1), we obtain

$$H_k{}^i(u^k u_h + v^k v_h) - (u^i u_k + v^i v_k)H_h{}^k = 0.$$

Transvecting u^h and v^h with the equation above, we have respectively

$$(1-\lambda^2)H_k{}^i u^k = H(u, u)u^i + H(u, v)v^i$$

and

$$(1-\lambda^2)H_k{}^i v^k = H(u, v)u^i + H(v, v)v^i,$$

from which, equations (2.4) and (2.5) respectively. Thus we conclude the proof.

Differentiating (2.4) covariantly, we get

$$\nabla_j H_{ik} u^k + H_{ik} \nabla_j u^k = \alpha_j u_i + \alpha \nabla_j u_i + \beta_j v_i + \beta \nabla_j v_i,$$

where $\alpha_j = \nabla_j \alpha$ and $\beta_j = \nabla_j \beta$. From this relation and the equation (1.7) of Codazzi, we have

$$H_{ik} \nabla_j u^k - H_{jk} \nabla_i u^k = \alpha_j u_i - \alpha_i u_j + \alpha(\nabla_j u_i - \nabla_i u_j) + \beta_j v_i - \beta_i v_j + \beta(\nabla_j v_i - \nabla_i v_j).$$

Substituting the second equation of (1.5) and (2.3) into the equation above, we have

$$\{2\phi(1-\beta) - 2\alpha\} f_{ji} = \alpha_j u_i - \alpha_i u_j + \beta_j v_i - \beta_i v_j, \tag{2.6}$$

which implies that vectors $\nabla\alpha$ and $\nabla\beta$ are linear combinations of u and v, that is, that α_j and β_j are expressed in the form

$$\alpha_j = A_1 u_j + A_2 v_j, \qquad \beta_j = B_1 u_j + B_2 v_j, \tag{2.7}$$

where A_1, A_2, B_1 and B_2 are differentiable functions in $N \cup N_0$. Consequently, the equation (2.6) is reduced to

$$\{2\phi(1-\beta) - 2\alpha\} f_{ji} = -(A_2 - B_1)(u_j v_i - u_i v_j).$$

Since the rank of the linear transformation f is equal to or greater than $2n-2$ and since M is finite dimensional, we have

LEMMA 2.2. *We have in* $N \cup N_0$

$$\alpha = \phi(1-\beta), \qquad A_2 = B_1. \tag{2.8}$$

By the similar method, we obtain from (2.5)

$$2H_{ik} f_h{}^k H_j{}^h = 2(\beta + \phi\gamma) f_{ji} + \beta_j u_i - \beta_i u_j + \gamma_j v_i - \gamma_i v_j, \tag{2.9}$$

where $\gamma_j = \nabla_j \gamma$. This means that the vector $\nabla\gamma$ is also a linear combination of vector u and v and hence γ_j is expressed in the form

$$\gamma_j = C_1 u_j + C_2 v_j,$$

where C_1 and C_2 are differentiable functions in $N \cup N_0$. Furthermore, we can prove

LEMMA 2.3. *The second fundamental tensor satisfies the following conditions in* $N\cup N_0$:

$$\begin{aligned}(2.10)\qquad & 2H_{jk}H_i{}^k-4\phi H_{ji}+2(\beta+\phi\gamma)g_{ji}\\ &=\{R+\lambda(B_2-C_1)\}u_ju_i+P(u_jv_i+u_iv_j)+\{Q+\lambda(B_2-C_1)\}v_jv_i\end{aligned}$$

and

$$(2.11)\qquad \lambda P=0,\qquad \lambda Q=\lambda R=(B_2-C_1)(1-\lambda^2),$$

where

$$P=2\beta(\alpha+\gamma)-4\phi\beta,$$

$$Q=2(\beta^2+\gamma^2+\beta-\phi\gamma),$$

$$R=2(\alpha^2+\beta^2-2\phi\alpha+\beta+\phi\gamma).$$

Proof. By means of (2.1), the equation (2.9) becomes

$$(2.12)\qquad \{2H_{ik}H_h{}^k-4\phi H_{ih}+2(\beta+\phi\gamma)g_{ih}\}f_j{}^h=(B_2-C_1)(u_jv_i-u_iv_j).$$

Transvecting u^j (resp. v^j) with the equation above, we get three relations in (2.10).

On the other hand, applying $f_l{}^j$ to (2.12) and interchanging indices l and j, we obtain the equation (2.10). Thus, this lemma is proved.

If we take account of (2.4) and (2.5), then we see that there exist, at an arbitrary point of $N\cup N_0$, two eigenvectors of the second fundamental tensor of M belonging to the plane section $P(u,v)$ spanned by u and v. Let τ_1 and τ_2 be eigenvalues corresponding to these two eigenvectors, respectively. Then the eigenvalues satisfy the quadratic equation

$$(2.13)\qquad \tau^2-(\alpha+\gamma)\tau+\alpha\gamma-\beta^2=0.$$

Moreover, (2.10) shows that $N\cup N_0$ has at most two distinct principal curvatures, say σ_1 and σ_2, associated with eigenvectors orthogonal to the plane section $P(u,v)$. First we prove

PROPOSITION 2.4. *N has at most four distinct principal curvatures* $\sigma_1, \sigma_2, \tau_1, \tau_2$ *such that*

$$\sigma_1=\phi+\sqrt{-\beta(1+\phi^2)},\qquad \sigma_2=\phi-\sqrt{-\beta(1+\phi^2)},$$

$$\tau_1=\phi+\sqrt{\beta^2(1+\phi^2)},\qquad \tau_2=\phi-\sqrt{\beta^2(1+\phi^2)}.$$

Proof. Transvecting u^jv_i with (2.1) and making use of (1.1), we get

$$\lambda\{H(u,u)+H(v,v)-2\phi(1-\lambda^2)\}=0,$$

from which,

(2.14) $$\alpha+\gamma=2\phi \quad \text{in } N.$$

According to (2.8) and the equation above, we have

(2.15) $$\gamma=\phi(1+\beta) \quad \text{in } N.$$

By making use of (2.8) and (2.15), we see that (2.13) implies

$$\tau_1=\phi+\sqrt{\beta^2(1+\phi^2)}, \qquad \tau_2=\phi-\sqrt{\beta^2(1+\phi^2)}.$$

On the other hand, the equation (2.10) is reduced to

$$H_{jk}H_i{}^k-2\phi H_{ji}+(\beta+\phi\gamma)g_{ji}=(\alpha^2+\beta^2-2\phi\alpha+\beta+\phi\gamma)(u_ju_i+v_jv_i)/(1-\lambda^2),$$

because P is equal to zero in N. Therefore, eliminating α and γ from the equation above, we have

(2.10)′ $$H_{jk}H_i{}^k-2\phi H_{ji}+\{\beta+\phi^2(1+\beta)\}g_{ji}=\beta(1+\beta)(1+\phi^2)(u_ju_i+v_jv_i)/(1-\lambda^2).$$

Thus, for an eigenvalue σ associated with an eigenvector orthogonal to the plane section $P(u, v)$ spanned by u and v, we have the quadratic equation

(2.16) $$\sigma^2-2\phi\sigma+\{\beta+\phi^2(1+\beta)\}=0.$$

Thus we conclude the proof.

Since principal curvatures are real, Proposition 2.4 implies that β is non-positive. This fact plays an important role not only in the proof of the following lemma but also in the later discussions.

LEMMA 2.5. *The function ϕ is constant in N.*

Proof. Differentiating the second equation of (2.7) covariantly, we have

$$\nabla_i\beta_j=\nabla_iB_1u_j+B_1\nabla_iu_j+\nabla_iB_2v_j+B_2\nabla_iv_j,$$

from which, taking the skew-symmetric part,

$$\nabla_iB_1u_j-\nabla_jB_1u_i+\nabla_iB_2v_j-\nabla_jB_2v_i=B_1(\nabla_ju_i-\nabla_iu_j)+B_2(\nabla_jv_i-\nabla_iv_j)=2(B_1+\phi B_2)f_{ji}.$$

Since f is of rank $2n-2$ or of rank $2n$, the coefficient $2(B_1+\phi B_2)$ vanishes identically in $N\cup N_0$, i.e.,

(2.17) $$B_1+\phi B_2=0 \quad \text{in } N\cup N_0.$$

In a similar way, we obtain

(2.18) $$C_1+\phi C_2=0 \quad \text{in } N\cup N_0.$$

Differentiating the first equation of (2.8) covariantly, we also have

$$\alpha_j=\phi_j(1-\beta)-\phi\beta_j.$$

Thus, putting $\Phi_1=u^j\phi_j/(1-\lambda^2)$ and $\Phi_2=v^j\phi_j/(1-\lambda^2)$, we have

$$\Phi_2(1-\beta)=A_2+\phi B_2.$$

By means of this relation, (2.17) and the second one of (2.8), we obtain

$$(1-\beta)\Phi_2=0.$$

Since β is non-positive in N, Φ_2 vanishes identically in N and hence

$$\phi_j=\Phi_1 u_j.$$

Differentiating the equation above covariantly and taking the skew-symmetric part, we get

$$\nabla_j\Phi_1 u_i-\nabla_i\Phi_1 u_j+2\Phi_1 f_{ji}=0,$$

from which,

$$\Phi_1=0.$$

Therefore the function ϕ is constant in N. This completes the proof.

Suppose that there exists a connected component of the set N_0, which has a non-empty open kernel W.

LEMMA 2.6. *The open kernel W has at most four distinct principal curvatures*

$$\sigma_1=\phi+\sqrt{\phi^2-\phi\gamma-1}, \qquad \sigma_2=\phi-\sqrt{\phi^2-\phi\gamma-1},$$

$$\tau_1=(\gamma+\sqrt{\gamma^2+4})/2, \qquad \tau_2=(\gamma-\sqrt{\gamma^2+4})/2,$$

where the multiplicities of σ_1, σ_2, τ_1 and τ_2 are $n-1$, $n-1$, 1 and 1, respectively.

Proof. Since $\lambda_j=v_j-H_{ji}u^i=0$ in the open kernel W, we get $H(u,u)=0$ and $H(u,v)=1$. Thus (2.4) and (2.5) are reduced to

$$H_k{}^i u^k=v^i, \tag{2.19}$$

$$H_k{}^i v^k=u^i+\gamma v^i, \tag{2.20}$$

respectively, where $\gamma=H(v,v)$. Consequently, for the coefficients α and β appearing in (2.4), we get $\alpha=0$ and $\beta=1$. Equations (2.19) and (2.20) show that two eigenvalues, say τ_1 and τ_2, corresponding to eigenvectors belonging to the plane section $P(u,v)$ are distinct and that they satisfy the quadratic equation

$$\tau^2-\gamma\tau-1=0,$$

from which, we obtain

$$\tau_1=(\gamma+\sqrt{\gamma^2+4})/2, \qquad \tau_2=(\gamma-\sqrt{\gamma^2+4})/2.$$

Since we have obtained $\alpha=0$ and $\beta=1$, (2.10) is simplified as follows:

$$2H_{jk}H_i{}^k-4\phi H_{ji}+2(1+\phi\gamma)g_{ji}=Ru_ju_i+P(u_jv_i+u_iv_j)+Qv_jv_i. \tag{2.21}$$

For the eigenvalue σ associated with an eigenvector perpendicular to $P(u,v)$, we get by (2.21)

$$\sigma^2-2\phi\sigma+1+\phi\gamma=0, \tag{2.22}$$

from which,

$$\sigma_1=\phi+\sqrt{\phi^2-\phi\gamma-1}, \qquad \sigma_2=\phi-\sqrt{\phi^2-\phi\gamma-1}.$$

Thus there exist at most two distinct principal curvatures, say σ_1 and σ_2, at each point of W.

The equation (2.1) implies $H(fX)=(2\phi-\sigma_1)fX$ for an eigenvector X corresponding to the eigenvalue σ_1. This mean that fX is also an eigenvector with an eigenvalue σ_2. Thus the multiplicities of σ_1 and σ_2 are equal to $n-1$. This completes the proof.

LEMMA 2.7. *On the open kernel W, the function γ is constant.*

Proof. Substituting $\beta=1$ into (2.12), we obtain

$$2\{H_{ik}H_h{}^k-2\phi H_{ih}+(1+\phi\gamma)g_{ih}\}f_j{}^h=-C_1(u_jv_i-u_iv_j).$$

Transvecting u^j with the equation above, we have $C_1=0$. Hence (2.18) implies

$$\phi C_2=0 \qquad \text{in} \quad W.$$

Suppose that there exists a point p in W such that $\phi(p)=0$, we have

$$2H_{jk}H_i{}^k+2g_{ji}=Ru_ju_i+P(u_jv_i+u_iv_j)+Qv_jv_i \qquad \text{at} \quad p,$$

because of (2.21). This means that there exist principal curvatures $\pm\sqrt{-1}$ at p. This is a contradiction. Consequently, ϕ vanishes nowhere in W and then the function γ is necessarily constant in W. Thus Lemma 2.7 is proved.

LEMMA 2.8. *The function ϕ is constant in the open kernel W, if $n\geqq 3$.*

Proof. Since W has at most four distinct principal curvatures σ_1, σ_2, τ_1 and τ_2 with multiplicities $n-1$, $n-1$, 1 and 1 respectively, we get, using Lemma 2.6, by a straightforward computation

$$H_j{}^j=2(n-1)\phi+\gamma, \tag{2.23}$$

$$H_{ji}H^{ji}=2(n-1)(2\phi^2-\phi\gamma-1)+(\gamma^2+2). \tag{2.24}$$

Differentiating (2.2) covariantly, we have

$$\nabla_l f_j{}^k H_{ki}+f_j{}^k\nabla_l H_{ki}+\nabla_l H_j{}^k f_{ki}+H_j{}^k\nabla_l f_{ki}=2\phi_l f_{ji}+2\phi\nabla_l f_{ji}.$$

Transvecting this equation with g^{li} and making use of (2.23) and the first equation of (1.5), we obtain

$$2(n-2)f_j{}^k\phi_k+2\{H_i{}^i-\gamma-2(n-1)\phi\}u_j+\{H_{ik}H^{ik}+(\gamma-2\phi)(H_i{}^i-\gamma)-\gamma^2+2(n-2)\}v_j=0.$$

Since $n\geqq 3$, from (2.23) and (2.24), we get

$$f_j{}^k\phi_k=0,$$

that is,

$$\phi_j=\Phi_1 u_j+\Phi_2 v_j. \tag{2.25}$$

By means of Lemma 2.3, coefficients P, Q and R in the equation (2.10) are given by

$$P=2\gamma-4\phi,\qquad Q=2(\gamma^2-\phi\gamma+2),\qquad R=2\phi\gamma+4,$$

because of $\alpha=0$ and $\beta=1$. Taking account of the second and the third equations of (1.5), we have

$$\nabla_i u^i=\nabla_i v^i=0,\qquad u^i\nabla_i u_j=u^i\nabla_i v_j=v^i\nabla_i u_j=v^i\nabla_i v_j=0.$$

Consequently, applying $\nabla^i=g^{ij}\nabla_j$ to (2.10) and taking account of the relations above, we have

$$\begin{aligned}&2\nabla^i H_{jk}H_i{}^k+2H_{jk}\nabla^i H_i{}^k-4\phi^i H_{ji}-4\phi\nabla^i H_{ji}+2\gamma\phi_j\\&=2\gamma\phi^i u_j u_i-4\phi^i(u_j v_i+u_i v_j)-2\gamma\phi^i v_j v_i,\end{aligned} \tag{2.26}$$

where $\phi^i=g^{ij}\phi_j$. By the equation (1.7) of Codazzi and (2.24), the first term in the left hand side of (2.26) is given by

$$2\nabla^i H_{jk}H_i{}^k=\nabla_j(H_{ik}H^{ik})=2(n-1)(4\phi-\gamma)\phi_j.$$

Substituting (2.23), (2.25) and this equation into (2.26), we have

$$(2\Phi_2-\gamma\Phi_1)u_j+(2\Phi_1+\gamma\Phi_2)v_j=0,$$

from which,

$$2\Phi_2-\gamma\Phi_1=0,\qquad 2\Phi_1+\gamma\Phi_2=0.$$

Therefore, $\Phi_1=\Phi_2=0$ and hence the function ϕ is constant. This completes the proof.

Summing up Lemmas 2.5 and 2.8 and taking account of the fact that N_1 is a bordered set, we have

PROPOSITION 2.9. *The function ϕ appearing in* (2.1) *is constant in M, if* $n\geqq 3$.

As a direct consequence of Lemmas 2.6, 2.7 and 2.8, we have

PROPOSITION 2.10. *If* $n \geqq 3$, *then a connected open kernel* W *of the set* N_0 *has at most four distinct constant principal curvatures*

$$\sigma_1=\phi+\sqrt{\phi^2-\phi\gamma-1}, \qquad \sigma_2=\phi-\sqrt{\phi^2-\phi\gamma-1},$$

$$\tau_1=(\gamma+\sqrt{\gamma^2+4})/2, \qquad \tau_2=(\gamma-\sqrt{\gamma^2+4})/2,$$

with multiplicities $n-1$, $n-1$, 1 *and* 1, *respectively.*

§ 3. Hypersurfaces of constant scalar curvature.

In this section, we shall concern with a hypersurface M of constant scalar curvature in $S^{2n+1}(1)$ satisfying the condition (2.1). We shall prove the following theorem, which has been, however, proved in a previous paper [4], provided that ϕ is constant.

THEOREM 3.1. *Let* M *be a hypersurface in* $S^{2n+1}(1)$ *satisfying* (2.1) *and being of constant scalar curvature. If* $n \geqq 3$, *then one of the following assertions* (1), (2), (3) *and* (4) *is true*:

(1) *M is totally umbilic*;

(2) *M has exactly two distinct constant principal curvatures* $\phi+\sqrt{1+\phi^2}$, $\phi-\sqrt{1+\phi^2}$ *with the same multiplicity* n;

(3) *M has exactly two distinct constant principal curvatures* ϕ *with multiplicity* $2n-1$ *and* $-1/\phi$ *with multiplicity* 1;

(4) *M has exactly four distinct constant principal curvatures* $\phi+\sqrt{\phi^2-\phi\gamma-1}$, $\phi-\sqrt{\phi^2-\phi\gamma-1}$, $(-1+\sqrt{1+\phi^2})/\phi$, $(-1-\sqrt{1+\phi^2})/\phi$ *with multiplicities* $n-1$, $n-1$, 1 *and* 1, *respectively.*

We shall give outlines of the proof of Theorem 3.1 for completeness. To prove this theorem, we need Lemmas 3.2, 3.3 and 3.4 which will be stated later.

By Lemma 2.1, the transforms Hu and Hv of the vectors u and v by the second fundamental tensor H are linear combinations of u and v, that is,

$$H_k{}^j u^k=\alpha u^j+\beta v^j, \tag{3.1}$$

$$H_k{}^j v^k=\beta u^j+\gamma v^j \tag{3.2}$$

in $N \cup N_0$, where the set N consists of points x such that $1>\lambda^2(x)>0$ and the set N_0 consists of points x such that $\lambda(x)=0$. First, we prove

LEMMA 3.2. *The functions* α, β *and* γ *are constant in* N.

Proof. By taking account of equations (3.1) and (3.2), there exist two eigenvalues τ_1 and τ_2 of the second fundamental tensor corresponding to eigenvectors belonging to the plane section $P(u, v)$, and τ_1, τ_2 satisfy the quadratic equation

$$\tau^2-(\alpha+\gamma)\tau+\alpha\gamma-\beta^2=0. \tag{3.3}$$

Consequently we find $\tau_1+\tau_2=2\phi$, because of (2.14). Let σ be an eigenvalue associated with an eigenvector X perpendicular to $P(u, v)$. Then the condition (2.1) shows that $2\phi-\sigma$ is also an eigenvalue associated with the transforms fX of X by the linear transformation f. On the other hand, since (2.10)′ is reduced to

$$H_{jk}H_i^k-2\phi H_{ji}+\{\beta+\phi^2(1+\beta)\}g_{ji}=\beta(1+\beta)(1+\phi^2)(u_ju_i+v_jv_i)/(1-\lambda^2), \tag{3.4}$$

the eigenvalue σ satisfies

$$\sigma^2-2\phi\sigma+\beta+\phi^2(1+\beta)=0. \tag{3.5}$$

Thus there exist at most two distinct eigenvalues, say σ and $2\phi-\sigma$, associated with eigenvectors perpendicular to the plane section $P(u, v)$. Their multiplicities are all equal to $n-1$. Hence we have

$$H_j{}^j=(n-1)\sigma+(n-1)(2\phi-\sigma)+\tau_1+\tau_2,$$

from which,

$$H_j{}^j=2n\phi. \tag{3.6}$$

Thus, the mean curvature is constant in N.

Now, transvecting g^{ji} to (3.4), we get

$$H_{ji}H^{ji}-2\phi H_j{}^j+2n\{\beta+\phi^2(1+\beta)\}=2\beta(1+\beta)(1+\phi^2).$$

Thus, by (1.9), (3.6) and the equation above, the scalar curvature K is given by

$$K=-2(1+\phi^2)\{\beta-(2n-1)\}(\beta+n). \tag{3.7}$$

Since K is constant and ϕ is also constant in N by Lemma 2.5, so is β in N. Thus, by (2.8) and (2.14), α and γ are constant in N. Thus, Lemma 3.2 is proved.

LEMMA 3.3. *Each point in N is umbilic or N has two distinct constant principal curvatures $\phi+\sqrt{1+\phi^2}$, $\phi-\sqrt{1+\phi^2}$ with the same multiplicity n.*

Proof. Making use of the second equation of (2.11) and (2.14), we have

$$2\beta^2+2\beta+\gamma^2-\alpha\gamma=0. \tag{3.8}$$

from which,

$$(\beta+\phi\gamma)-(\alpha\gamma-\beta^2)=(2\beta^2+2\beta+\gamma^2-\alpha\gamma)/2=0,$$

that is,

$$\beta+\phi\gamma=\alpha\gamma-\beta^2. \tag{3.9}$$

Consequently, equation (3.3) coincides with equation (3.5), and therefore there exist at most two distinct principal curvatures τ_1 and τ_2 at each point in N, where

$$\tau_1=\phi+\sqrt{\beta^2(1+\phi^2)}, \qquad \tau_2=\phi-\sqrt{\beta^2(1+\phi^2)}.$$

Substituting (2.8) and (2.15) into (3.9), we have

$$\beta(1+\beta)(1+\phi^2)=0.$$

This implies that $\beta=0$ or $\beta=-1$. Thus it is evident that, in the case where $\beta=0$ in N, each point in N is umbilic and that, in the case where $\beta=-1$ in N, N has distinct constant principal curvature $\phi+(1+\phi^2)^{1/2}$ and $\phi-(1+\phi^2)^{1/2}$ with the same multiplicity n. This completes the proof.

LEMMA 3.4. *If $\phi^2-\phi\gamma-1>0$ in a connected open kernel W of N_0, then W has exactly four distinct constant principal curvatures*

$$\phi+\sqrt{\phi^2-\phi\gamma-1}, \qquad \phi-\sqrt{\phi^2-\phi\gamma-1},$$

$$(-1+\sqrt{1+\phi^2})/\phi, \qquad (-1-\sqrt{1+\phi^2})/\phi$$

with multiplicities $n-1$, $n-1$, 1 and 1, respectively.

If $\phi^2-\phi\gamma-1=0$ in a connected open kernel W of N_0, then W has exactly two distinct constant principal curvatures

$$\phi, \qquad -1/\phi$$

with multiplicities $2n-1$ and 1, respectively.

Proof. The eigenvalue σ associated with an eigenvector orthogonal to the plane section $P(u, v)$ satisfies the equation (2.22). This implies that

$$\phi^2-\phi\gamma-1\geqq 0.$$

By proposition 2.10, for eigenvalues $\sigma_1, \sigma_2, \tau_1$ and τ_2 obtained in Lemma 2.6, we have

$$\sigma_1=\sigma_2=\phi, \qquad \tau_1=\phi, \qquad \tau_2=-1/\phi,$$

or

$$\sigma_1=\sigma_2=\phi, \qquad \tau_1=-1/\phi, \qquad \tau_2=\phi,$$

if $\phi^2-\phi\gamma-1=0$.

Next, we consider the case where $\phi^2-\phi\gamma-1>0$. In this case, assuming $\sigma_1=\tau_1$, we obtain

$$\sqrt{\gamma^2+4}\sqrt{\phi^2-\phi\gamma-1}=0,$$

which contradicts $\phi^2-\phi\gamma-1>0$. Thus we have $\sigma_1\neq\tau_1$. In a similar way, we have

$\sigma_1 \neq \tau_2$, $\sigma_2 \neq \tau_1$ and $\sigma_2 \neq \tau_2$, if $\phi^2 - \phi\gamma - 1 > 0$. This implies that W has four distinct constant principal curvatures, if $\phi^2 - \phi\gamma - 1 > 0$. By Lemma 2.6, the multiplicities of σ_1 and σ_2 are equal to $n-1$. On the other hand, by virtue of a formula due to Cartan [2] for the hypersurface with constant principal curvatures in a sphere, we get

$$\frac{1+\tau_1\sigma_1}{\tau_1-\sigma_1} + \frac{1+\tau_2\sigma_1}{\tau_2-\sigma_1} + (n-1)\frac{1+\sigma_2\sigma_1}{\sigma_2-\sigma_1} = 0,$$

from which,

$$(\phi\gamma+2)(\sigma_1{}^2 - \gamma\sigma_1 - 1) = 0.$$

Since τ_1 and τ_2 are different from σ_1, we get

$$\phi\gamma + 2 = 0,$$

from which,

$$\tau_1 = (-1+\sqrt{1+\phi^2})/\phi, \qquad \tau_2 = (-1-\sqrt{1+\phi^2})/\phi.$$

This completes the proof.

Proof of Theorem 3.1. The function $\beta = H(u, v)/(1-\lambda^2)$ is defined and differentiable in $N \cup N_0$. We now see, from Proposition 2.4 and Lemma 3.2, that β is non-positive constant in N. On the other hand, (2.19) implies that β is equal to 1 in W. Therefore, W is necessarily empty or identical with M itself.

When W is empty, as consequences of Lemma 3.3, the assertions (1) and (2) stated in Theorem 3.1 are true. When $W = M$, as consequences of Lemma 3.4, the assertions (3) and (4) in Theorem 3.1 are true. Thus, Theorem 3.1 is proved completely.

Following Theorem 3.1, we now prove

THEOREM 3.5. *Let M be a complete hypersurface in $S^{2n+1}(1)$ satisfying* (2.1) *and being of constant scalar curvature. If $n \geqq 3$, then one of the following two assertions* (a) *and* (b) *is true*:

(a) *M is isometric to one of the following spaces*:

(1) *the great sphere $S^{2n}(1)$*;

(2) *the small sphere $S^{2n}(c)$, where $c = 1+\phi^2$*;

(3) *the product manifold $S^{2n-1}(c_1) \times S^1(c_2)$, where $c_1 = 1+\phi^2$ and $c_2 = 1+1/\phi^2$*;

(4) *the product manifold $S^n(c_1) \times S^n(c_2)$, where $c_1 = 2(1+\phi^2+\phi\sqrt{1+\phi^2})$ and $c_1 = 2(1+\phi^2-\phi\sqrt{1+\phi^2})$*;

(b) *M has exactly four distinct constant principal curvatures $\phi \pm \sqrt{1+\phi^2}$, $(-1 \pm \sqrt{1+\phi^2})/\phi$ of multiplicities $n-1$, $n-1$, 1 and 1, respectively.*

Proof. Suppose that the open kernel of any connected component of the set N_0 consisting of points x such that $\lambda(x)=0$ is empty. Then, Lemma 3.3 implies that each point in N is umbilic or that N has two distinct principal curvatures $\tau_1=\phi+(1+\phi^2)^{1/2}$, $\tau_2=\phi-(1+\phi^2)^{1/2}$ with the same multiplicity n. Thus, the principal curvatures of M itself has the same property as that stated above, because of continuity of principal curvatures. In the case where there exist two distinct ones, we have two distinct distributions D_1 and D_2 on M which assign the eigenspaces $D_1(x)$ and $D_2(x)$ to each point x in M, where $D_1(x)$ and $D_2(x)$ are eigenspaces of τ_1 and τ_2 respectively. The distributions D_1 and D_2 are of the same dimension n, and mutually orthogonal. Since each eigenvalue is constant, each distribution is involutive and parallel with respect to the Riemannian connection in M. Let M_i $(i=1,2)$ be a maximal integral manifold of D_i. Then M_i is totally geodesic, and M is locally Riemannian product of M_1 and M_2. Thus, integrating the equations of Gauss and Weingarten, we can verify that M is isometric to the product space $S^n(c_1)\times S^n(c_2)$, where $c_1=1+[\phi+(1+\phi^2)^{1/2}]^2$ and $c_2=1+[\phi-(1+\phi^2)^{1/2}]^2$. Thus, in the present case, only the case (4) of the assertion (a) occurs. In the other case, where each point in N is umbilic, only the cases (1) and (2) of the assertion (a) occur.

Next, suppose that there exists a connected component of N_0 which contains an interior point. Then it was proved in Theorem 3.1 that an open kernel is the hypersurface M itself. In the case where there are exactly two distinct constant principal curvatures, using similar divices as those developed above, we can verify that the case (3) of the assertion (a) occurs, if $\phi^2-\phi\gamma-1=0$, and the assertion (b) is true, if $\phi^2-\phi\gamma-1>0$. Thus Theorem 3.5 is proved.

§ 4. Compact hypersurfaces.

We prove in this section the following

THEOREM 4.1. *Let M be a compact hypersurface in $S^{2n+1}(1)$ satisfying* (2.1). *If $n\geqq 3$, then one of the following two assertions* (a) *and* (b) *is true*:

(a) *M is isometric to one of the following spaces*:

(1) *the great sphere $S^{2n}(1)$*;

(2) *the small sphere $S^{2n}(c)$, where $c=1+\phi^2$*;

(3) *the product manifold $S^{2n-1}(c_1)\times S^1(c_2)$, where $c_1=1+\phi^2$ and $c_2=1+1/\phi^2$*;

(4) *the product manifold $S^n(c_1)\times S^n(c_2)$, where $c_1=2(1+\phi^2+\phi\sqrt{1+\phi^2})$ and $c_2=2(1+\phi^2-\phi\sqrt{1+\phi^2})$*;

(b) *M has exactly four distinct constant principal curvatures $\phi\pm\sqrt{1+\phi^2}$, $(-1\pm\sqrt{1+\phi^2})/\phi$ with multiplicities $n-1$, $n-1$, 1 and 1, respectively.*

As is already seen in § 2 and § 3, a connected open kernel W of N_0 is empty or is identical with M itself and, when $W=M$, W has exactly two distinct constant principal curvatures ϕ, $-1/\phi$ or exactly four distinct constant principal curvatures $\phi\pm(1+\phi^2)^{1/2}$, $[-1\pm(1+\phi^2)^{1/2}]/\phi$. Consequently, by the proof of Theorem 3.5, in order to prove this theorem, it suffices to show that the function β is equal to 0 or -1 in the case where W is empty.

Now, in the sequel, suppose that W is empty. Thus in the following Lemmas 4.2, 4.3 and 4.4, we restrict ourselves to the case where W is empty. When the assumptions stated in Theorem 4.1 are satisfied, the function ϕ in the condition (2.1) must be constant by means of Lemmas 2.5 and 2.8. From Lemma 2.1, we see that the transforms Hu and Hv of u and v by the transformation H are linear combinations of u and v, i.e., in $N\cup N_0$

$$H_k{}^j u^k=\alpha u^j+\beta v^j, \tag{4.1}$$

$$H_k{}^j v^k=\beta u^j+\gamma v^j. \tag{4.2}$$

Moreover, we have already obtained in (2.8)

$$\alpha=\phi(1-\beta),\quad A_2=B_1 \qquad \text{in} \quad N\cup N_0. \tag{4.3}$$

The functions α, β and γ are defined and differentiable in $N\cup N_0$. We have also obtained in (2.15)

$$\gamma=\phi(1+\beta) \qquad \text{in} \quad N.$$

However, this equation is satisfied also in $N\cup N_0$, that is,

$$\gamma=\phi(1+\beta) \qquad \text{in} \quad N\cup N_0, \tag{4.4}$$

since N_0 is a bordered set. By means of Proposition 2.4, we get at most four distinct principal curvatures σ_1, σ_2, τ_1 and τ_2 such that

$$\begin{aligned} &\sigma_1=\phi+\sqrt{-\beta(1+\phi^2)}, \qquad \sigma_2=\phi-\sqrt{-\beta(1+\phi^2)},\\ &\tau_1=\phi+\sqrt{\beta^2(1+\phi^2)}, \qquad \tau_2=\phi-\sqrt{\beta^2(1+\phi^2)} \end{aligned} \tag{4.5}$$

at each point in N, and hence, N_0 being a bordered set, also in $N\cup N_0$ because of the continuity of principal curvatures. Under the condition (2.1), the multiplicities of σ_1 and σ_2 are $n-1$ and those of τ_1 and τ_2 are 1. Thus, the mean curvature is equal to $2n\phi$, which is constant. By means of (3.7), it follows from this fact that the scalar curvature K satisfies

$$K=-2(1+\phi^2)\{\beta-(2n-1)\}(\beta+n) \qquad \text{in} \quad N\cup N_0. \tag{4.6}$$

Since β is non-positive, solving the quadratic equation above, we have

$$\beta=(n-1-\sqrt{\Lambda})/2 \qquad \text{in} \quad N\cup N_0,$$

where

$$\Lambda=(n-1)^2-4\{K/2(1+\phi^2)-n(2n-1)\}\geqq 0 \qquad \text{in} \quad N\cup N_0.$$

We have $\Lambda>0$ in M, because N_1 is a bordered set. Hence we can define a function $\bar{\beta}$ in M by

$$\bar{\beta}=(n-1-\sqrt{\Lambda})/2. \tag{4.7}$$

Then, the function $\bar{\beta}$ thus defined is an extension of the function β which is defined only in $N\cup N_0$. Without fear of confusion, we denote the extended function by the same letter β. Thus we prove

LEMMA 4.2. *The function β is differentiable in M.*

On the set $N\cup N_0$, differentiating equation (4.4) covariantly and taking account of constantness of ϕ, we get $\gamma_j=\phi\beta_j$, and hence

$$C_1=\phi B_1. \tag{4.8}$$

By Lemma 2.3, we have

$$\lambda R=(B_2-C_1)(1-\lambda^2)=\lambda\{2(\alpha^2+\beta^2)-4\phi\alpha+2(\beta+\phi\gamma)\}$$

from which,

$$B_2=2\lambda\beta(1+\beta)/(1-\lambda^2) \qquad \text{in} \quad N\cup N_0 \tag{4.9}$$

because of (2.17), (4.3), (4.4) and (4.8). Taking account of (4.9), we prove

LEMMA 4.3. *$\beta(x)$ is equal to 0 or -1 at each point x in N_1.*

Proof. Since N_1 is also a bordered set, for an arbitrary but fixed point x in N_1, we can choose a sequence $\{x_j\}$ of points belonging to N such that x_j converges to x. Substituting (4.9) into the equation $\beta_j v^j=B_2(1-\lambda^2)$, we have

$$\beta_j v^j=2\lambda\beta(1+\beta) \qquad \text{in} \quad N\cup N_0. \tag{4.10}$$

Since the functions β, v and λ are differentiable in M and $v=0$ in N_1, from (4.10), we see that

$$\lim_{j\to\infty} 2\lambda\beta(1+\beta)(x_j)=\pm 2\beta(1+\beta)(x)=0.$$

This completes the proof.

Next, we shall show that β is equal to 0 or -1 in M. As is already shown, there exist at most four distinct principal curvatures $\sigma_1, \sigma_2, \tau_1$ and τ_2 at each point in M. Using (4.5), we obtain

$$\begin{aligned}&(1+\sigma_1\sigma_2)(\sigma_1-\sigma_2)^2+(1+\sigma_1\tau_1)(\sigma_1-\tau_1)^2+(1+\tau_1\sigma_2)(\tau_1-\sigma_2)^2\\ &\quad+(1+\tau_2\sigma_1)(\tau_2-\sigma_1)^2+(1+\sigma_2\tau_2)(\sigma_2-\tau_2)^2+(1+\tau_1\tau_2)(\tau_1-\tau_2)^2\\ &=-4(1+\phi^2)^2\beta(1+\beta)(1-\beta)(2-\beta).\end{aligned}$$

Denoting by $\kappa_1, \kappa_2, \cdots, \kappa_{2n}$ all of principal curvatures of M, we see that the equation above is equivalent to

$$\sum_{i<j}(1+\kappa_i\kappa_j)(\kappa_i-\kappa_j)^2=-4(1+\phi^2)^2\beta(1+\beta)(1-\beta)(2-\beta). \tag{4.11}$$

By a formula of Simon's type for the hypersurface of constant mean curvature in a sphere [5], we obtain

$$\frac{1}{2}\Delta(H_{ji}H^{ji})=\nabla_k H_{ji}\nabla^k H^{ji}+\sum_{i<j}(1+\kappa_i\kappa_j)(\kappa_i-\kappa_j)^2,$$

where Δ is denoted the Laplacian, i.e., Beltrami operator. Thus we have

$$\frac{1}{2}\Delta(H_{ji}H^{ji})=\nabla_k H_{ji}\nabla^k H^{ji}-4(1+\phi^2)^2\beta(1+\beta)(1-\beta)(2-\beta). \tag{4.12}$$

On the other hand, by (2.17) we get $B_1+\phi B_2=0$ and hence

$$\begin{aligned}\beta_j&=B_2(-\phi u_j+v_j),\\ \lambda_j&=(1-\beta)(-\phi u_j+v_j).\end{aligned} \tag{4.13}$$

Differentiating (4.9) covariantly and making use of the above equations, we have

$$\nabla_j B_2=\frac{2\beta(1+\beta)}{(1-\lambda^2)^2}\{3\lambda^2(1+\beta)+(1-\beta)\}(-\phi u_j+v_j), \tag{4.14}$$

from which, by simple computations,

$$\Delta\beta=\frac{2(1+\phi^2)\beta(1+\beta)}{1-\lambda^2}\{\lambda^2(3+3\beta-2n)+(1-\beta)\}. \tag{4.15}$$

Since the mean curvature is constant, using (1.9) and (4.6), we get by a straightforward calculation,

$$\begin{aligned}&\Delta(H_{ji}H^{ji})=-\Delta K\\ &=\frac{4(1+\phi^2)^2\beta(1+\beta)}{1-\lambda^2}[(2\beta-n+1)\{2\lambda^2(\beta-n+2)+(1-\beta)(1-\lambda^2)\}+4\lambda^2\beta(1+\beta)].\end{aligned}$$

Combining (4.12) and the equation above, we find

$$\begin{aligned}\nabla_k H_{ji}\nabla^k H^{ji}=\frac{2(1+\phi^2)^2\beta(1+\beta)}{1-\lambda^2}&[2\lambda^2\{4\beta^2-(3n-7)\beta+(n-1)(n-2)\}\\ &-(n-5)(1-\lambda^2)(1-\beta)].\end{aligned} \tag{4.16}$$

Making use of (4.16), we prove the following lemma which is required to prove Theorem 4.1.

LEMMA 4.4. *β is equal to* 0 *or* -1 *in M.*

Proof. In the case where $n \geqq 6$, since the left hand side of (4.16) is non-negative in M, so is the right hand side in M and hence in N_0. This implies

$$-2(n-5)(1+\phi^2)^2\beta(1+\beta)(1-\beta)(x) \geqq 0 \quad \text{at} \quad x \in N_0,$$

from which, we get

$$-1 \leqq \beta(x) \leqq 0 \quad \text{at} \quad x \in N_0,$$

because the function β is non-positive. Since M is compact, the function β has the minimum at a point p in M. Supposing $\beta(p) < -1$, we see by Lemma 4.3 that p belongs to N. Let U be a suitable neighbourhood of p in N such that $\beta(x) < -1$ for any point x in U. Since $\beta(p)$ is the minimum and $\beta(1+\beta)$ is positive in U, (4.9) shows that $\lambda=0$ at p, that is, p belongs to N_0. This is a contradiction. Thus we have

$$-1 \leqq \beta \leqq 0 \quad \text{in} \quad M.$$

Then the right hand side of (4.12) is non-negative and hence, by the well-known theorem of Green (cf. [8]), we have

$$4(1+\phi^2)^2\beta(1+\beta)(1-\beta)(2-\beta)=0 \quad \text{in} \quad M.$$

This implies that $\beta(1+\beta)$ vanishes identically in M. Consequently, in the case where $n \geqq 6$, the assertion of Lemma 4.4 is true.

When $5 \geqq n \geqq 3$, since the quadratic polynomial $4\beta^2-(3n-7)\beta+(n-1)(n-2)$ is non-negative, taking account of the right hand side of (4.15), we see that

$$\beta(1+\beta) \geqq 0.$$

By the continuity of β, it follows that β vanishes identically or that β is not greater than -1.

Suppose that β is not greater than -1. Since M is compact, there exists a point q in M such that $\beta(q)$ is the maximal value on M. Furthermore, suppose that q is the point in the set $N \cup N_0$. We now define a linear and elliptic differential operator L of the second order in $N \cup N_0$ defined by

$$L = g^{ji}\frac{\partial^2}{\partial x^j \partial x^i} + h^k \frac{\partial}{\partial x^k},$$

$\{x^h\}$ being local coordinates of M, where

$$h^k = \frac{\lambda}{1-\lambda^2} \cdot \frac{3k(1-\lambda^2)-3\sqrt{D}-4n}{k(1-\lambda^2)-\sqrt{D}-4}\lambda^k - g^{ji}\left\{ {k \atop j\ i} \right\},$$

$$D = k^2(1-\lambda^2)^2 - 4k(1-\lambda^2),$$

and $\{{}_j{}^k{}_i\}$ is the Christoffel's symbol formed with the Riemannian metric tensor g in M, k is a non-positive constant as will be stated later. Then the function β satisfies the equation

$$L(\beta)=\frac{2(1+\phi^2)}{1-\lambda^2}\beta(1+\beta)(1-\beta). \tag{4.17}$$

In fact, using (4.9) and (4.13), we get the differential equation

$$(1-\beta)\beta_j=\frac{2\lambda\beta(1+\beta)}{1-\lambda^2}\lambda_j \qquad \text{in} \quad N\cup N_0, \tag{4.18}$$

from which,

$$(1+\beta)^2=k\beta(1-\lambda^2) \qquad \text{in} \quad N\cup N_0. \tag{4.19}$$

By the definition of h^k, we see that the first and the last terms of $L(\beta)$ is reduced to $\Delta\beta$. Next, we consider the second term of $L(\beta)$. By (4.19), we may suppose that k is a negative constant, because β is equal to -1 if k is assumed to be zero. Then, it follows from (4.19) that

$$\frac{3k(1-\lambda^2)-3\sqrt{D}-4n}{k(1-\lambda^2)-\sqrt{D}-4}=-\frac{3(1+\beta)-2n}{1-\beta}.$$

Thus we have

$$L(\beta)=\Delta\beta-\frac{\lambda}{1-\lambda^2}\,\frac{3(1+\beta)-2n}{1-\beta}\lambda^i\beta_i.$$

Since (4.18) implies, together with (1.5), $\lambda^i\beta_i=B_2(1-\beta)(1+\phi^2)(1-\lambda^2)$, the equation above becomes

$$L(\beta)=\Delta\beta-\frac{2(1+\phi^2)}{1-\lambda^2}\lambda^2\beta(1+\beta)(3+3\beta-2n).$$

By virtue of this equation and (4.15), we have (4.17).

Combining $\beta<-1$ and (4.17), we get

$$L(\beta)\geqq 0 \qquad \text{in} \quad U.$$

By a theorem due to Hopf [3, 7] this means that β is constant in U, so that B_2 is equal to 0 in U. By (4.9), we get

$$\lambda\beta(1+\beta)=0.$$

Hence, in the case where q is a point in $N\cup N_0$, β must be equal to -1, because the set N_0 has no interior points.

Next, suppose that q is a point belonging to N_1. Then, taking account of the fact that N_1 is a bordered set, we can choose a sequence $\{x_j\}$ of points belonging to N such that x_j converges to the point q. We may treat the subject in the case where $\beta(x_j) < -1$ for arbitrary points x_j. By (4.14), we obtain

$$\nabla_j B_2 \nabla^j B_2 = \frac{4(1+\phi^2)}{(1-\lambda^2)^3} \beta^2(1+\beta)^2\{3\lambda^2(1+\beta)+(1-\beta)\}^2,$$

Combining the equation above with (4.19), we have

$$(\nabla_j B_2 \nabla^j B_2)(x_i) = \frac{4k^3(1+\phi^2)}{(1+\beta)} \beta^5\{3\lambda^2(1+\beta)+(1-\beta)\}^2(x_i).$$

On the other hand, $(\nabla_j B_2 \nabla^j B_2)(x_i)$ converges to $(\nabla_j B_2 \nabla^j B_2)(q)$, because β is differentiable. Thus the right hand side of the equation above should converge. Therefore, because of $\beta(q) = -1$ given in Lemma 4.3, we obtain

$$\lim_{i\to\infty} k^3\beta^5\{3\lambda^2(1+\beta)+(1-\beta)\}^2(x_i) = 0,$$

from which,

$$-4k^3 = 0.$$

This implies that β must be equal to -1 in M, because of (4.19). We now conclude the proof of Theorem 4.1 completely.

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TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY.